CI-Beam-105

Lattice Design and Computational Dynamics I

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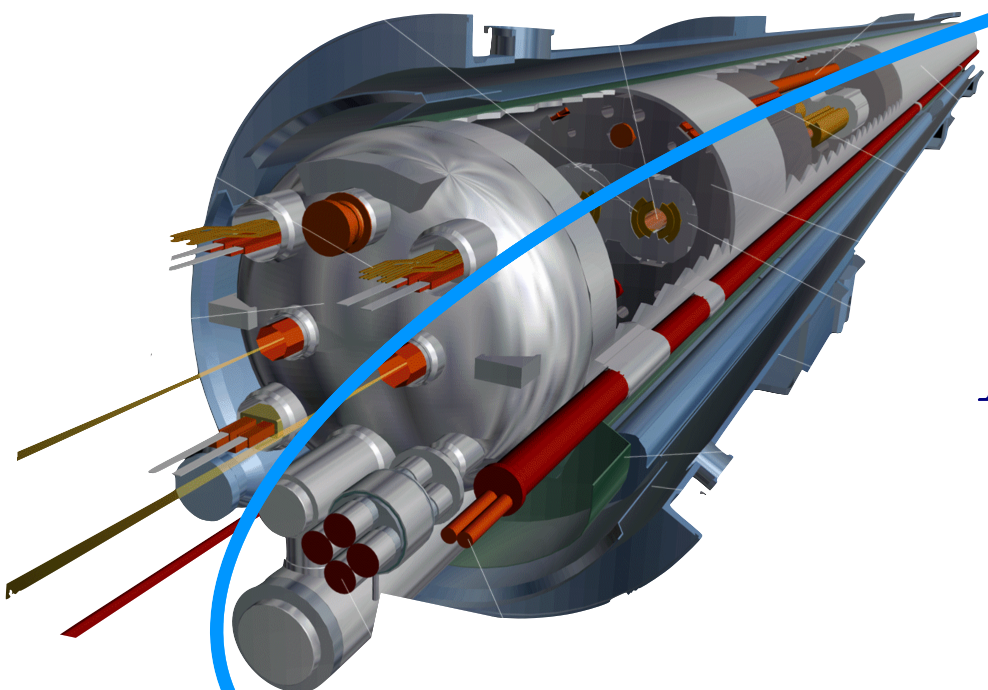
Beam Rigidity

Particles travel following a designed circular orbit in ring-type accelerators.

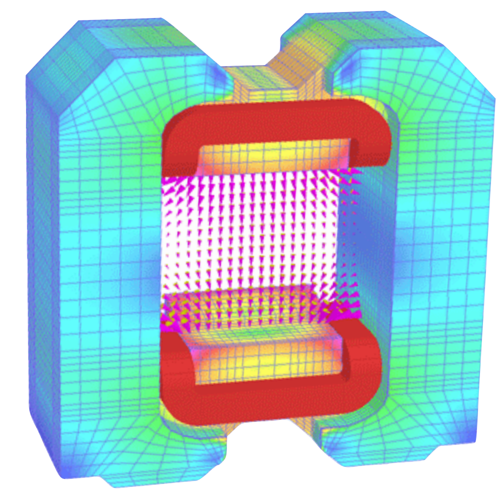
- By means of dipole magnets,
- Relying on the balance between the centripetal force and Lorentz force.



$$F_{Lorentz} = q\mathbf{E} + q[\mathbf{v} \times \mathbf{B}]$$

$$F_{centripetal} = \frac{mv^2}{\rho}$$

A Dipole Magnet



$$B.\rho = \frac{p}{q}$$

Magnetic guide field

Transverse size of the beam is negligible in comparison to the radius of its trajectory (accelerator circumference), therefore one can approximate the magnetic field about the particle trajectory using Taylor series.

Taylor Series: $$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$$

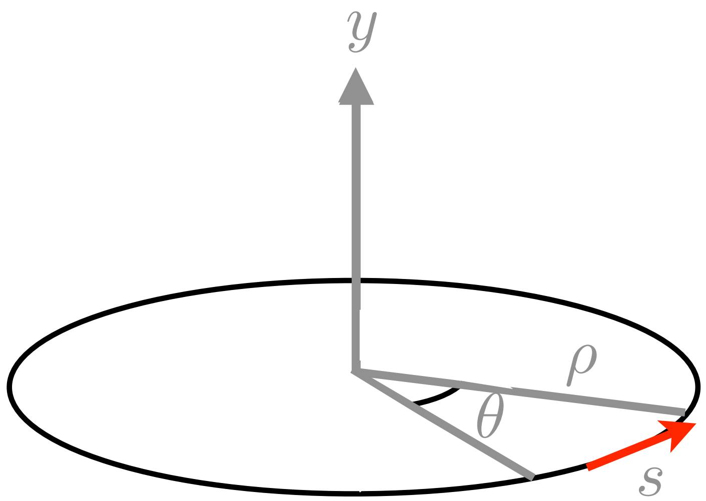
Taylor expansion of the magnetic field B in horizontal axis perpendicular to the axis of movement of the beam:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{d^3 B_y}{dx^3} x^3 + \ldots$$

normalise with respect to momentum, p/e

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} + \frac{g}{p/e} x + \frac{1}{2!} \frac{g'}{p/e} x^2 + \frac{1}{3!} \frac{g''}{p/e} x^3 + \ldots$$

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + kx + \frac{1}{2!} m x^2 + \frac{1}{3!} o x^3 + \ldots$$



circular coordinate system

Number of Magnetic Poles	Magnetic Strength	Effect
2 (Dipole)	$\frac{1}{\rho} = \frac{e}{p} B_{z0}$	Steering
4 (Quadrupole)	$k = \frac{e}{p} \frac{dB_z}{dx}$	Focusing
6 (Sextupole)	$m = \frac{e}{p} \frac{d^2 B_z}{dx^2}$	Chromaticity compensation
8 (Octupole)	$o = \frac{e}{p} \frac{d^3 B_z}{dx^3}$	Compensation of field errors
etc.	...	...

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + kx + \frac{1}{2!} m x^2 + \frac{1}{3!} o x^3 + \ldots$$

Magnetic field strength of a dipole magnet

$$\frac{1}{\rho} = \frac{e}{p}B \dashrightarrow \frac{1}{\rho} = \frac{e}{\gamma m v}B \dashrightarrow \frac{1}{\rho} = \frac{ec}{\gamma m \beta c c}B \dashrightarrow \boxed{\frac{1}{\rho} = \frac{ec}{E\beta}B}$$

$$B\rho[T.m] = \frac{1}{ec}\beta E \dashrightarrow B\rho[T.m] = \frac{1}{0.2998}\beta E[GeV]$$

reminder

$$p = \gamma m v$$
$$v = \beta c$$

Normalised field strength $\frac{1}{\rho}[m^{-1}] = \frac{0.2998.B_0(T)}{p(GeV/c)}$

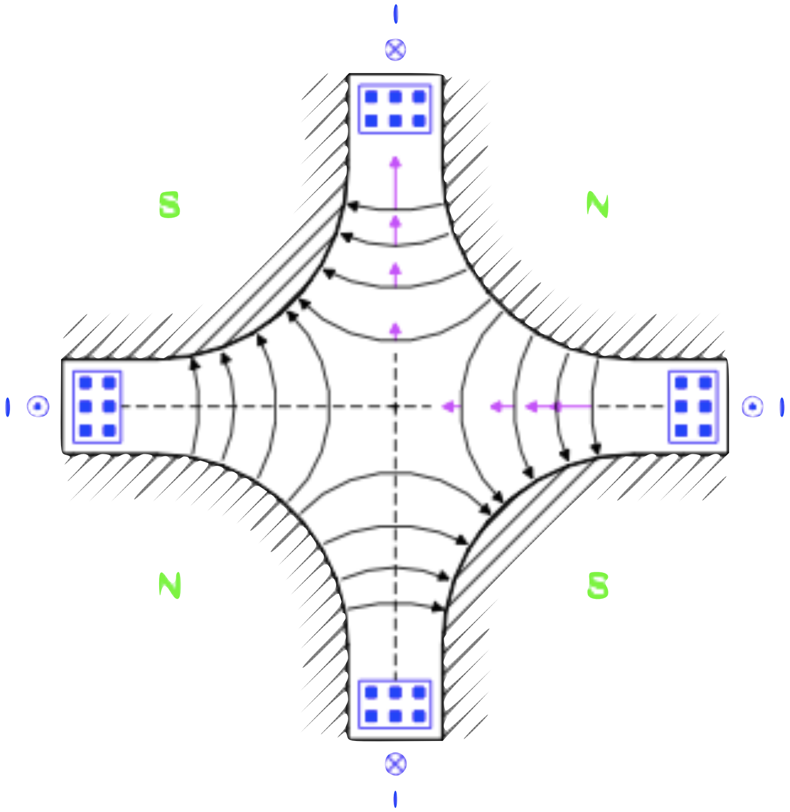
Magnetic field strength of a quadrupole magnet

