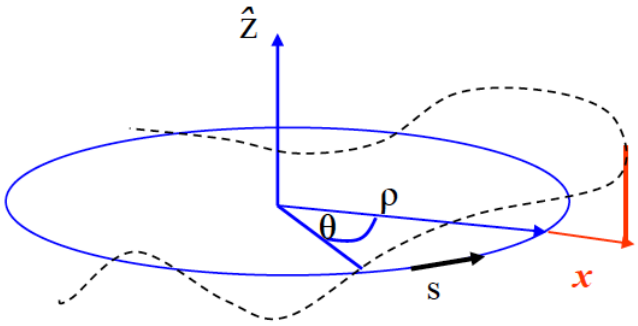
PART THREE- TRANSFER MATRICES, STABILITY AND TUNES

Recall - Hill's Equation and Solutions



$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$
$$z'' - kz = 0$$

Piecewise Solution

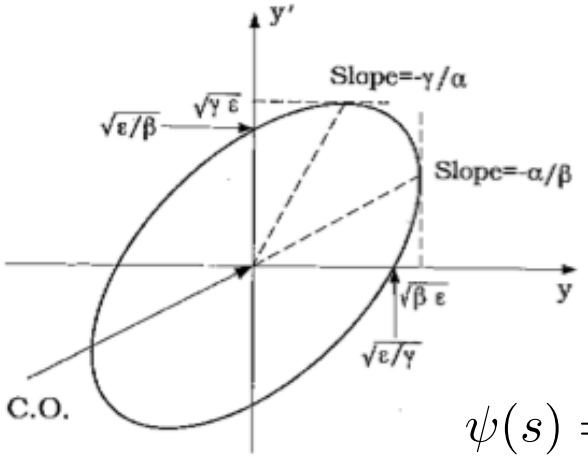
$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Courant-Snyder Solution

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$



$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

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The beta function

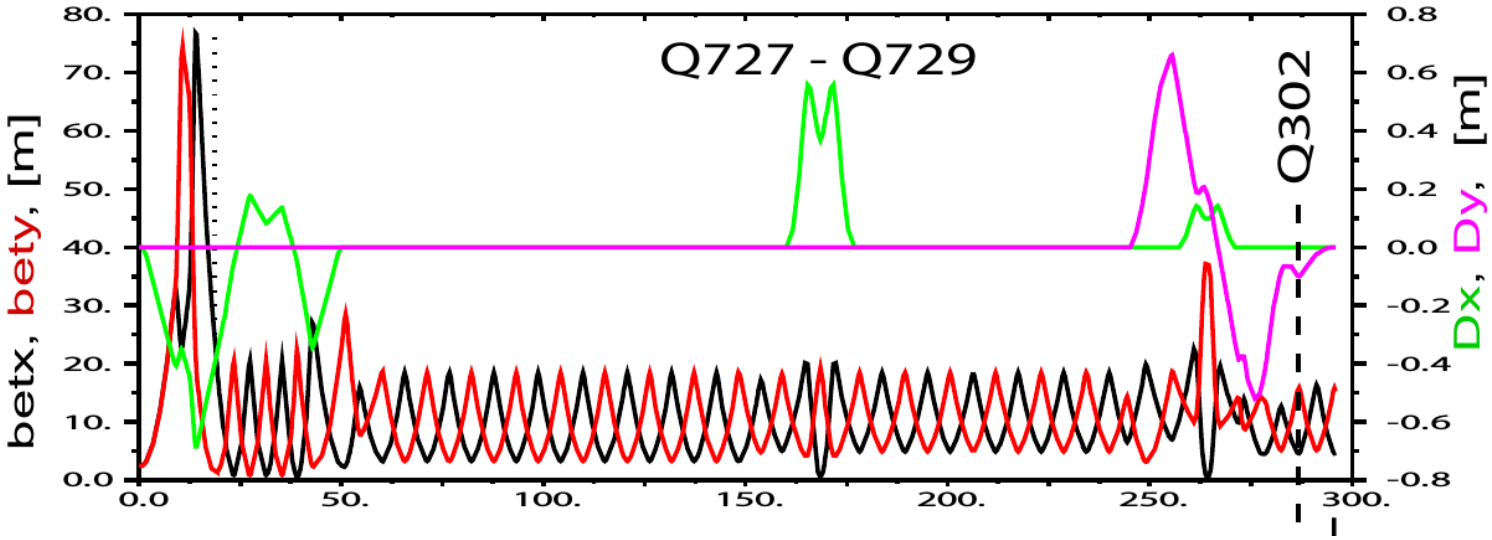
The beta function is a central quantity in the Courant-Synder formalism

It is a positive function of position in the machine, and has the same periodicity as the lattice itself.

It is determined only by the focusing properties of the lattice.

It is maximised in a focusing quadrupole and minimized in a defocusing quadrupole.

Below is a typical example from a transfer line at the g-2 experiment at Fermilab showing betatron oscillations.



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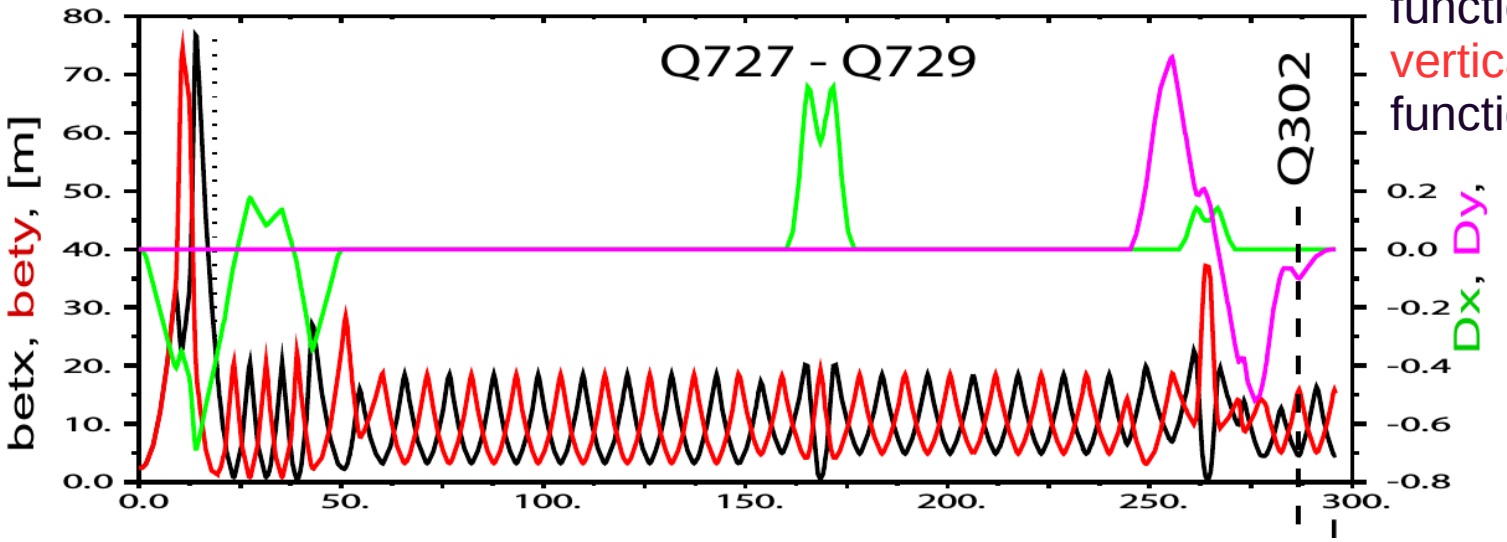

Note that there is a horizontal beta function and a vertical beta function.

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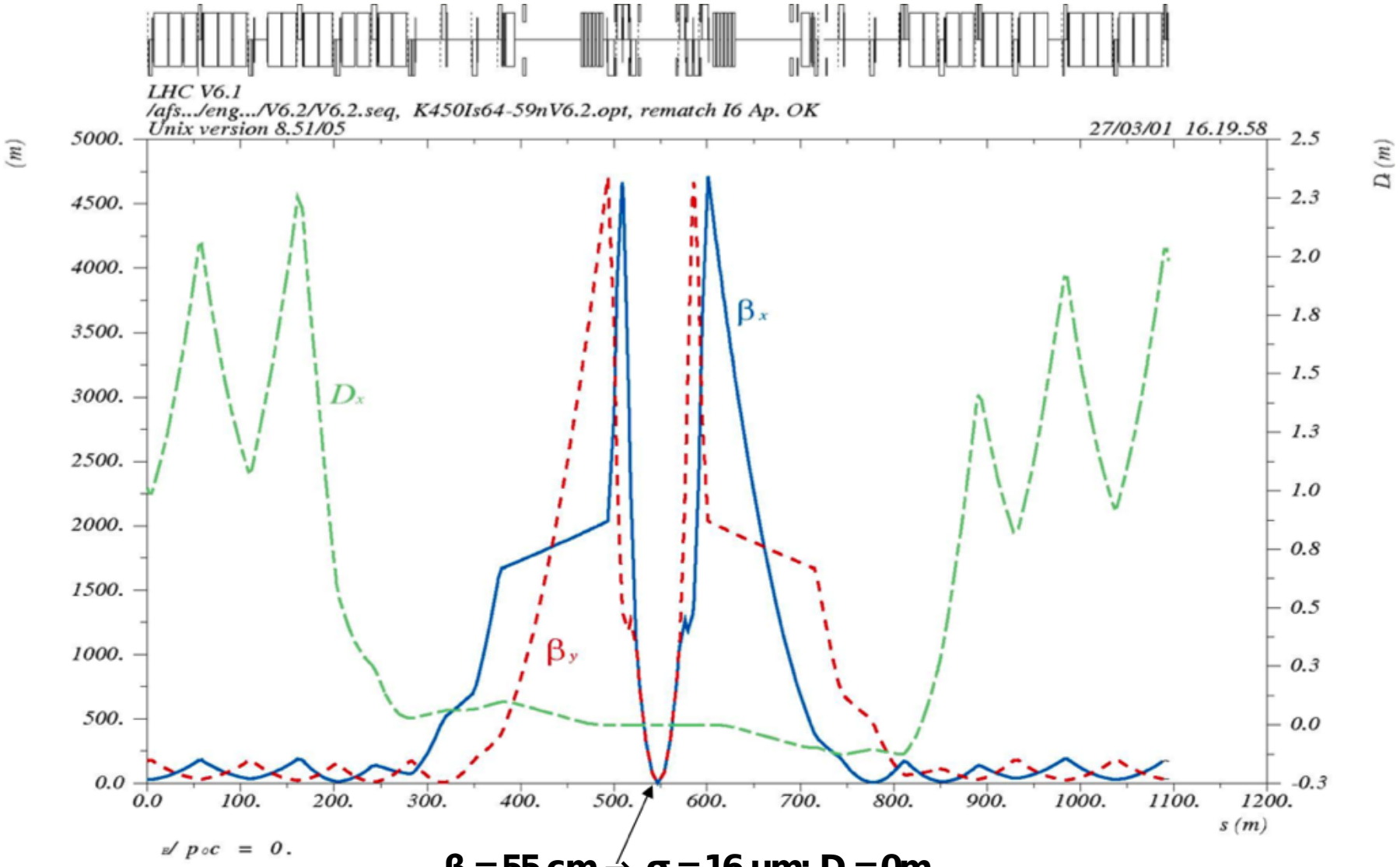
The beta functions of the LHC



β = 55 cm → σ = 16 μm; D = 0m

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The transfer matrix in terms of Courant-Snyder parameters / functions / lattice functions

Can we write a general transfer matrix between any two points in terms of the ‘lattice functions’?

The transfer matrix in terms of Courant-Snyder parameters / functions / lattice functions

Can we write a general transfer matrix between any two points in terms of the lattice functions? To begin with, we return to the Courant-Snyder form of the solution to Hill's equation, but written slightly differently

$$x(s) = c_1\sqrt{\beta(s)}\cos\psi(s) + c_2\sqrt{\beta(s)}\sin\psi(s)$$

where c_1 and c_2 are constants yet to be determined. If we define the initial conditions at the point '0' to be

$$\beta(0) = \beta_0 \quad \alpha(0) = \alpha_0 \quad \psi(0) = 0$$

and write the initial particle coordinates to be x_0 and x_0' then we can fix the unknown constants.

We also need to recall

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)} \qquad \alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds}$$

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$$c_1 = \frac{x_0}{\sqrt{\beta_0}} \qquad c_2 = \sqrt{\beta_0}x_0' + \frac{\alpha_0}{\sqrt{\beta_0}}x_0$$

And so we can write x(s) in the form

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0\sin\psi(s)\right]x_0 + \sqrt{\beta_0\beta(s)}x_0'\sin\psi(s)$$

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The transfer matrix in terms of the Courant-Snyder parameters / functions / lattice

As in the piecewise solutions, we see the expression for x(s) is linear in x_0 and x_0'.

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0 \sin\psi(s)\right] x_0 + \sqrt{\beta_0\beta(s)}\, x_0' \sin\psi(s)$$

Taking the derivative of this expression, we can cast this equation into a convenient matrix form (as it's linear)

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M(s_1|s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

where

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

$$\psi = \psi(s_1) - \psi(s_0)$$

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$$\psi = \psi(s_1) - \psi(s_0)$$

The subscripts 0 and 1 refer to the beginning and end of the transfer map.

This means the transfer matrix between two points is purely determined by the lattice functions at each point and the phase advance between the points!

The one-turn map

The one turn (one period) map is a very useful quantity (we mentioned it previously for piece-wise solutions to Hill's equation).

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

The map for one turn of the ring means we come back to the same s position, and so

$$\beta_1 = \beta_0 = \beta \quad \alpha_1 = \alpha_0 = \alpha \quad \gamma_1 = \gamma_0 = \gamma$$

And so the one turn map is

$$M(s+C|s) = \begin{pmatrix} \cos(\Psi) + \alpha\sin(\Psi) & \beta\sin(\Psi) \\ -\gamma\sin(\Psi) & \cos(\Psi) - \alpha\sin(\Psi) \end{pmatrix}$$

where we have used $\gamma(s) = \dfrac{1+\alpha^2(s)}{\beta(s)}$

and the phase advance for one turn is:

$$\psi_1 - \psi_0 = \Psi$$

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The one-turn map

If we know this map we can determine the lattice functions.

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We can multiply all the piece-wise matrices for all the elements in the ring together to obtain the total matrix for one turn of the machine

$$M\left(s+C|s\right)=\prod_{i}M\left(s_{i+1}|s_{i}\right)$$

$$M(s+C|s)=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

which we can compare with

$$M(s+C|s)=\begin{pmatrix} \cos(\Psi)+\alpha\sin(\Psi) & \beta\sin(\Psi) \\ -\gamma\sin(\Psi) & \cos(\Psi)-\alpha\sin(\Psi) \end{pmatrix}$$

The one-turn map

We now have

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We can get the one-turn phase from the trace of this matrix!

$$\Psi = \arccos\left(\frac{m_{11} + m_{22}}{2}\right)$$

We can get the lattice function from the other matrix elements.

$$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi} \qquad \gamma = -\frac{m_{21}}{\sin\Psi}$$

Note for the phase advance to be real-valued and hence **stable**, we need

$$|\mathrm{Tr}M| \leq 2$$

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The one-turn map at a different location

If we know the one-turn map at one location, s, is there a way to calculate it at another location, s', provided we know the transfer matrix M for s to s'?

The answer is yes. They are related to each other by the similarity transform

$$M(s'+C|s') = M(s'|s) \cdot M(s+C|s) \cdot M^{-1}(s'|s)$$

Similarity transforms come from matrix theory. They preserve eigenvalues, traces, etc.

We'll now denote the matrix M(s'|s) (i.e. the map from s to s') by

$$M(s'|s) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

(redefining m_{11}, etc). Let's use this to calculate how the lattice functions transform from place to place if we know the transfer matrix.

The transformation of the lattice functions

Starting with the similarity transform,

$$M(s'+C|s') = M(s'|s)\cdot M(s+C|s)\cdot M^{-1}(s'|s)$$

We can express the one-turn maps in terms of the lattice functions at the locations s and s'

$$M\,(s+C|s) = \begin{pmatrix} \cos(\Psi_0)+\alpha_0\sin(\Psi_0) & \beta_0\sin(\Psi_0) \\ -\gamma_0\sin(\Psi_0) & \cos(\Psi_0)-\alpha_0\sin(\Psi_0) \end{pmatrix}$$

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and after a page of algebra we obtain the lattice functions at point s' (or 1) in terms of the lattice functions at point s (or 0) and the elements of the matrix M. The answer is

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} m_{11}m_{22}+m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

Knowing M, we can transform the lattice functions to any point in the beam line.

Courant-Snyder parameter evolution in a drift

E.g. In a drift space of length L we have

$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

And so

$$m_{11} = 1 \quad m_{12} = L \quad m_{21} = 0 \quad m_{22} = 1$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -L \\ -2L & 1 & L^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

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The lattice functions evolve

$$\beta_1 = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha_1 = \alpha_0 - \gamma_0 L$$

$$\gamma_1 = \gamma_0$$

A particle evolves

$$x(L) = x_0 + L x'_0$$

$$x'(L) = x'_0$$

The phase advance and tune

Several times we have used the phase advance for one turn of the closed orbit (or any period of a periodic structure). It is

$$\Psi = \int_s^{s+C} \frac{ds}{\beta(s)}$$

We often call the phase advance for one complete turn of a ring the **tune**, and express 2π units of

$$\nu = \frac{\Psi}{2\pi} = \frac{1}{2\pi} \int_s^{s+C} \frac{ds}{\beta(s)}$$

(or Q)

There is one tune for each plane, including the longitudinal plane.
It's an important function for beam stability.
Note we can evaluate the tune at any point in the ring and always get the same answer (a property not shared by α, β and γ)

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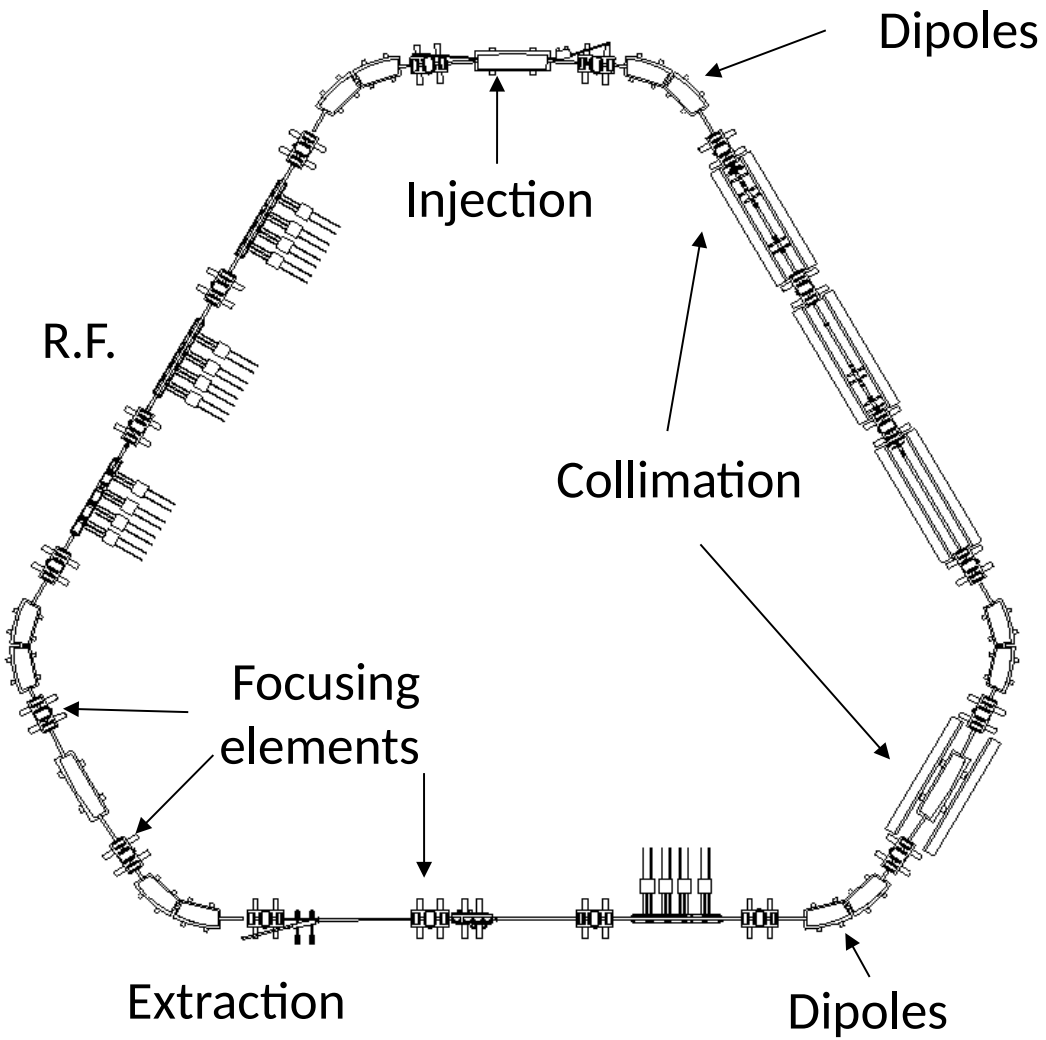
A simple approximation to the tune can come from the average value of the beta function $\bar{\beta}$ and average radius R.

$$\nu \approx \frac{1}{2\pi} \frac{C}{\bar{\beta}} \approx \frac{R}{\bar{\beta}}$$

Applying beam dynamics tools to a lattice

Let's apply the tools we've developed to a storage ring.



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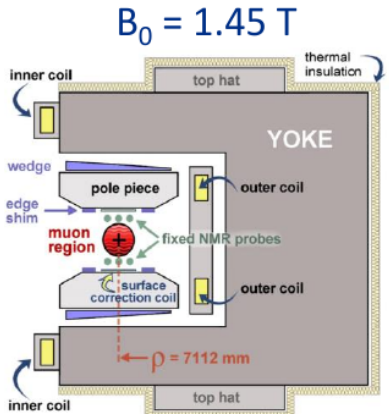
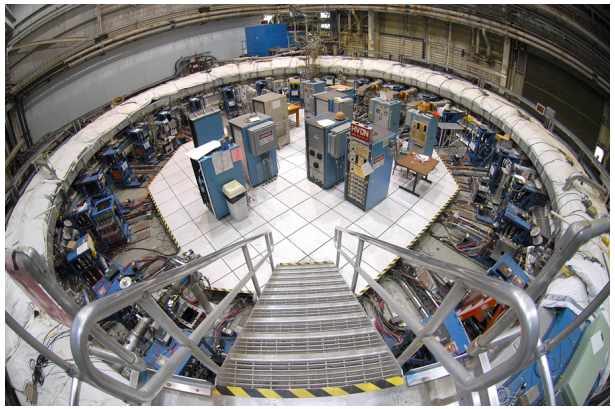
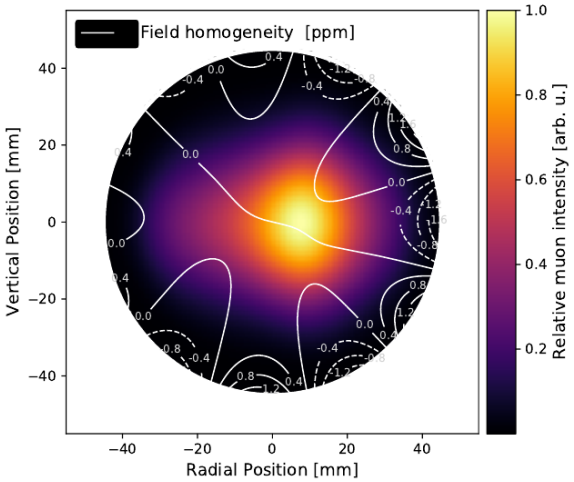
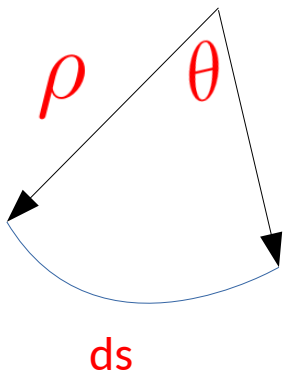
Bending