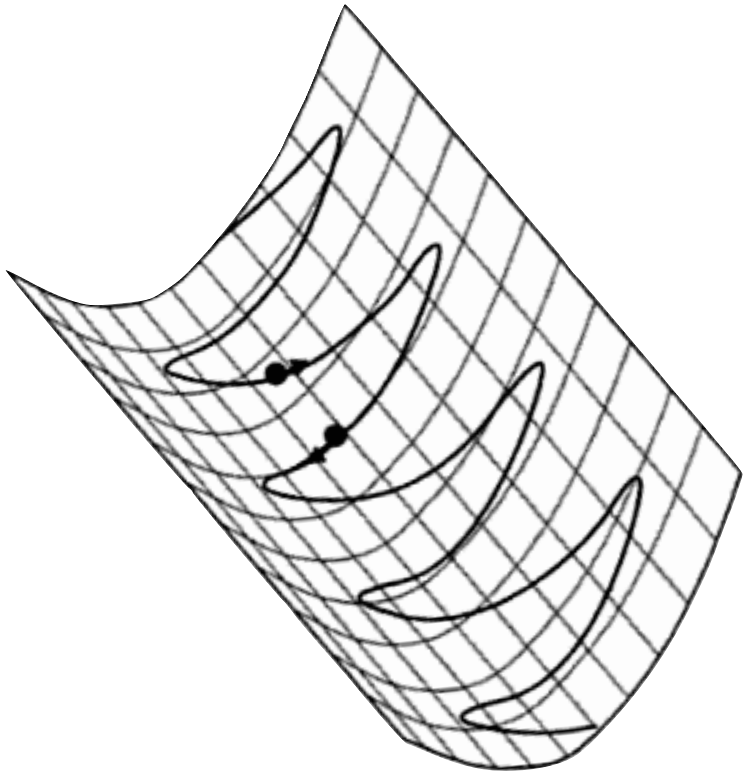
A linearly increasing Lorentz force: $B_x = gy \quad B_y = gx$

Normalised field strength: $$k[m^{-2}] = \frac{0.2998.g}{p(GeV/c)}$$

Focal length of a quadrupole: $$f = \frac{1}{k \cdot l_q}$$

Quadrupole Magnet






CERN, PS 1959

CERN, SPS 1976

General approximations for upcoming slides

- Calculations are done using the reference particle moving on the design orbit unless otherwise is stated.

- For all other particles must satisfy the below condition to be considered within the beam.

$$x,\, y << \rho$$

- Only linear terms of x and y components of the magnetic guide field will be considered.

Radial acceleration

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit

$$\rho = constant \qquad \frac{d\rho}{dt} = 0$$

$$F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$$

$$F = mv^2/\rho = m\rho\omega^2$$

General trajectory

$$\rho \to \rho + x$$

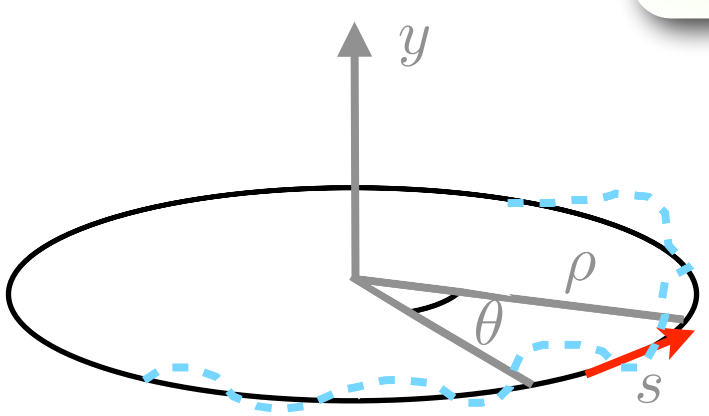
$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = eB_y v$$

$$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$

$$x \approx mm \qquad \rho \approx m$$

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = eB_y v$$



Circular coordinate system

Equation of motion

- **Linear terms of the guide field**

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev(B_0 + x\frac{\partial B_y}{\partial x})$$

:m

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{evB_0}{m} + \frac{evxg}{m}$$

- **Change of independent variable: t -> s**

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(\frac{dx}{ds}\frac{ds}{dt}) = \frac{d}{ds}(\frac{dx}{ds}\frac{ds}{dt})\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''v^2 + \frac{dx}{ds}\frac{dv}{ds}v$$

0, no acceleration

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{evB_0}{m} + \frac{evxg}{m}$$

:v^2

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

mv=p

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{xg}{p/e}$$

g/(p/e)=k

$$x'' + x(\frac{1}{\rho^2} - k) = 0$$

Previously,

- Radial accelerations on and off the orbit,
- Consider linear terms of the guide field,
- Change independent variable,
- Normalise to particle momentum.

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = eB_y v$$

$$x'' + x(\frac{1}{\rho^2} - k) = 0$$

$$k = 0, x'' = -\frac{1}{\rho^2}x$$

Note: There is a restoring force provided by dipole magnets on a ring in the absence of quadrupole magnets: "Weak focusing".

Previously,

- Radial accelerations on and off the orbit,
- Consider linear terms of the guide field,
- Change independent variable,
- Normalise to particle momentum.

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = eB_y v$$

$$x'' + x(\frac{1}{\rho^2} - k) = 0$$

Equation for the vertical motion

$\frac{1}{\rho^2} = 0$ no dipoles...in general...

$k \leftrightarrow -k$ quad field changes sign

$$y'' + ky = 0$$

$$x'' + x(\frac{1}{\rho^2} - k) = 0$$

Definition:

In the horizontal plane: $K = \frac{1}{\rho^2} - k$

In the vertical plane: $K = k$

$$x'' - Kx = 0$$

Equation of motion of an harmonic oscillator!

Equation of Motion

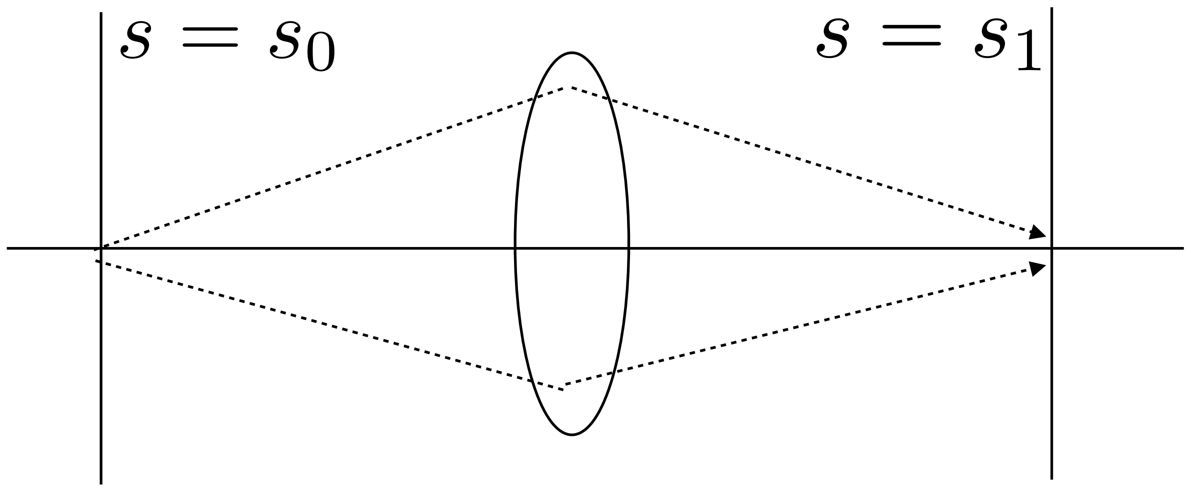
General solution of the harmonic oscillator:

$$x(s) = x_0 cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K}sin(\sqrt{K}s) + x_0' cos(\sqrt{K}s)$$

$$M = \begin{pmatrix} cos\sqrt{|K|}s & \frac{1}{\sqrt{|K|}} sin\sqrt{|K|}s \\ -\sqrt{|K|}sin\sqrt{|K|}s & cos\sqrt{|K|}s \end{pmatrix}$$