The first task is to define the curved reference orbit using a layout of dipole magnets. This forms the fundamental footprint of the machine and defines our coordinate system for future analysis. E.g. Fermilab g-2 storage ring









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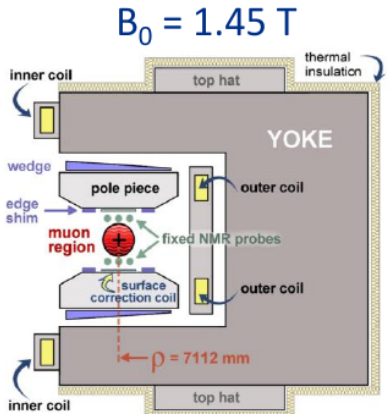
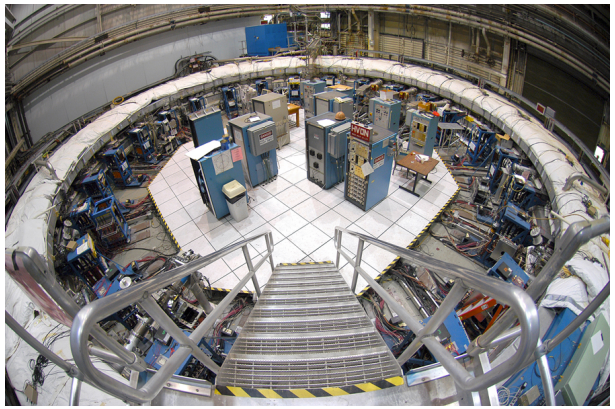
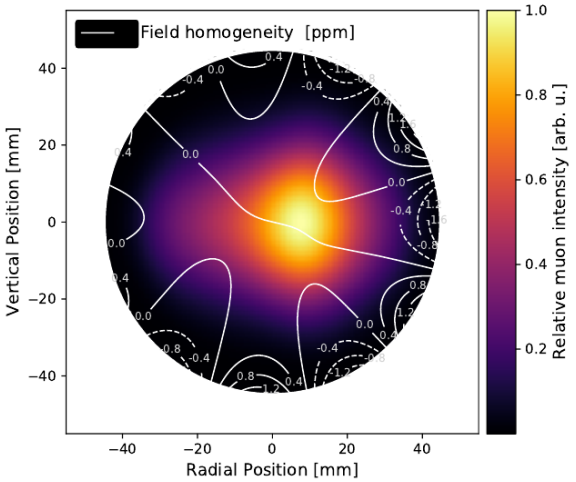
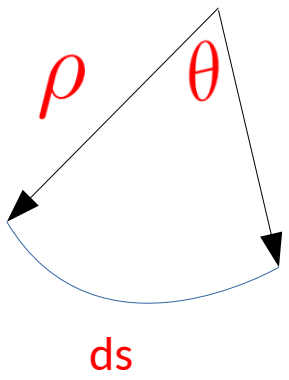




$$d\theta = \frac{ds}{\rho(s)} \approx \frac{dl}{\rho(l)} \approx \frac{B(l)q}{p} dl$$

$$\theta \approx \frac{\int B(l)dl}{(p/q)} = \frac{\int B(l)dl}{B_0\rho_0} = 2\pi$$





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Example – the LHC

$$N_{\mathrm{dipoles}} = 1232$$

$$l_{\mathrm{dipole}} = 15\mathrm{m}$$

$$[B] = Vsm^{-2}$$

$$\int Bdl = NlB = 2\pi\frac{p}{q}$$

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$$[B] = Vsm^{-2}$$

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$$B = \frac{2\pi \cdot 7000 \cdot 10^9[eV]}{1232 \cdot 15\text{m} \cdot 3 \times 10^8[m/s]q} = 8.3\text{T}$$

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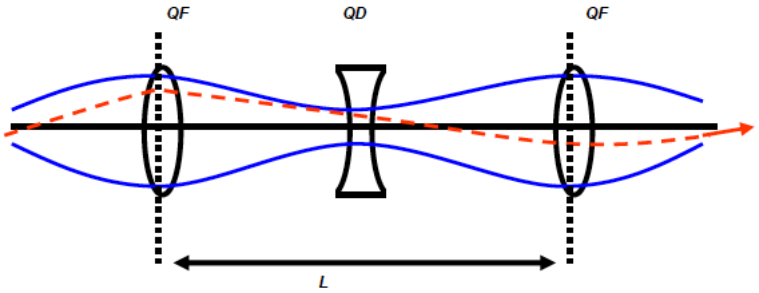
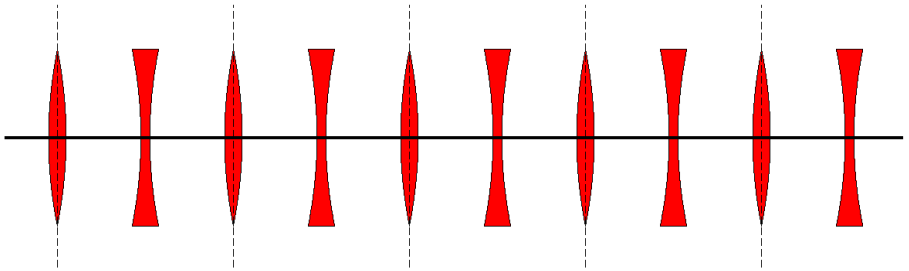
The FODO cell

Recall that two quadrupoles of opposite polarity could provide focusing in both planes at the same time. This is the fundamental building block of the FODO lattice.

The basic building block of this periodic structure is the FODO cell, consisting of a horizontally focusing quadrupole (F), a space (O), a defocusing quadrupole (D) and a space (O).



We can repeat the FODO cell to make a FODO channel. Note the drift space (O) can contain nothing, a bend, some diagnostics, an RF cavity or even a whole particle physics experiment!



The dynamics in a FODO cell

To understand the beam dynamics in a FODO cell we need to compute the one-period map.

To do this we simply multiply the matrices of the components of the cell together, **conventionally starting in the middle of one of the quadrupoles,** which means we start and end with a quadrupole matrix of half strength (length)

Recall

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

First element!

and we multiply these matrices in sequence

$$M_{\text{FODO}} = M_{\text{Fh}} \cdot M_{\text{drift}} \cdot M_{\text{D}} \cdot M_{\text{drift}} \cdot M_{\text{Fh}}$$

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The FODO cell

Let's be concrete and take some real numbers

K = +/- 0.541244 m^{-2}

l_q = 0.5 m

L = 2.5 m

Multiplying out the matrices

$$\begin{pmatrix} 0.983134 & 0.248593 \\ -0.134549 & 0.983134 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2.5 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1.06842 & 0.511352 \\ 0.276767 & 1.06842 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2.5 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.983134 & 0.248593 \\ -0.134549 & 0.983134 \end{pmatrix}$$

We obtain

$$M_{\mathrm{FODO}} = \begin{pmatrix} 0.707107 & 8.20716 \\ -0.0609224 & 0.707107 \end{pmatrix}$$

The FODO cell

Let's be concrete and take some real numbers

K = +/- 0.541244 m^{-2}

l_q = 0.5 m

L = 2.5 m

$$M_{\mathrm{FODO}} = \begin{pmatrix} 0.707107 & 8.20716 \\ -0.0609224 & 0.707107 \end{pmatrix}$$

This is the one period map of the FODO cell, and so has the form

$$M(s+L|s) = \begin{pmatrix} \cos\Psi + \alpha\sin\Psi & \beta\sin\Psi \\ -\gamma\sin\Psi & \cos\Psi - \alpha\sin\Psi \end{pmatrix}$$

Recall

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s)+\psi_0) \qquad \psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} \qquad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

Properties of our FODO cell

Is the FODO cell stable? For this we need the trace of the one-turn map to be less than or equal to 2. Here it is 1.414. So this FODO cell will give stable dynamics in this plane.

What is the phase advance per cell? Recall

$$\Psi = \arccos\left(\frac{m_{11}+m_{22}}{2}\right)$$

The phase advance per cell is 45 degrees. This is a “45 degree cell”.

What are the lattice functions at the middle of the focusing quadrupole? We use

$$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11}-m_{22}}{2\sin\Psi}$$

And find that β=9.645 m and α=0.
i.e. The beam size is at a maximum.

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A thin lens FODO cell

We can also make our life easier and compute the matrix for our FODO cell using the thin lens matrices. Again, starting from the middle of QF we have

$$M_{\mathrm{FODO}} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix}$$

We end up with the matrix in terms of L and f

$$M_{\mathrm{FODO}} = \begin{bmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{bmatrix}$$

We can ask for what parameters the FODO cell gives stable motion. This means

$$\mathrm{Tr}(M) \leq 2 \rightarrow |f| > \frac{L}{2}$$

We can also write the cell phase advance in terms of the parameters:

$$\cos(\Psi) = \frac{1}{2} Tr(M) = 1 - \frac{L^2}{2f^2} \qquad \sin\left(\frac{\Psi}{2}\right) = \frac{L}{2|f|}$$

Transfer line stability

Our stability equation from the previous slide seems slightly odd at first

$$\mathrm{Tr}(M) \leq 2 \rightarrow |f| > \frac{L}{2}$$

It seems to say motion is stable when focusing is weak…

This makes sense though. If the focussing is too strong then the periodicity of beta can't match the periodicity of the lattice.

Beta in a FODO cell

Finally…

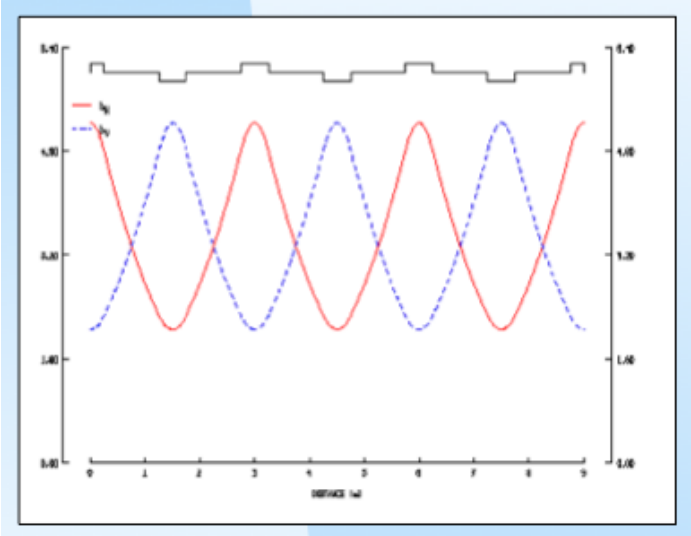
We can now compute all the lattice functions for a FODO cell.

Note that β_x is maximised in the middle of the focusing quadrupoles, and this maximum depends solely on the cell length and phase advance.