Focusing $K > 0$

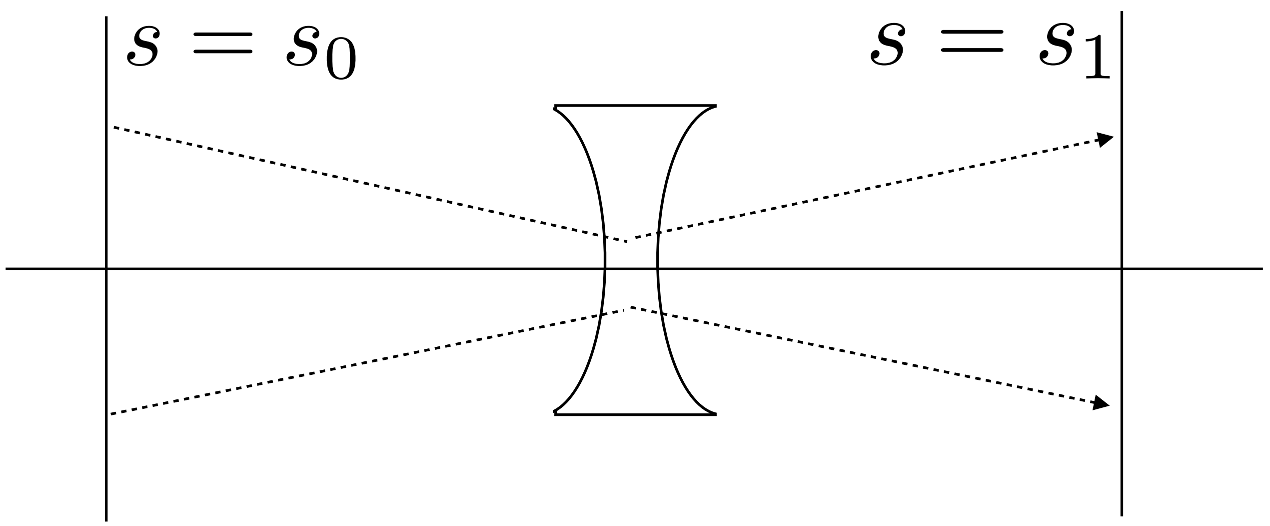


One can calculate (x_1,x_1') points for a particle at position S_1 using that particle's initial coordinates (x_0,x_0') at S_0 and the "transfer matrix" between S_0 and S_1.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$M = \begin{pmatrix} cos\sqrt{|K|}s & \frac{1}{\sqrt{|K|}} sin\sqrt{|K|}s \\ -\sqrt{|K|} sin\sqrt{|K|}s & cos\sqrt{|K|}s \end{pmatrix}$$

Defocusing $K < 0$



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Thin lens approximation

For practicality…

$$f = \frac{1}{kl_q} >> l_q$$

$$l_q \rightarrow 0$$

$$kl_q = constant$$

Generally a magnet length is an order of magnitude smaller than its focal length.

$$\begin{pmatrix} cos(\sqrt{|K|}s) = 1 - \frac{s^2|K|}{2} + ... & \frac{1}{\sqrt{|K|}} sin(\sqrt{|K|}s) = s - \frac{s^3|K|}{6} \cdots \\ -\sqrt{|K|} sin(\sqrt{|K|}s) = -s|K| + \frac{s^3|K|^2}{6} - ... & cos(\sqrt{|K|}s) = 1 - \frac{s^2|K|}{2} + ... \end{pmatrix}$$

Reminder

Trigonometric functions [edit] https://en.wikipedia.org/wiki/Taylor_series

The usual trigonometric functions and their inverses have the following Maclaurin series:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \quad \text{for all } x$$

For practicality...

$$f = \frac{1}{kl_q} >> l_q$$

$$l_q \rightarrow 0$$

$$kl_q = constant$$

Generally a magnet length is an order of magnitude smaller than its focal length.

matrix of a defocusing quadrupole

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

matrix of a focusing quadrupole

$$M_{QD} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} cos\sqrt{|K|}s & \frac{1}{\sqrt{|K|}} sin\sqrt{|K|}s \\ -\sqrt{|K|} sin\sqrt{|K|}s & cos\sqrt{|K|}s \end{pmatrix}$$

If there is no magnets along the trajectory...

$$K = 0$$

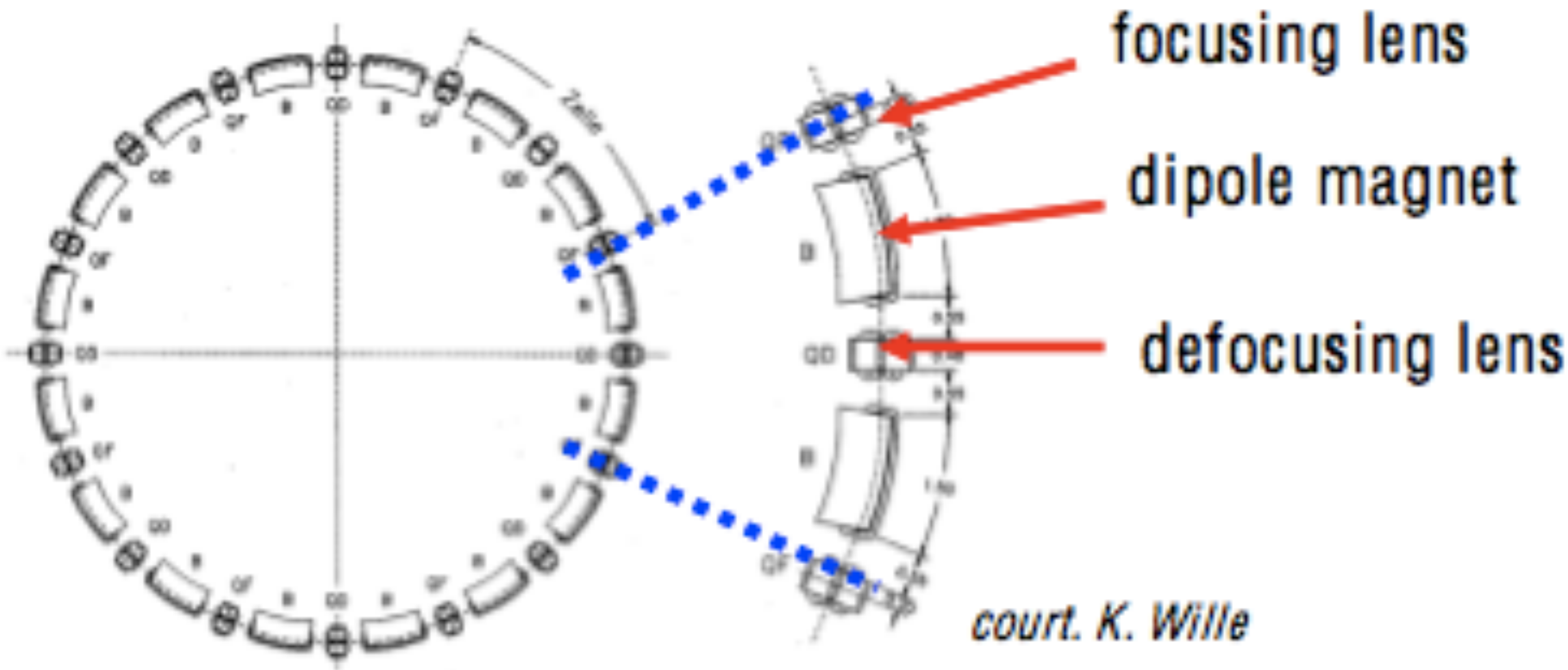
$$\lim_{k \to 0} \frac{sin\sqrt{|k|}x}{\sqrt{|k|}} = x$$

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Transfer matrix for a drift space.