Using $$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi}$$

We get $$\beta_F = \frac{2L(1+\frac{L}{2f})}{\sin(\Psi)} \qquad \alpha_F = 0 \qquad \gamma_F = \frac{1}{\beta_F}$$

In the D quad $$f \rightarrow -f \qquad \beta_D = \frac{2L(1-\frac{L}{2f})}{\sin(\Psi)}$$



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ERROR AND RESONANCES

A reminder of some of the terms we ignored...

$$
\begin{aligned}
x'' + (\kappa_x^2 + k)\, x = {} & \kappa_x\delta - \kappa_x\delta^2 + \kappa_x\delta^3 - (\underline{k} + \kappa_x\kappa_y)\, y \qquad (3.75)\\
& - (\underline{m} + 2\kappa_x\underline{k} + 2\kappa_y k + 2\kappa_x^2\kappa_y)\, xy - \tfrac{1}{2}m\,(x^2 - y^2)\\
& - (\kappa_x^3 + 2\kappa_x k)\, x^2 - (\kappa_x\kappa_y^2 - \tfrac{1}{2}\,\kappa_x k + \tfrac{2}{3}\kappa_y\,\underline{k} - \tfrac{1}{2}\,\kappa_x'')\, y^2\\
& + \tfrac{1}{2}\,\kappa_x(x'^2 - y'^2) + \kappa_x'\,(xx' + yy') + \kappa_y'\,(x'y - xy') + \kappa_y x'y'\\
& - \tfrac{1}{6}\,r\,x(x^2 - 3y^2) + \tfrac{1}{6}\,\underline{r}\,y\,(y^2 - 3x^2)\\
& + \tfrac{1}{12}(\kappa_y\underline{m} - 11\kappa_x m + 2\kappa_y^2 k - 10\kappa_x^2 k + k'' - \kappa_y\kappa_y''\\
& \qquad + \kappa_x\kappa_x'' - \kappa_y'^{\,2} + \kappa_x'^{\,2})\, x^3\\
& - (2\kappa_x\underline{m} + \kappa_y m + \kappa_x^2\underline{k} + 2\kappa_x\kappa_y k)\, x^2 y\\
& + \tfrac{1}{4}(5\kappa_x m - 7\kappa_y\underline{m} + 6\kappa_x^2 k + k'' - \kappa_y\kappa_y'' - 2\kappa_y^2 k\\
& \qquad + 5\kappa_x\kappa_x'' + \kappa_x'^{\,2} - \kappa_y'^{\,2} - \kappa_x\kappa_y\underline{k})\, xy^2\\
& + \tfrac{1}{6}(10\kappa_x\kappa_y k + 8\kappa_x''\kappa_y + \kappa_x\underline{m} + 4\kappa_y^2\underline{k} + \underline{k}'' + 2\kappa_x'\kappa_y' + 5\kappa_y m)\, y^3\\
& - (2\kappa_x^2 + \tfrac{3}{2}k)\, xx'^2 - (\kappa_x'\kappa_y + \kappa_x\kappa_y')\, xx'y - \kappa_x\kappa_x'\, x^2x'\\
& - \tfrac{1}{2}\underline{k}'x^2y' - \kappa_y\kappa_y' x'y^2 - \kappa_x\kappa_y xx'y' - \tfrac{1}{2}(\underline{k} + 3\kappa_x\kappa_y)\, x'^2y\\
& + k'xyy' - \tfrac{1}{2}(k + \kappa_x^2)\, xy'^2 - (2\kappa_y^2 - \underline{k})x'yy' + \tfrac{1}{2}\underline{k}'y^2y' - \tfrac{1}{2}\underline{k}\,yy'^2\\
& + (2\kappa_x^2 + k)\, x\,\delta + (2\kappa_x\kappa_y + \underline{k})\, y\,\delta - \kappa_x' yy'\delta + \kappa_y'\, xy'\delta\\
& + \tfrac{1}{2}\kappa_x(x'^2 + y'^2)\delta + (\tfrac{3}{2}\kappa_y\underline{k} + \kappa_x\kappa_y^2 - \tfrac{1}{2}\kappa_x k - \tfrac{1}{2}\kappa_x'' - \tfrac{1}{2}m)\, y^2\delta\\
& + (\tfrac{1}{2}m + 2\kappa_x k + \kappa_x^3)\, x^2\delta + (\underline{m} + 2\kappa_x^2\kappa_y + 2\kappa_y k + 2\kappa_x\underline{k})\, xy\delta\\
& - (k + 2\kappa_x^2)\, x\delta^2 - (\underline{k} + 2\kappa_x\kappa_y)\, y\delta^2 + \mathcal{O}(4)\,.
\end{aligned}
$$

Real life and field errors

Recall that we started with an arbitrary magnetic field and made an expansion

$$\frac{e}{p}B_z(x) = \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}ox^3 + \ldots$$

where the first term is the dipole (steering) field, the second is the quadrupole(focussing) term, the next is the sextupole term, etc.

To create these fields we build magnets with a specified field quality. These magnets will never be perfect. Therefore any magnet will have small contributions of higher-order field components.

In addition, magnets will not be perfectly aligned. E.g. if a quadrupole is displaced it will apply an additional dipole field to the beam.

Finally, the magnet strength may differ from the design value and may vary with time. E.g. a power supply may deliver too much or too little current to an electromagnet.

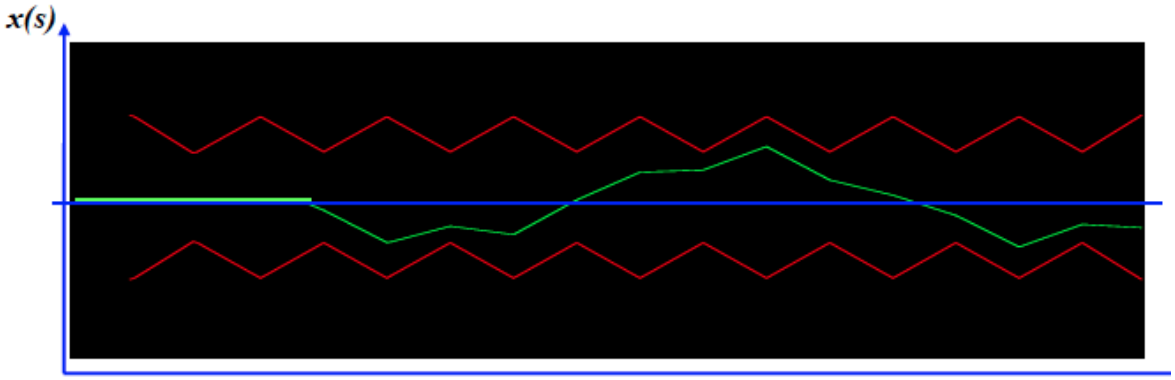
In this lecture we include the effects of some of these 'field errors' into our solutions to Hill's equation.

Closed orbit distortion

The design orbit defined by all of the dipoles in the ring is sometime refered to as the "closed design orbit".

This is the orbit a reference particle would follow in a perfect situation.

If there is a small additional dipole kick – the orbit will distort, and this distortion will affect the orbit around the entire ring. i.e. the effects of a small kick at any location are not localised; they will be seen everywhere in the ring!



This closed orbit distortion defines a position-dependent orbit offset around the ring. In effect the particles no longer undergo betatron oscillations around the design orbit but around a new closed orbit

$$x(s) = x_\beta(s) + x_{co}(s)$$

Closed orbit distortion

Imagine we have a “dipole kick” error of strength ΔB and length l, at some location s_0

$$\theta = \frac{\int Bdl}{B\rho} \Rightarrow \Delta\theta = \frac{\Delta Bl}{B\rho}$$

Recall the transport map in terms of the Twiss parameters

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

- The M_{12} element shows how a horizontal angular kick (Δx’) will translate into a horizontal displacement at another point in the ring.
- On each “turn” we experience another kick.
- We need to apply this map for many turns, summing over the kicks, to see how the displacement accumulates.

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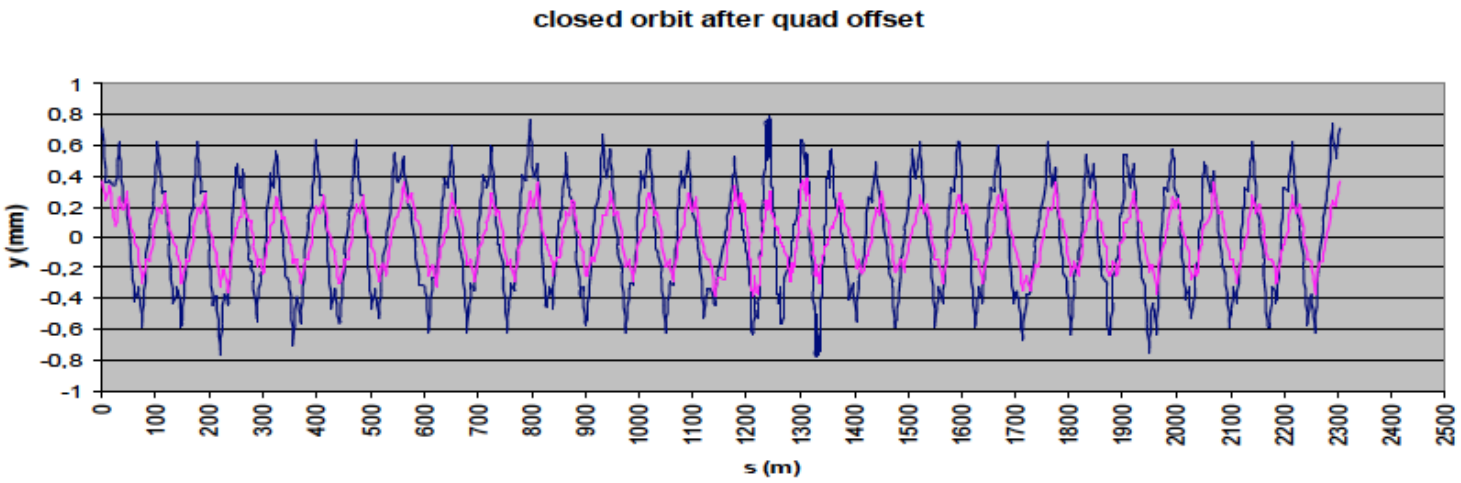
- On each “turn” we experience another kick.
- We need to apply this map for many turns, summing over the kicks, to see how the displacement accumulates.

Closed orbit distortion

A short(ish) analysis adding up the effects from each kick after a large number of turns gives the distorted closed orbit in terms of the beta function at s_0 and the beta function at any other position in the ring s.

$$x_{co}(s) = \Delta\theta \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin(\pi\nu)} \cos(\pi\nu - |\psi(s) - \psi(s_0)|) \qquad \nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$

We can minimise this distortion by monitoring the position of the beam and using orbit correction magnets.



Resonances

Our expression for the closed orbit distortion has an overall factor of

$$\frac{1}{2\sin\pi\nu}$$

This means that every time the tune becomes an integer, the argument of the sine becomes a multiple of π, and this factor diverges.

This is an example of resonance.

Imagine the tune was 1 in a machine.

- Then the particle would encounter a dipole error at the same point in the machine and at the same phase in its betatron oscillation on every turn.
- This means the effect of the dipole error accumulates constructively on every turn.
- We avoid this by minimising magnet errors and staying away from dangerous values of the tune.

NB Consider what happens to the one turn map in this case.

Resonances

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More generally resonances occur when

$$m\nu_x + n\nu_y = p \qquad m, n, p = \text{integer}$$

Quadrupole errors

Imagine we have an extra quadrupole in our ring (or a quadrupole field error) of strength k and length L at location s_0. Unlike the dipole error, this will change the focussing properties of the lattice causing:

1)A change in the beta function
2)A change in the tune.

Recall that the tune is given by:

$$\nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$

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1)A change in the beta function
2)A change in the tune.

Change in the tune: $$\Delta\nu = \frac{\beta(s_0)kL}{4\pi} = \frac{\beta(s_0)}{4\pi f}$$

- The perturbed tune increases if k > 0, which corresponds to a focusing quadrupole i.e. focussing more means more oscillations. So we get a positive tune shift for increased particle focusing.

- This means a pure quadrupole field error would shift the tune one way in one plane and the other way in the other plane

- However, we can also get tune shifts from space-charge, beam-beam effects and electron clouds, which can cause same-sign tune shift in both planes

- The effect of the quadrupole error is proportional to the local beta function. This is a common feature - the beta function magnifies local field errors.

A distribution of quadrupole errors

If we have a distribution of quadrupole errors around the ring, the approximate tune shift can be calculated from

$$\Delta\nu = \frac{1}{4\pi}\oint ds\beta(s)k(s)$$

This effect can also be used deliberately to measure the beta functions.

- We vary the strength of a single quadrupole in the ring.
- We measure the tune.
- The response is proportional to the beta function at the quadrupole.

In general the beta function tells you how sensitive the beam is to perturbations.

Beta beat

The change in the beta function is itself a function of s, and oscillates twice as fast as the original beta function:

$$\frac{\Delta\beta(s)}{\beta} = -\frac{k\beta(s_0)}{2\sin(2\pi\nu)}\cos(2\pi\nu - 2|\psi(s) - \psi(s_0)|)$$

This is why it's called a 'beta beat'.