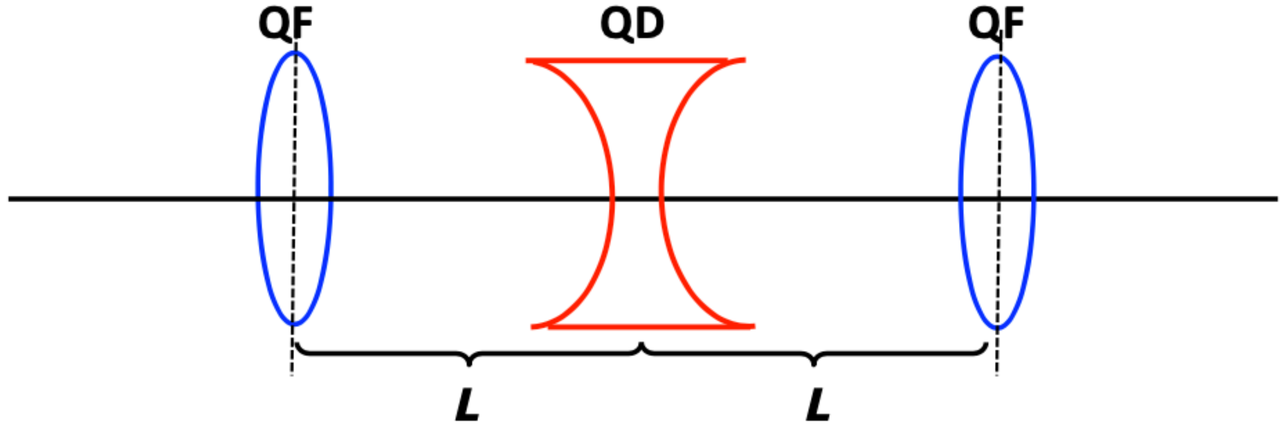
Transfer matrix for a section of many elements is found by multiplying the transfer matrices of individual elements in this section.

$$M_{total} = M_{QF} * M_D * M_{Bend} * M_D * M_{QD} * M_D * M_{Bend} * M_{D^*} \ldots$$



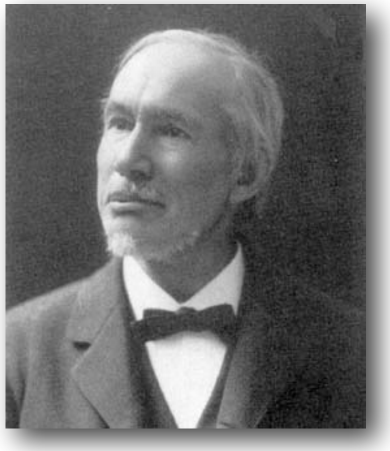
Symmetric transfer matrix with respect to the centre of quadrupole magnets.

$$M_{FODO} = M_{HQF} \cdot M_{Drift} \cdot M_{QD} \cdot M_{Drift} \cdot M_{HQF}$$



$$M_{HQF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \qquad M_{Drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \qquad M_{QD} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$



Equation of motion under periodic focusing conditions...

George William Hill (1838 - 1914) Mathematician - Astronomer

http://www-history.mcs.st-andrews.ac.uk/Biographies/Hill.html

Hill's Equation

$$x''(s) - k(s)x(s) = 0$$

k(s) indicates that the focusing properties change as a function of position along the lattice.

General Solution

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}cos(\psi(s) + \phi)$$

(1)

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}cos(\psi(s)+\phi) \qquad x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}[\alpha(s)cos(\psi(s)+\phi)+sin(\psi(s)+\phi)]$$

(2)

$$cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\epsilon}\sqrt{\beta(s)}} \qquad sin(\psi(s)+\phi) = -\frac{\beta x' + x\alpha}{\sqrt{\beta(s)}\sqrt{\epsilon}}$$

(3)

$$cos^2(\psi(s)+\phi) = \frac{x^2(s)}{\epsilon\beta(s)}$$

$$sin^2(\psi(s)+\phi) = \frac{1}{\epsilon\beta}(\beta^2(s)x'^2(s) + 2\beta(s)\alpha(s)x'(s)x(s) + \alpha^2(s)x^2(s))$$

(4)

$$sin^2(\psi(s)+\phi) + cos^2(\psi(s)+\phi) = 1$$

(5)

$$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Parametric representation of beam emittance in terms of Twiss parameters, α, β, γ.

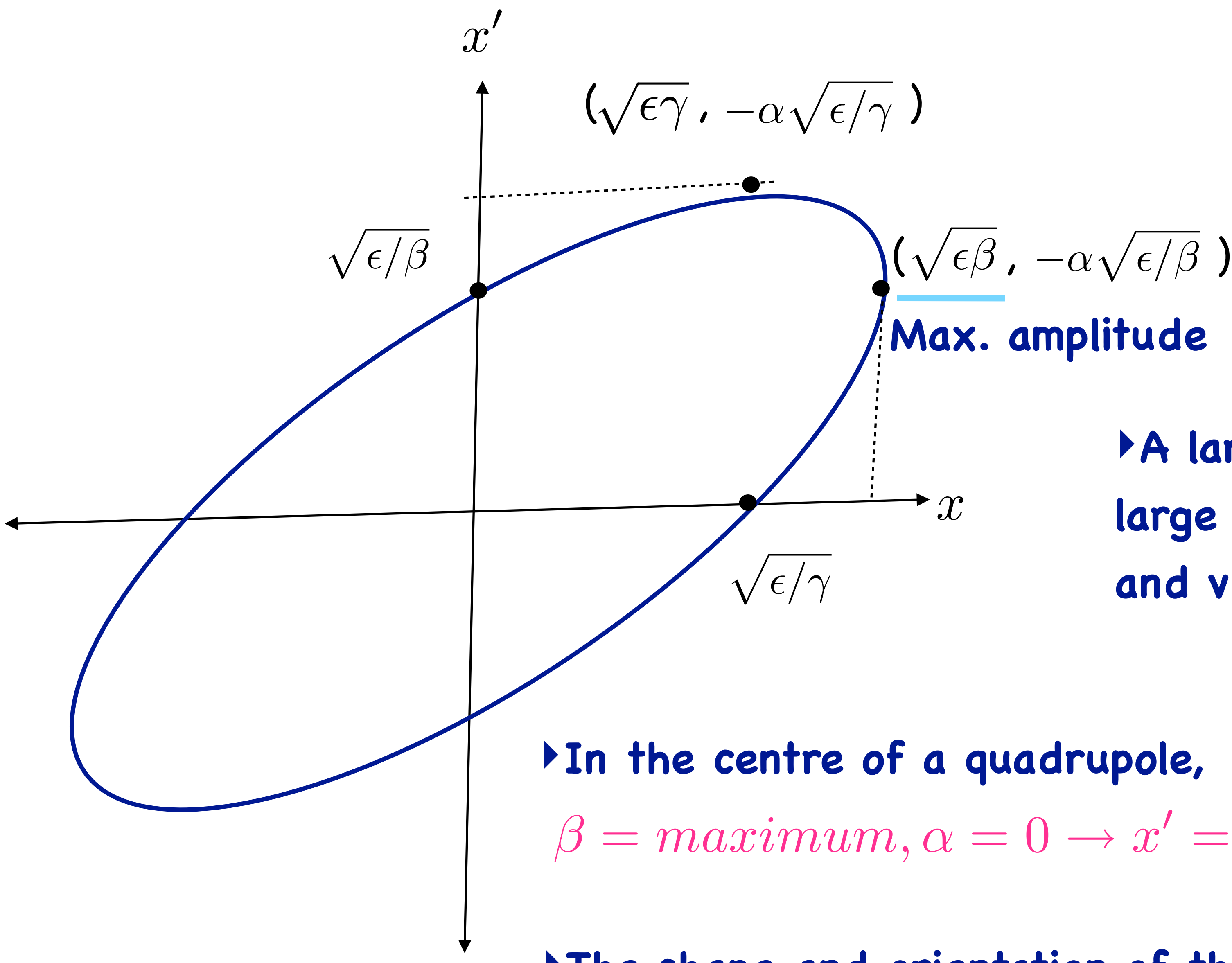
Note:

$$\alpha(s) = -\frac{1}{2}\beta'(s) \qquad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

The area that the beam occupies in the phase space is a conserved quantity.