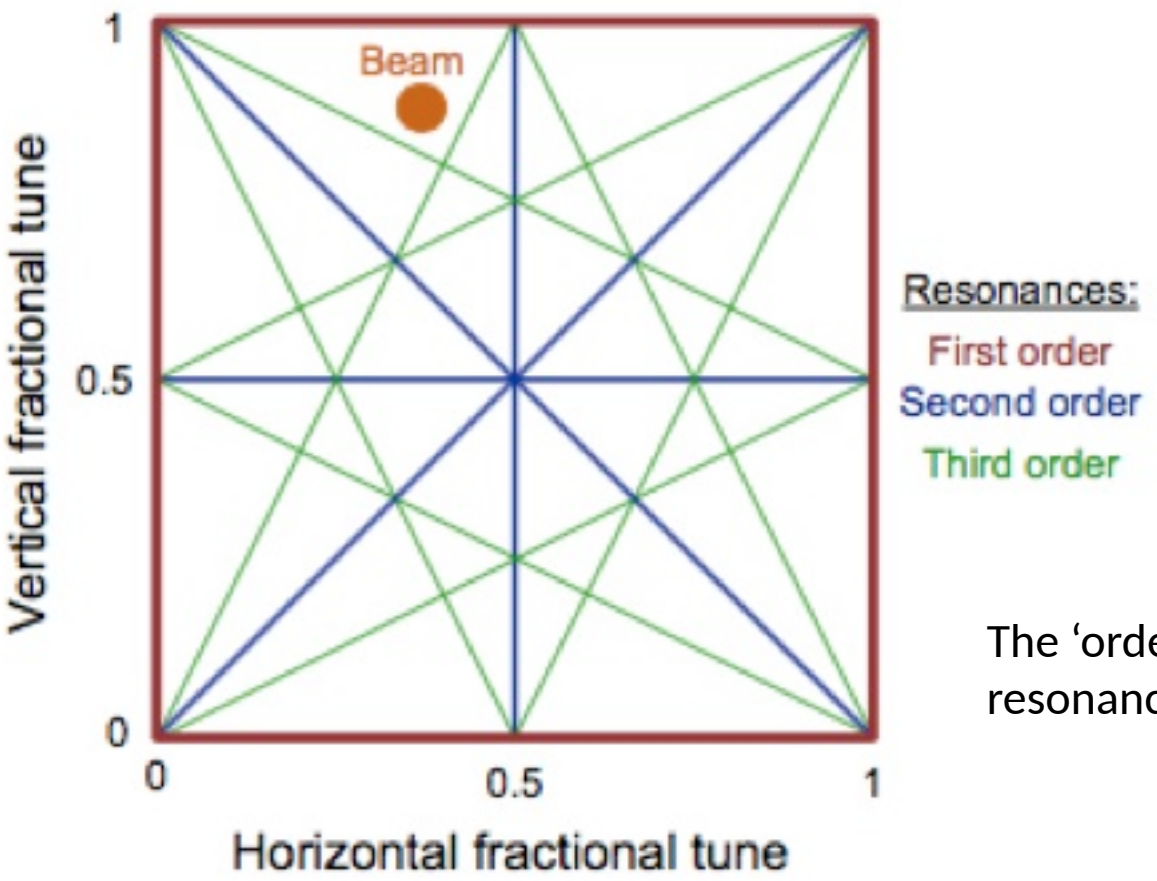
The strength of the distortion is proportional to the quadrupole error (k) and to the beta function at the position of the error s_0.

As before we have a sinusoidal term in the demoninator that depends on the tune.

This gives us a 'half-integer resonance'.

Resonance diagram

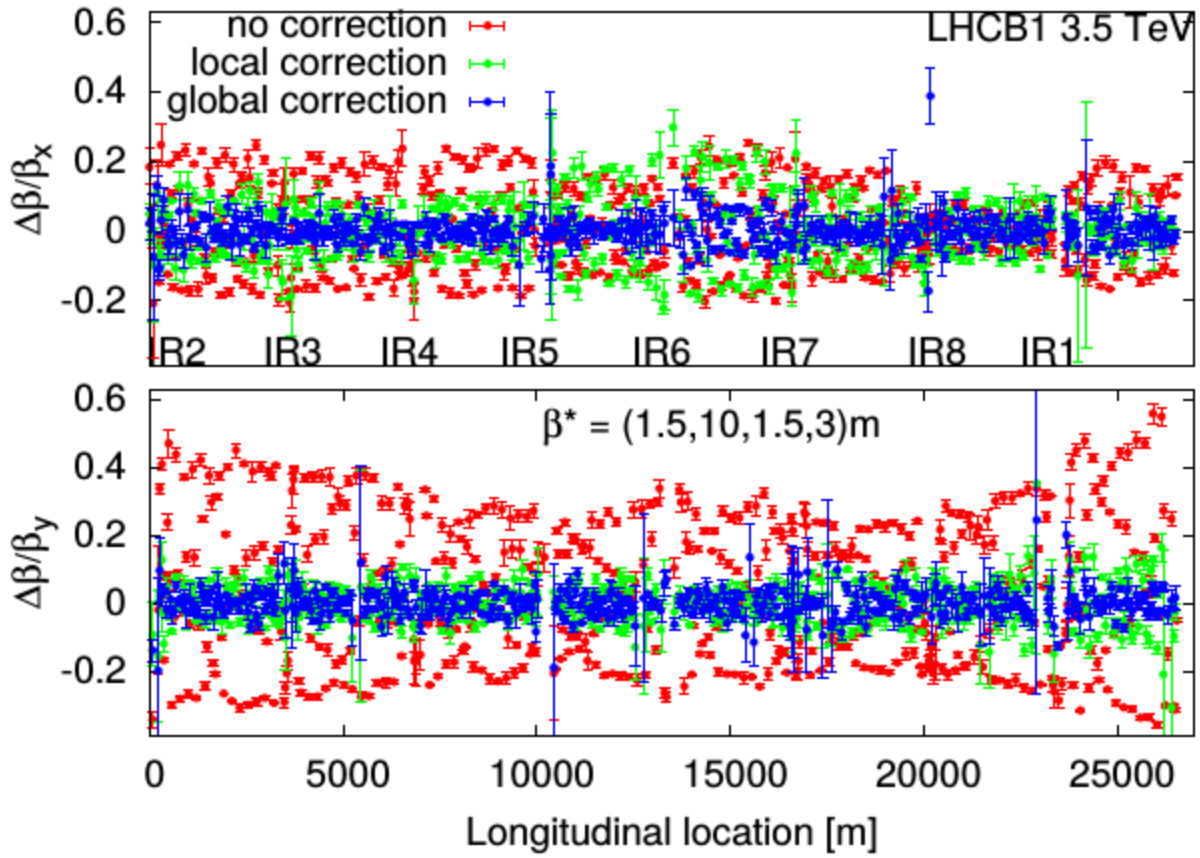
Similarly, higher-order resonances are generated by errors in high-order multipoles.



The 'order' of the resonance is m+n

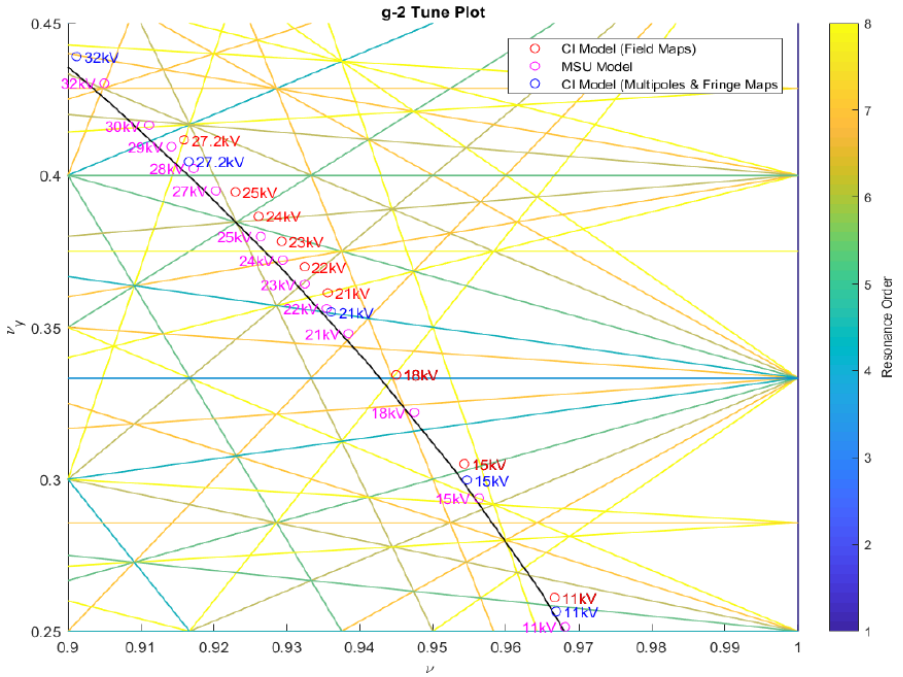
$$m, n, p = \text{integer} \qquad m\nu_x + n\nu_y = p$$

Beta beat at the LHC



Summary

- Dipole Errors
  - Introduce closed orbit distortion
  - Betatron oscillations occur around the new orbit
  - Give resonance on integer values of the tune
- Quadrupole Errors
  - Introduce a ‘beta beat’
  - Introduce a tune shift
  - Give resonances on half-integer values of the tune.
- Detailed simulation needed to calculate tune and avoid all resonances.



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DISPERSION, CHROMATICITY, EMITTANCE

Off-momentum particles

So far we have considered beam motion where all particles have the design momentum p. We refer to these particles as **on-momentum** particles.

In general, a particle's momentum will be p + Δp

$$p + \Delta p = p(1 + \delta) \qquad \delta = \Delta p / p$$

Recall in our solutions to Hill's equation we had

$$\left(\frac{\rho}{\rho + x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho + x} = -\frac{eB_y}{m\gamma v_s}$$

Which we can write as

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{(\rho + x)^2}{\rho^2} \frac{eB_y}{m\gamma v_s}$$

Expanding the right hand side and dropping small terms we obtained

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{eB_y}{m\gamma v_s}\left(1 + \frac{2x}{\rho}\right)$$

$$(\rho + x)^2 = \rho^2 + x^2 + 2x\rho$$

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the deviation 'delta'

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

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Homogeneous and inhomogeneous Hill's equations

Now we see a little more structure to Hill's equation

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

It looks like the homogeneous version (below) apart from a term linear in δ

$$x'' + K_x(s)x = 0$$

The extra term on the RHS will drive the x motion of an off-momentum particle, which we shall call horizontal dispersion, or simply ***dispersion***.

The general solution for the horizontal motion of a particle is given by the sum of two terms : the betatron motion term (soln to homogeneous eqn) and an off-momentum dispersion term (soln to inhomogeneous eqn):

$$x(s) = x_h(s) + x_i(s)$$

We can think of $x_i(s)$ as a closed orbit term, around which $x_h(s)$ oscillates.

Let's define a special orbit, D(s), which is followed by a particle with δ=1

$$D(s) = \frac{x_i(s)}{\Delta p/p}$$

Dispersion

Our newly-defined dispersion function D(s):

- Is the orbit of a particle with δ=Δp/p=1.
- Obeys Hill's equation.
- Determines the orbit of any (slightly) off-momentum particle

$$x(s) = x_h(s) + D(s)\delta$$

This is similar to a dipole error closed-orbit distortion.