- Envelope of this area in x-x' space is a an ellipse parametrised as a function of s coordinate.

$$\epsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$



- A large beta function might indicate a large beam radius and a small divergence and vice verse.

- In the centre of a quadrupole,

$$\beta = maximum, \alpha = 0 \rightarrow x' = 0$$

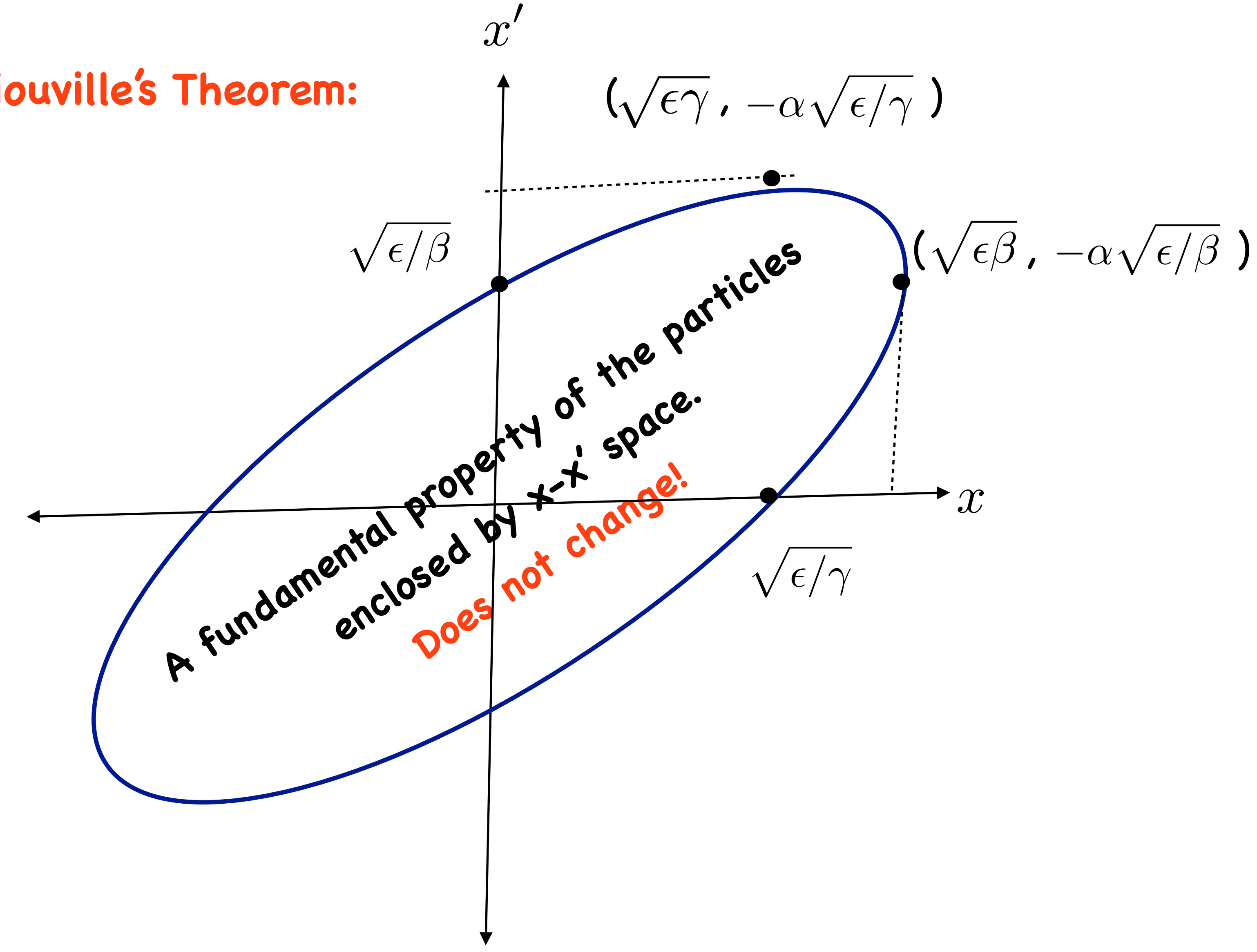
- The shape and orientation of this ellipse is determined by the Twiss parametes, α, β, γ.

The area that the beam occupies in the phase space is a conserved quantity.

- Envelope of this area in x-x' space is a an ellipse parametrised as a function of s coordinate.

$$\epsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

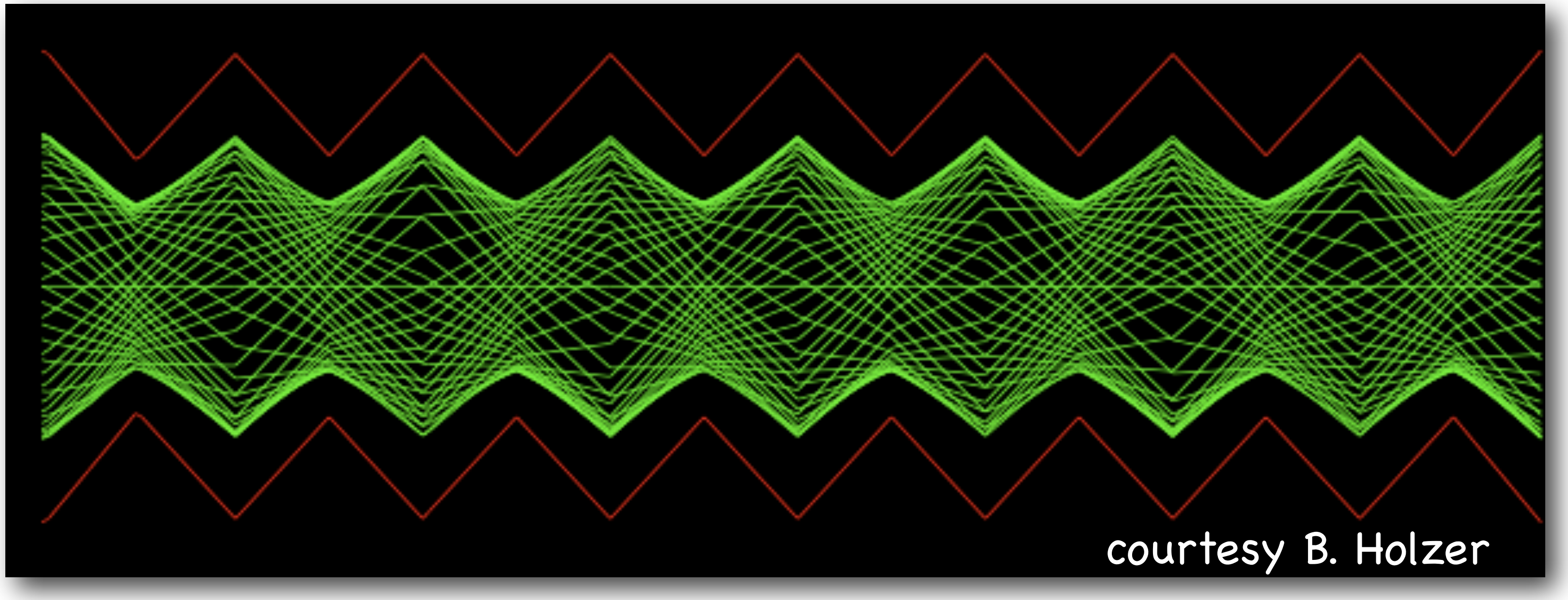
Liouville's Theorem:



Normalised emittance:

$$\epsilon^* = (\gamma_r\beta_r)\epsilon$$

The matrix formalism given in earlier slides could not provide information about the collective behaviour of the beam particles.



Transverse beam envelope oscillations, x(s), about the ideal beam orbit is called "Betatron oscillations".

Beta function is a periodic function defined by the properties of the magnetic lattice across the accelerator.

$$\beta(s+L) = \beta(s)$$

Phase advance between "0" and "s"

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Number of betatron oscillations per a full turn around the machine is called the "tune" of the machine.

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Betatron tune is important to be able to define the particle movement in the transverse plane. In an ideal accelerator (ideal magnets and perfect alignment) and for a monochromatic beam, betatron tune can be at any value depending on the quadrupole strengths in the lattice. However, in reality, small errors in magnetic fields and the alignment of the elements are unavoidable. Therefore, in order to prevent the instabilities caused by such errors betatron tune of a machine has to be selected very carefully.

A simple example case: Let's consider a ring working at an integer tune and having dipoles with certain magnet field errors. In this case particles will arrive at the perturbation region with the same phase relation at every turn. Therefore, the kick due to the field error will add up systematically at every turn and the amplitude of the betatron oscillation will increase until the particles are lost on the machine apertures.

Tune Resonances

- A horizontal and a vertical tune value are defined for accelerator rings: Qx and Qy.
- For high order magnets, the field strength in one plane is related to the field strength in the other transverse plane. Hence, the betatron oscillations are coupled in these two planes.

$$pQ_x + qQ_y = m$$

m, p, q: integer numbers

- The degree of the resonance is given as |p| + |q|.
- Optical resonances and the multipole fields that caused them.

Source Field	Resonance Condition
Dipole	Q=p
Quadrupole	2Q=p
Sextupole	3Q=p
Octupole	4Q=p
etc.	...

- Qx and Qy pair defined for an accelerator is called the working point of that machine.
- As the strength of a resonance significantly decreases by its degree, generally, only resonances up to 5th degree are considered.

Resonance Diagram

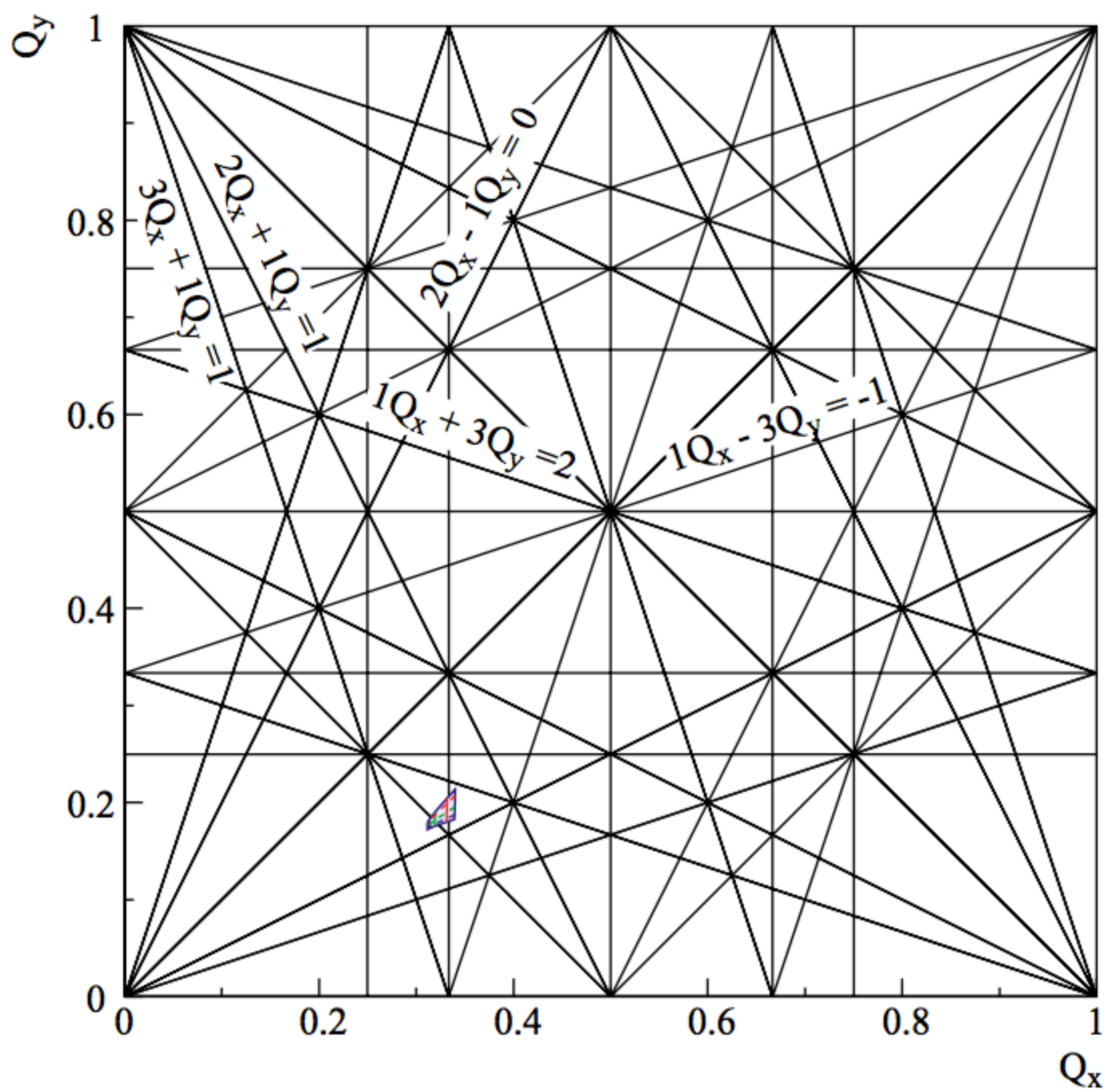


Figure 7: Illustration of a tune diagram for resonances up to 4th order. The typical tune area, occupied by a colliding beam at LEP1 is also shown as shaded area ($Q_x \approx 0.31 - 0.34$ and $Q_y \approx 0.17 - 0.214$).

CERN-SL-2000-037-DI

https://jwenning.web.cern.ch/jwenning/documents/lepmain_sl.pdf

$$pQ_x + qQ_y = m$$

m, p, q: integer numbers

Tune combinations that cause unwanted resonances can be shown in a tune diagram. The area occupied in the tune space by a beam is called the **"tune footprint"** of that beam.

Performance of an accelerator and the particle background in a collider are related to the tune footprint of that accelerator.