Typical values:

$$x_h \sim 2 - 5\mathrm{mm}$$

$$D(s) \sim 1 - 2\mathrm{m}$$

$$\delta \sim 10^{-3}$$

Dispersion

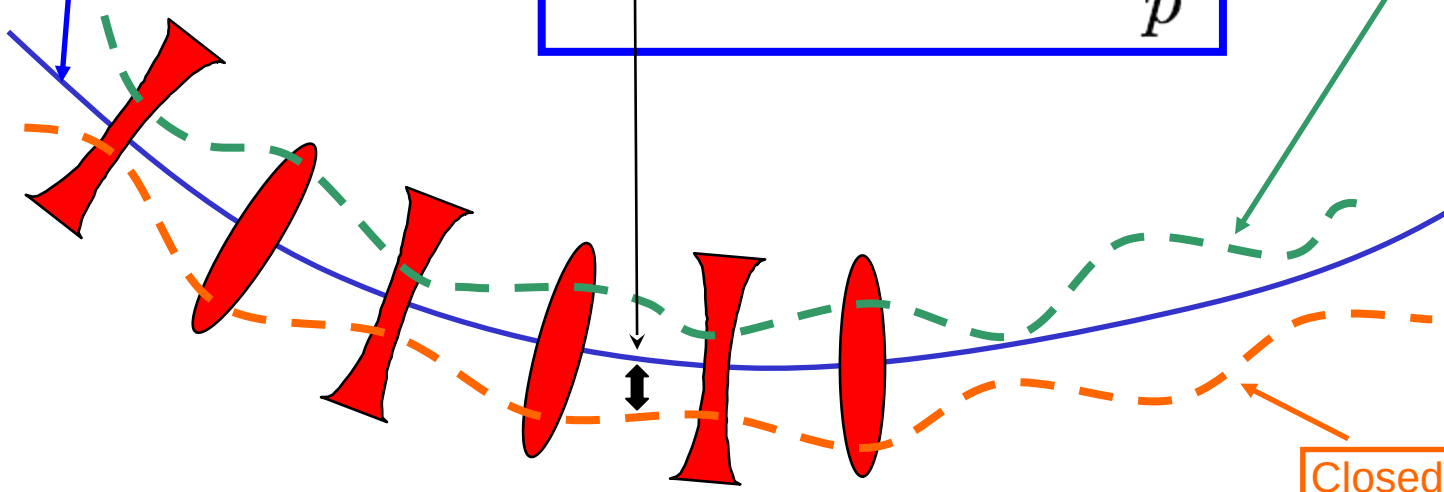
Particle's momentum error

Lattice property

Central design orbit is closed for $p=p_0$

$$\Delta x(s) = D(s) \times \frac{\Delta p}{p}$$

Closed orbit for $p < p_0$



Closed orbit for $p > p_0$

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

Calculating D(s)

We need to find a solution to the inhomogeneous Hill's equation and add it to the general solution of the homogeneous equation.

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

For D(s) on the closed orbit we assume the only field is the dipole field. This means that D(s) is a solution of

$$D(s)'' + D(s)\frac{1}{\rho^2} = \frac{1}{\rho}$$

We have already solved the homogeneous equation (below).

$$D''(s) + D(s)\frac{1}{\rho^2} = 0 \qquad M_{\text{dipole,x}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

We only need to find a particular solution of the inhomogeneous equation and add this solution to the solution of the homogeneous equation. If the RHS is just a constant, then a valid choice of a particular solution is also a constant

$$D_p = C = \text{const}$$

Calculating D(s)

$$D(s)'' + D(s)\frac{1}{\rho^2} = \frac{1}{\rho}$$
$$D_p = C = \text{const}$$

Inserting this solution for D into the inhomogeneous equation above immediately gives

$$\frac{C}{\rho^2} = \frac{1}{\rho} \Rightarrow C = \rho$$

And so our general solution for D(s) is

$$D(s) = A\cos(s/\rho) + B\sin(s/\rho) + \rho$$
$$D'(s) = -\frac{A}{\rho}\sin(s/\rho) + \frac{B}{\rho}\cos(s/\rho)$$

The matrix equation for D(s)

As before we determine A and B using the initial conditions at s=0

$$D(0) = D_0 \qquad D'(0) = D'_0$$

Inserting these into our general solution yields

$$A = D_0 - \rho \quad B = \rho D'_0$$

Hence we can write the dispersion function as

$$D(s) = D_0 \cos(s/\rho) + D'_0 \sin(s/\rho) + \rho(1 - \cos(s/\rho))$$
$$D'(s) = -\frac{D_0}{\rho} \sin(s/\rho) + D'_0 \cos(s/\rho) + \sin(s/\rho)$$

Which we can write as a matrix equation

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(s/\rho) & \rho \sin(s/\rho) & \rho(1 - \cos(s/\rho)) \\ -\frac{1}{\rho} \sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

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The dispersion

Note the upper-left 2x2 matrix is just the transfer matrix for a dipole we have already derived.

The additional terms in th dipole transfer matrix produce or ‘drive’ the dispersion.

As the motion is given as the sum of the betatron motion and the dispersion

$$x(s) = x_\beta(s) + D(s)\delta$$

We can write the general motion as a matrix equation

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & D \\ m_{21} & m_{22} & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

Dispersion in a short 'sector' dipole and a quadrupole

For a short sector dipole with a small bending angle θ

$$\theta = \frac{l}{\rho} \ll 1$$

we can find a simplified matrix from its entrance to its exit

$$\begin{pmatrix} \cos(s/\rho) & \rho\sin(s/\rho) & \rho\left(1-\cos(s/\rho)\right) \\ -\frac{1}{\rho}\sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & l & l\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

This is useful for quick calculations and corresponds to a thin-lens kick for an off-momentum particle.

In a quadrupole the dispersion function is focussed/defocussed, but there is no driving term for the dispersion and so the 3x3 map is given by

$$\begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion in a FODO cell

Consider a FODO cell with thin lens quadrupoles and dipoles in the drift sections. We can calculate dispersion in the same way we computed the beta functions in a FODO cell previously.

Start at the middle of the F quad, so we have a magnetic arrangement

$$\frac{\mathrm{QF}}{2} \quad \mathrm{B} \quad \mathrm{QD} \quad \mathrm{B} \quad \frac{\mathrm{QF}}{2}$$

Looking at only the x motion in the thin-lens and small angle approximations we find

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which evaluates to

$$M = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L(1+\frac{L}{2f}) & 2L\theta(1+\frac{L}{4f}) \\ -\frac{L}{2f}(1-\frac{L^2}{2f^2}) & 1-\frac{L^2}{2f^2} & 2\theta(1-\frac{L}{4f}-\frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix}$$

Here L is the length of each dipole, θ is the bend angle and f is the quadrupole focal length. The upper 2x2 is the same as previously calculated, but now we've added the dispersion.

Dispersion in a FODO cell

By symmetry, the dispersion in the middle of QF must satisfy the closed orbit condition

$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = M \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

and if we solve the resulting equation, noting that in a FODO cell the phase advance is given by

$$\sin\left(\frac{\psi}{2}\right) = \frac{L}{2|f|}$$

we get the dispersion in the middle of the QF

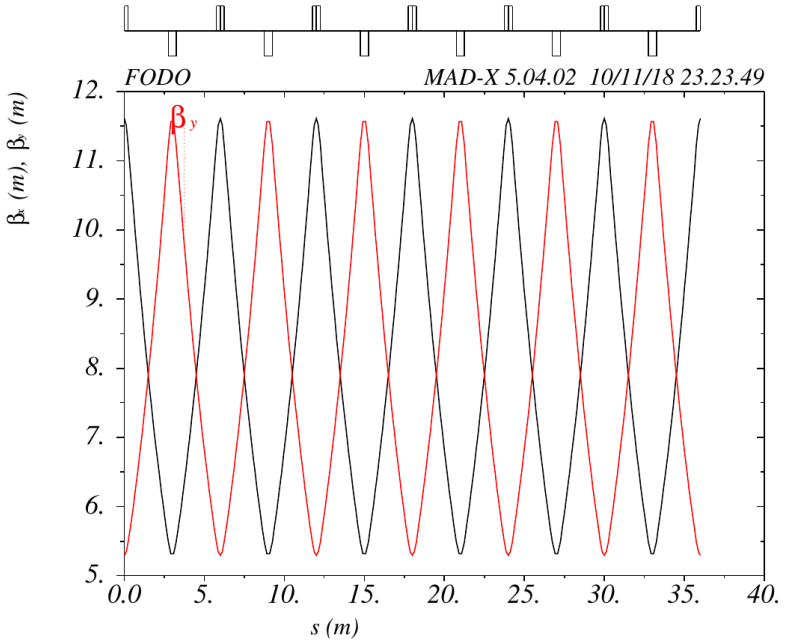
$$D_F = \frac{L\theta\left(1+\frac{1}{2}\sin\left(\frac{\psi}{2}\right)\right)}{\sin^2\left(\frac{\psi}{2}\right)} \qquad D'_F = 0$$

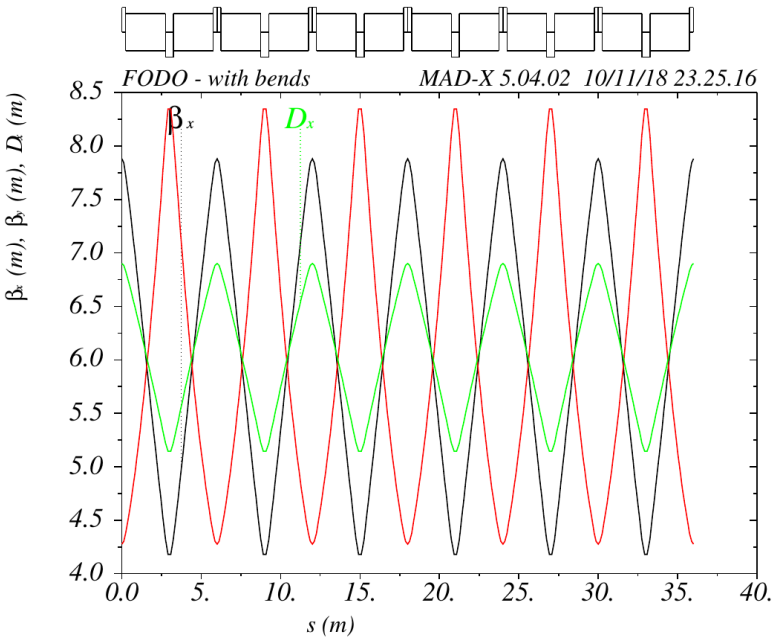
we can get the dispersion elsewhere by transforming this vector using our 3x3 maps. For example in the middle of QD we get

$$D_D = \frac{L\theta\left(1-\frac{1}{2}\sin\left(\frac{\psi}{2}\right)\right)}{\sin^2\left(\frac{\psi}{2}\right)} \qquad D'_D = 0$$

Simulating dispersion

D(s) is created (or driven) by dipoles, focused by quadrupoles and will grow in a drift if the angular dispersion D' is non-zero





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Controlling dispersion

Dispersion-free lattices are important in many applications. These allow bending of the beam without generating dispersion.

Examples are: Chasman-green, triple-bend achromat,…

We also can build a dispersion suppressor, which matches the periodic dispersion in an arc (perhaps made of FODO cells) into a dispersion-free straight section.

We can also displace the beam transversely without generating dispersion using a sequence of bends, sometimes called a geometrical achromat.

These will be covered in later courses….
But let’s look briefly at some examples

The double bend achromat (DBA)

If the dispersion function is non-zero the orbits of particles depend on the particles' momenta. An "achromatic system" means the beam positions at each end do not depend on momentum.

i.e. we require an arrangement of magnets, including bends, which does not generate any dispersion through the structure.

A single bend is not achromatic. In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet.

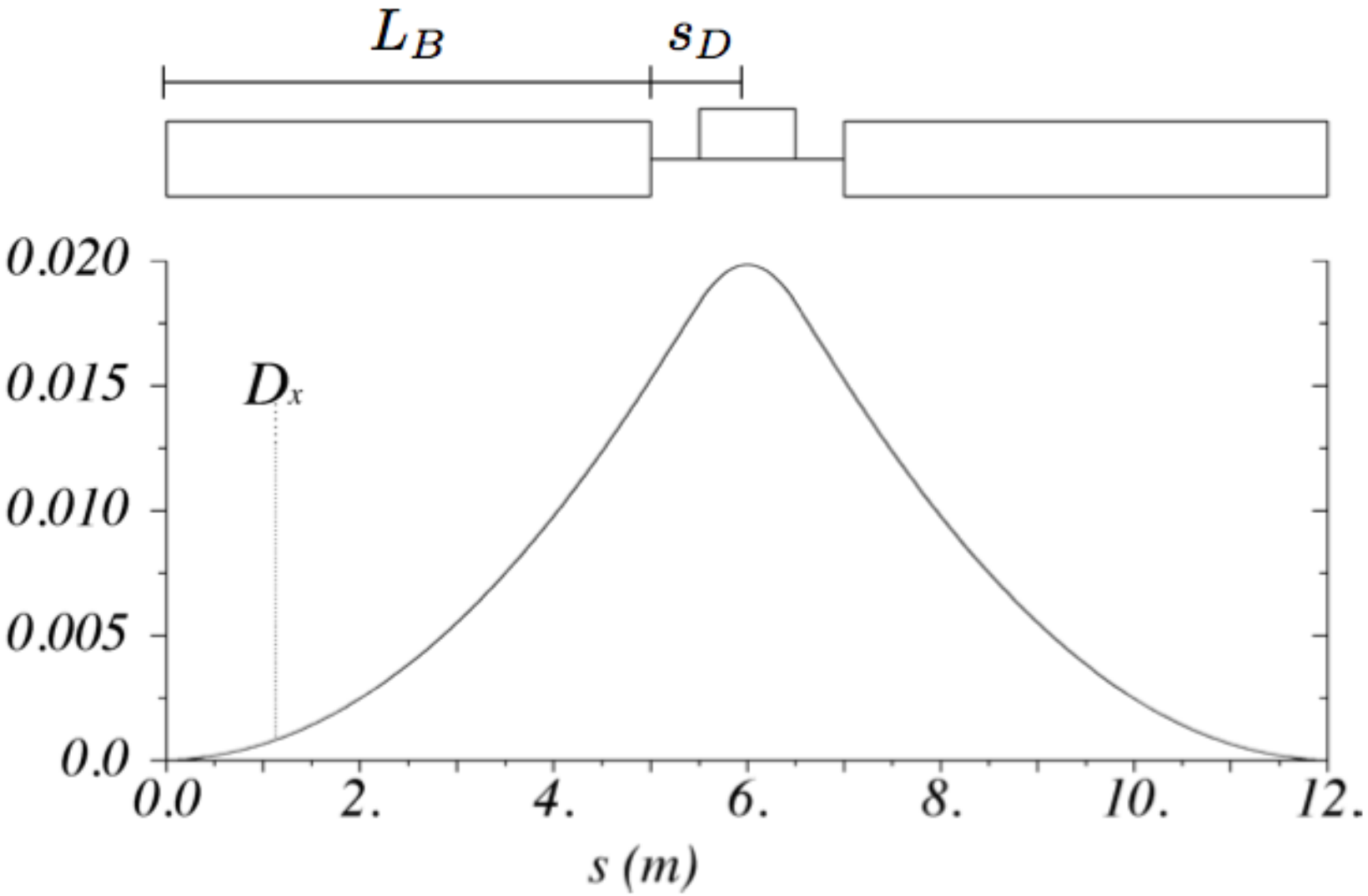
With one focusing quad in the middle between two dipoles, one can get the achromat condition, which means no additional dispersion is driven by the structure.

Due to the mirror symmetry of the lattice w.r.t. to the middle quadrupole D' should be zero in the centre of the lattice. This is the so-called double bend achromat (DBA) structure.

We generally need further quads outside DBA section to match the betatron functions, tunes, etc.

Similarly, one can design triple bend achromat (TBA), quadrupole bend achromat (QBA), and multi-bend achromat (MBA or nBA) structures.

DBA structure with a single quadrupole (sometimes called Chasman-Green)



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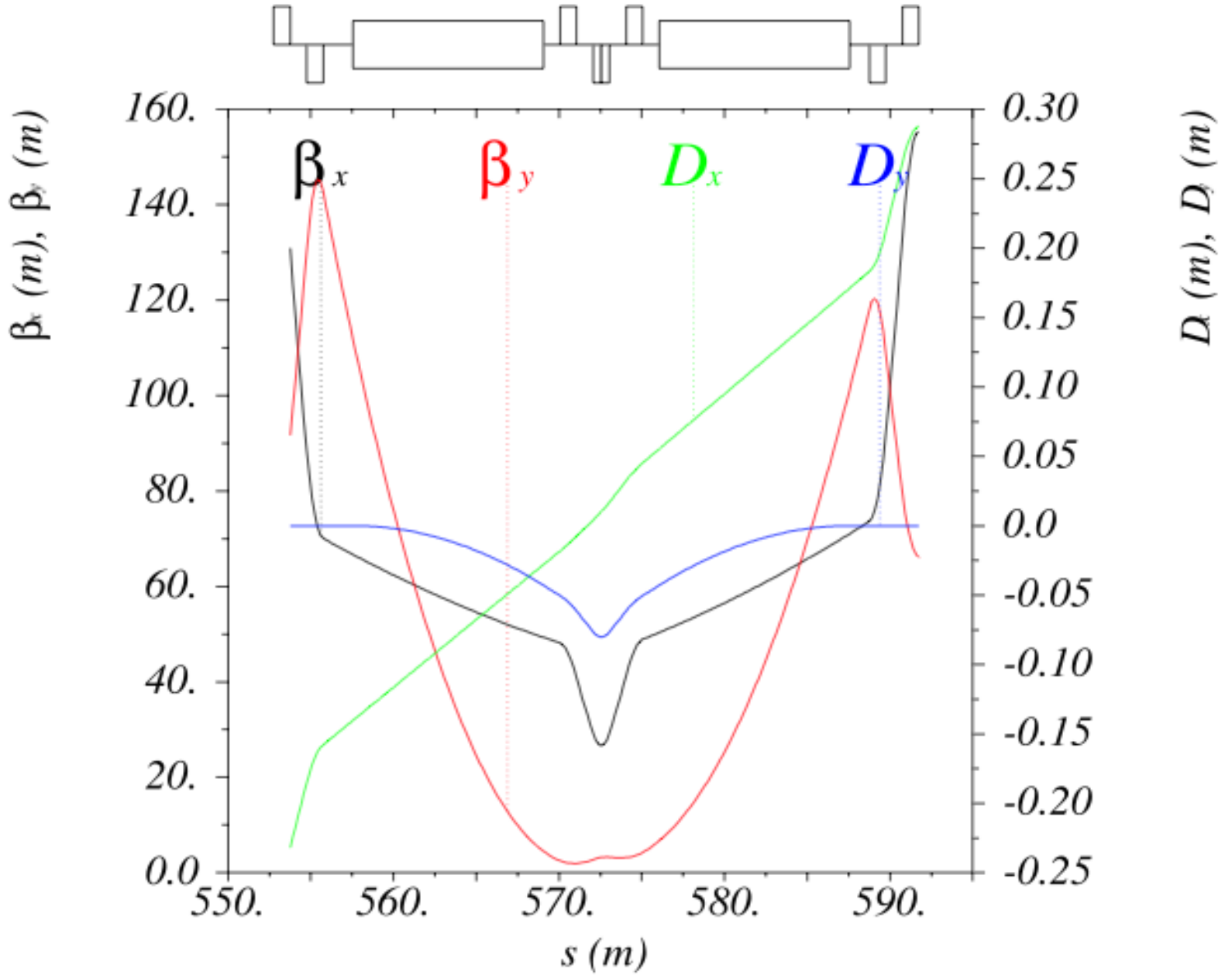
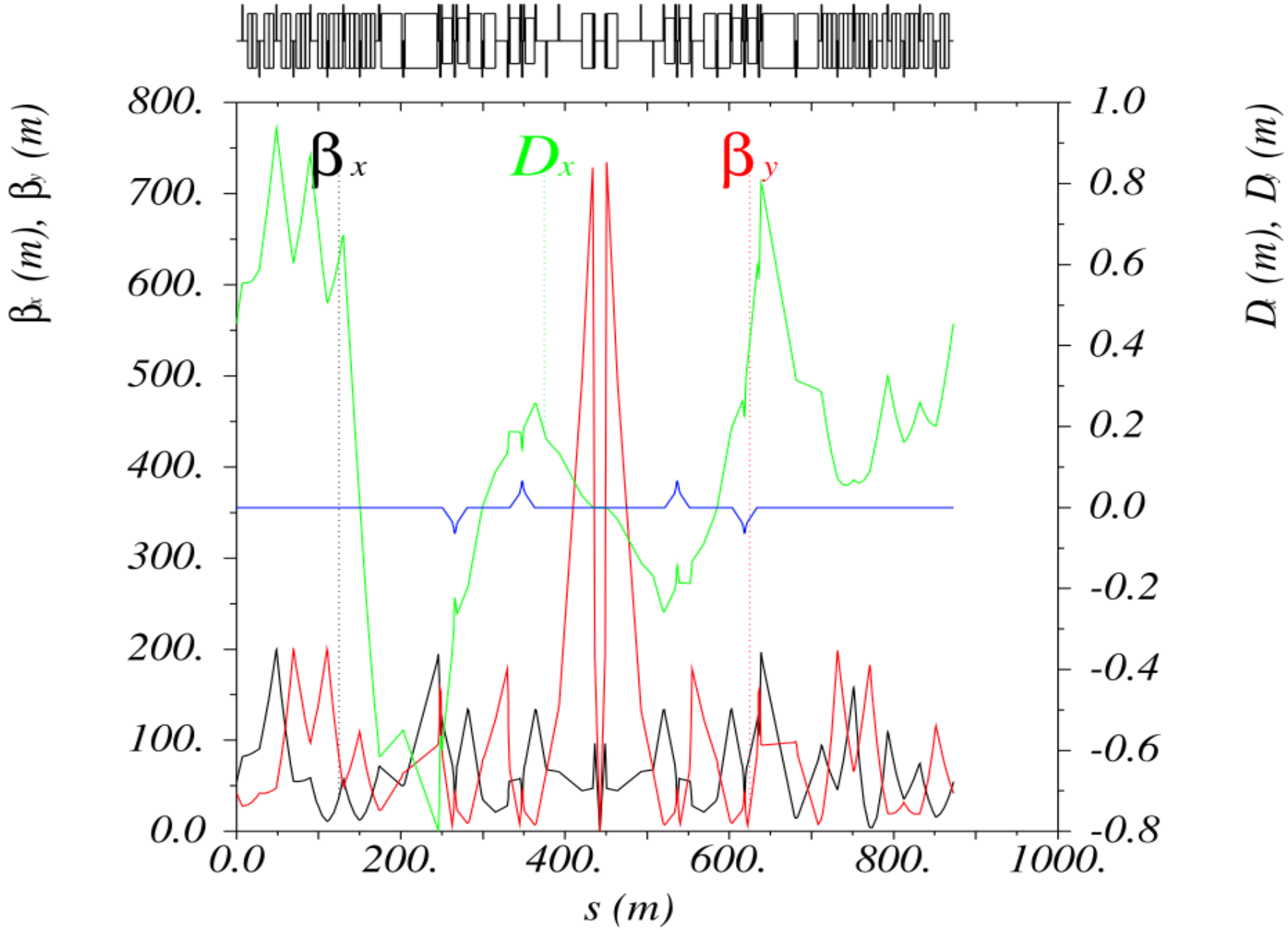
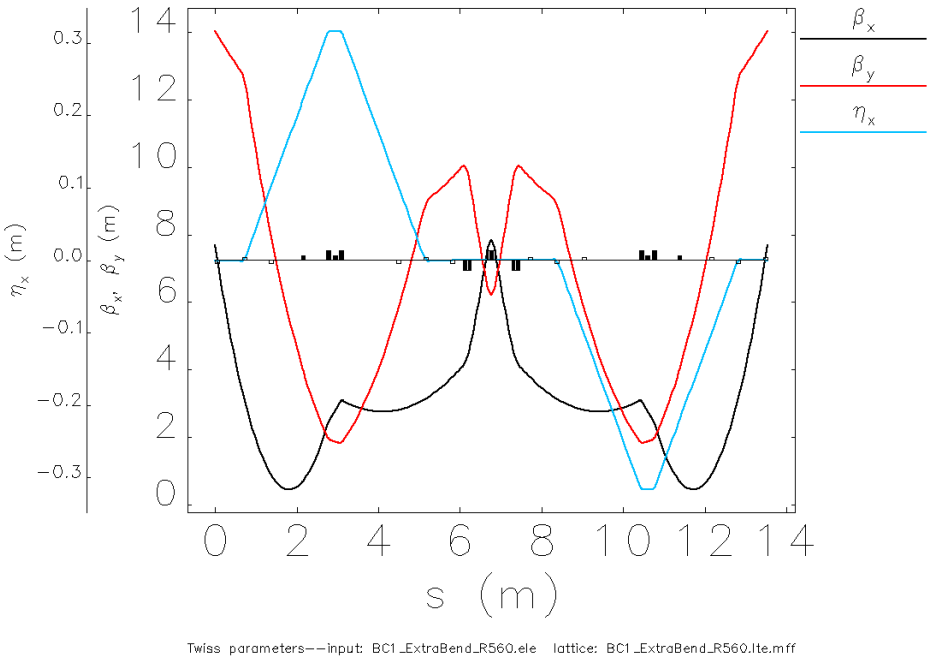
A DBA structure with a quadrupole triplet (vertical)



The long straight section of the LHeC collider (optical work done by CI)



MAX-IV (Gus Perez-Segurana CI)



In addition to affecting the transverse motion, dispersion also has longitudinal effects...