Particle Trajectory for off momentum particles

From page 11:
$$x'' - \frac{1}{\rho}(1 - \frac{x}{\rho}) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$p = p_0 + \Delta p$

Repeat the calculation taking into account a small momentum error:

$$\Delta p << p_0 \rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

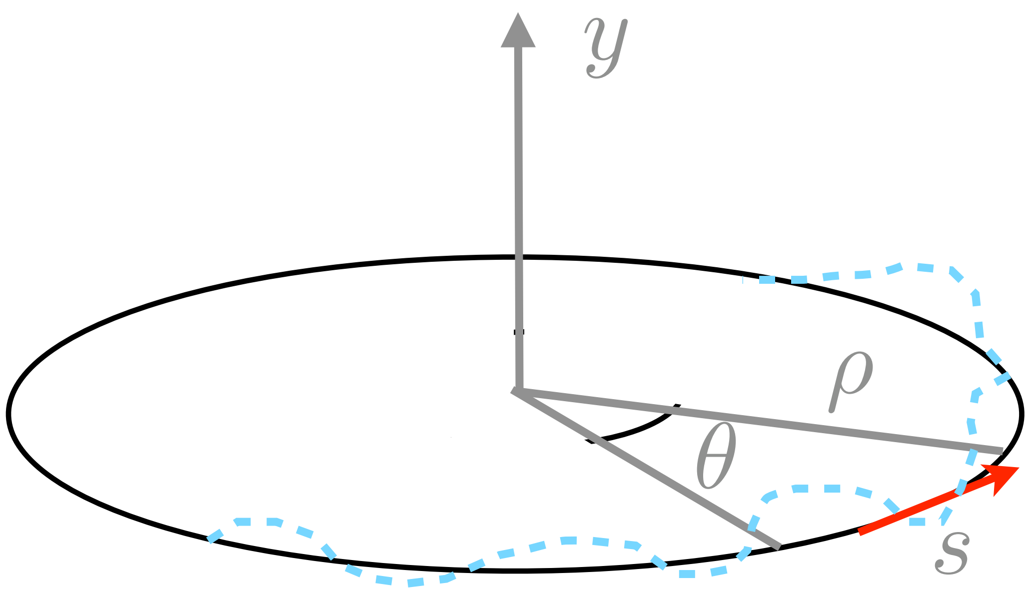
$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{eB_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{exg}{p_0}}_{k * x} - \underbrace{xeg \frac{\Delta p}{p_0^2}}_{\approx 0 \ (x, \Delta p \rightarrow small)}$$

$$x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

Particle Trajectory for off momentum particles

$$\Delta p/p \neq 0$$

- Let's investigate the case where the momentum spread of the beam is nonzero.



$$x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

Solution to the inhomogeneous equation of motion:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

Particle Trajectory for off momentum particles

Matrix Formalism

$$x(s) = x_\beta(s) + D(s) \cdot \Delta p/p$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \Delta p/p$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

- The orbit of an ideal particle is defined for dp/p = 0.
- The orbit of an arbitrary particle is defined by considering an additional term due to the momentum spread of the beam.
- Therefore, D(s) defines an orbit according to the focusing properties of the lattice.
- Dispersion is caused by the dipole magnets.
- And it needs to be zero, for example, at the interaction point of a collider.

Particle Trajectory for off momentum particles

Momentum compaction factor

It relates the particles momentum spread to the lengthening of the orbit through the dispersion function of the beam.

Orbit lengthening for off-momentum particles.

$$\alpha_{cp} = \frac{1}{L} \oint \frac{D(s)}{\rho(s)} ds$$

$$\frac{\delta l_{\epsilon}}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

Particle Trajectory for off momentum particles

Quadrupole Errors and Chromaticity

- Quadrupole errors cause tune shift
- ΔQ, is proportional to the beta function in a quadrupole.
- Chromaticity is a quantity which relates the tune shift and momentum spread.

$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

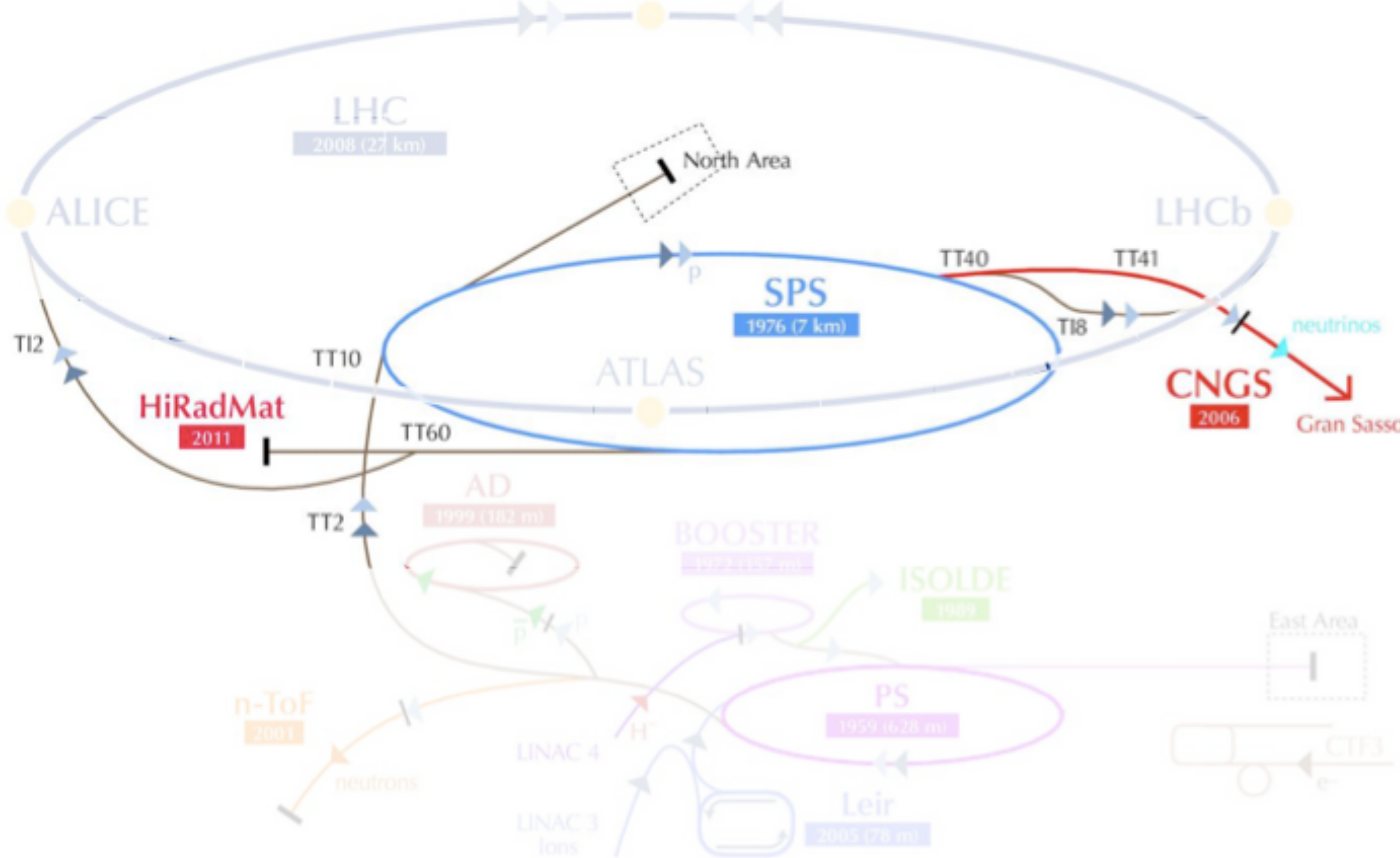
$$Q' = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$

$$\Delta Q = Q'\frac{\Delta p}{p}$$

Total chromaticity in a lattice including the contribution from sextupoles.

$$Q' = \mp\frac{1}{4\pi}\oint \beta(s)\ [k(s)\ +\ m(s)D(s)]\ ds$$

Calculation of the beam size at CERN's SPS extraction point before the beam is extracted into the LHC.



Calculation of the beam size at CERN's SPS extraction point before the beam is extracted into the LHC.

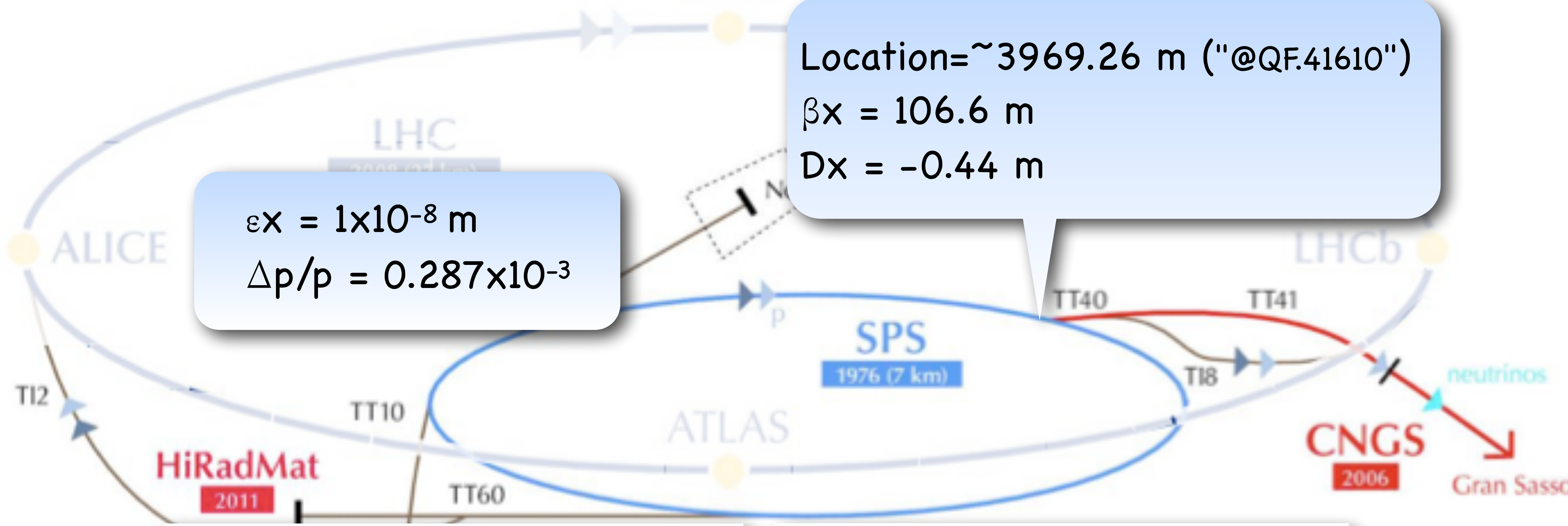
$\varepsilon x = 1\times10^{-8}$ m

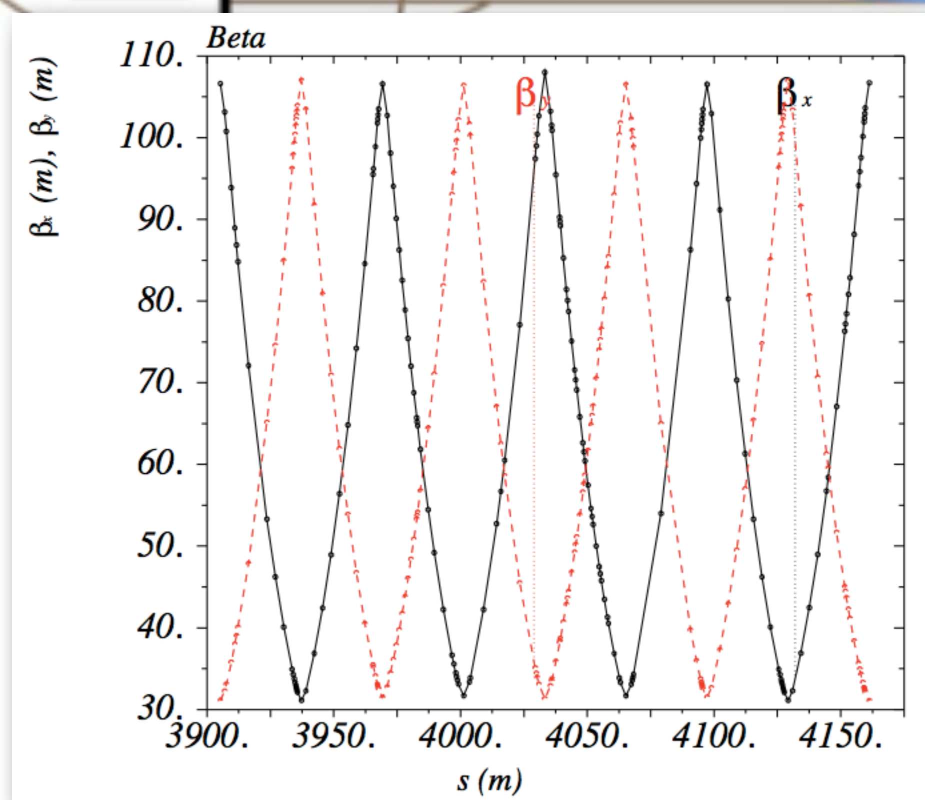
$\Delta p/p = 0.287\times10^{-3}$

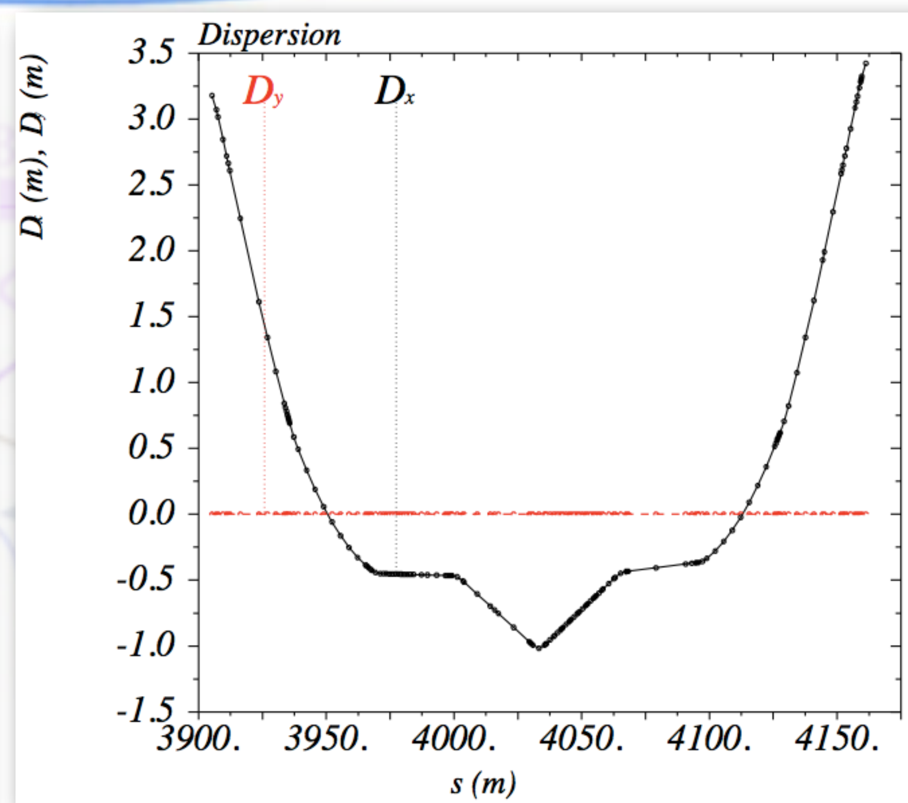
Location=~3969.26 m ("@QF.41610")

βx = 106.6 m

Dx = -0.44 m







Exercise

Calculation of the beam size at CERN's SPS extraction point before the beam is extracted into the LHC.

Location=~3969.26 m ("@QF.41610")
βx = 106.6 m
Dx = -0.44 m

εx = 1×10^{-8} m
$\Delta p/p$ = 0.287×10^{-3}

Quadratic Sum

$$\sigma_{x,y} = \sqrt{\epsilon_{x,y}\beta_{x,y} + (D_x \frac{\Delta p}{p})^2}$$

Linear Sum

$$\sigma_{x,y} = \sqrt{\epsilon_{x,y}\beta_{x,y}} + |D_x \frac{\Delta p}{p}|$$

Tolerances

$$\sigma_{x,y} = \tau\sqrt{\epsilon_{x,y}\beta_{x,y}} + \tau|D_x \frac{\Delta p}{p}| + c$$

Consider the error of τ = 1.1 and a mechanical tolerance of c = 100 μm and calculate the beam size as linear and quadratic sum.