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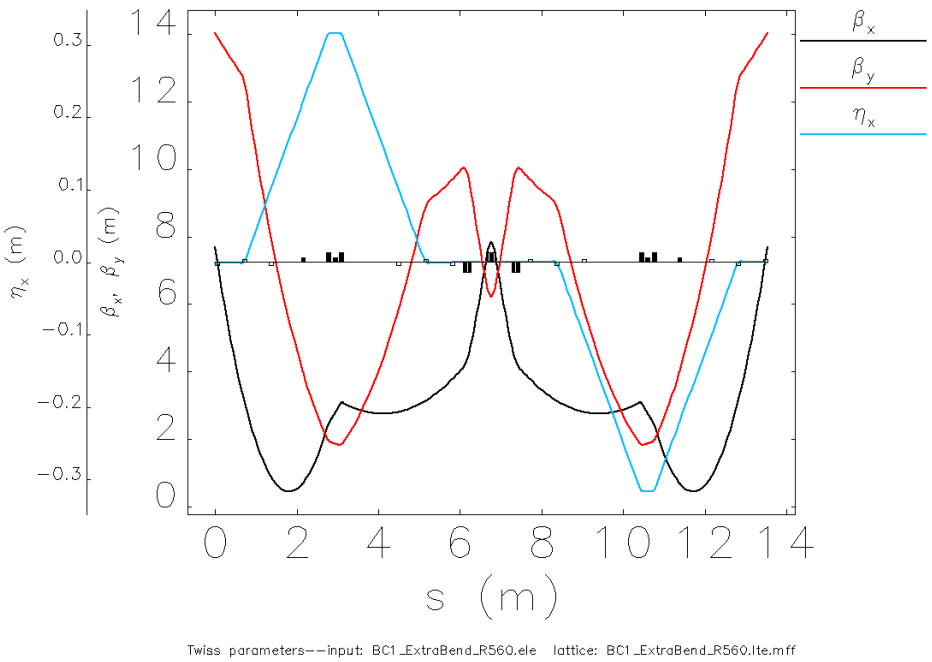
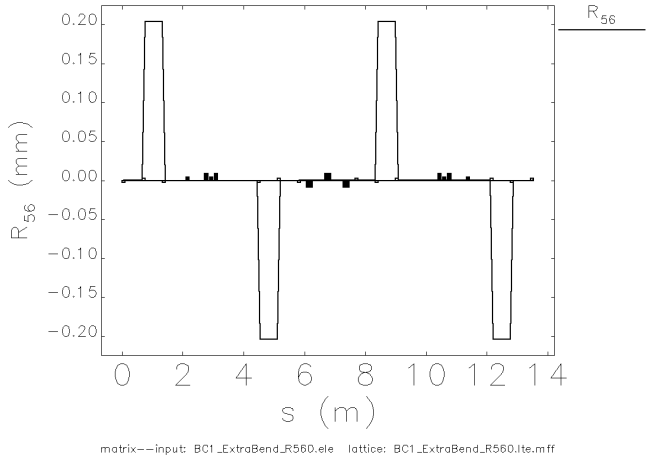
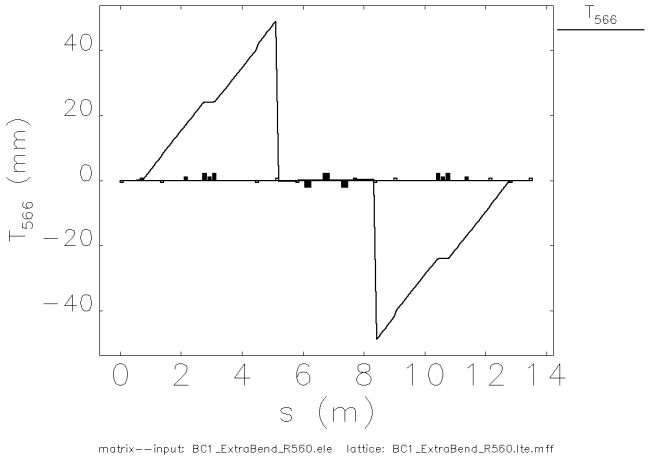
MAX-IV (Gus Perez-Segurana CI)

Higher-order terms







In addition to affecting the transverse motion, dispersion also has longitudinal effects...

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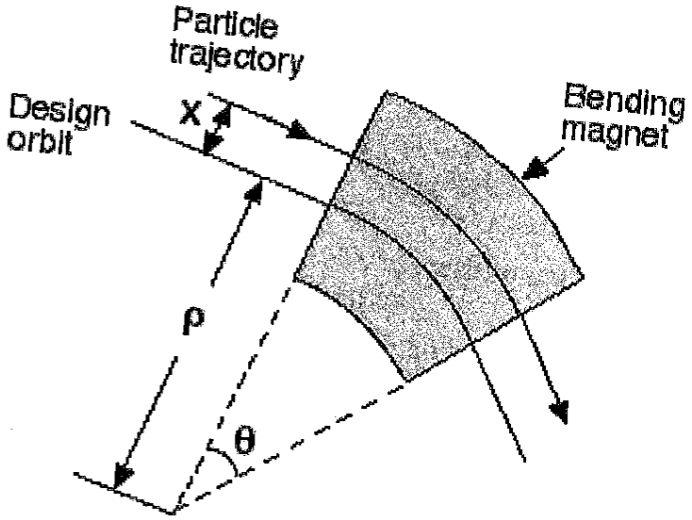
Momentum compaction

A momentum offset changes the horizontal orbit of a particle through dispersion.

Ideally, a machine with only horizontal bends does not generate any vertical dispersion

However, dispersion does generate a longitudinal effect, as the total circumference of an off-momentum particle's trip around the machine will be different to the reference particle.

Let's calculate the path length difference. Consider this situation:



$$\Delta C = (\rho + x)\theta - \rho\theta = x\theta$$

Momentum compaction

The path length deviation is given by

$$\Delta C = (\rho + x)\theta - \rho\theta = x\theta$$

The change in circumference of the machine is given by an integral over the whole ring

$$\Delta C = \oint \frac{x_{\mathrm{CO}}(s)}{\rho(s)} ds \qquad ds = \rho\, d\theta$$

For the case where the closed orbit distortion is given by a momentum error

$$\Delta C = \delta \oint \frac{D(s)}{\rho(s)} ds \qquad x_{\mathrm{CO}}(s) = D(s)\delta$$

We define the **linear** momentum compaction factor

$$\alpha_c = \frac{1}{\delta}\frac{\Delta C}{C} \quad \text{so} \quad \frac{\Delta C}{C} = \alpha_c \delta = \alpha_c \frac{\Delta p}{p}$$

Therefore the linear momentum compaction factor is given by

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Typical lattices and momentum compaction

The momentum compaction factor is an important lattice design parameter

If the orbit is exactly circular we get

$$\alpha_C = \frac{1}{2\pi\rho}\frac{1}{\rho}\oint D(s)ds = \frac{\langle D \rangle}{\rho}$$

A large value means the path length varies a lot for off-momentum particles. This means the particles tend to spread out and the bunch length becomes long.

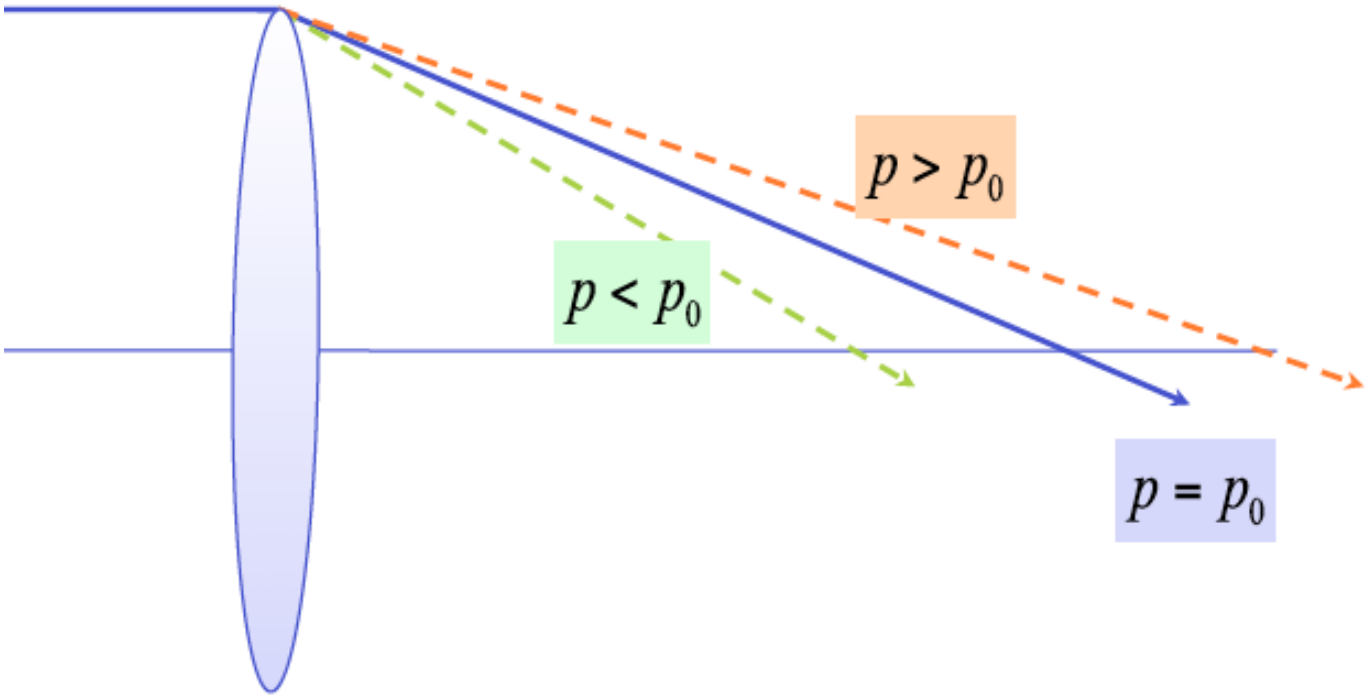
Similarly, a small value means a shorter bunch length.

Typically <D> > 0. In this case the momentum compaction factor is > 0 and the orbit gets longer for positive momentum deviations.

An **isochronous** lattice is designed to counter this natural tendency, i.e. path length doesn't depend on momentum.

Chromaticity

Consider some particles with slightly different momenta passing through a FODO cell



Higher momentum particles have a higher rigidity, so experience weaker effects when passing through magnetic fields. This means focusing is momentum-dependent and so the machine tune will depend on momentum deviation.

Chromaticity

If a machine's tunes depend linearly on the momentum deviation then

$$\nu_{x,y} = \nu_{x,y}(0) + \xi_{x,y}\delta$$

where the linear chromaticity is ξ. To analyse this we return to the equations of motion, but this time keeping all terms linear in x and δ. Recall

$$\left(\frac{\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho+x} = -\frac{eB_y}{m\gamma v_s}$$

$$x'' - \frac{\rho+x}{\rho^2} \approx -\frac{q}{p}(B_{y0}+gx)(1-\delta)(1+\frac{2x}{\rho})$$

This time, we keep the term (x.δ) we previously dropped. After dropping higher order terms we obtain the equation of motion with both a dispersion term and a 'chromatic' term

$$x'' + K_x(s)x = \frac{\delta}{\rho} + \left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)x\delta$$

As usual

$$K_x(s) = \frac{1}{\rho^2} + \frac{g}{B\rho}$$

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As usual

$$K_x(s) = \frac{1}{\rho^2} + \frac{g}{B\rho}$$

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Chromaticity

We can think of this chromatic term as a quadrupole field error of strength

$$\Delta K_x = -\left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)\delta$$

A similar analysis in the vertical plane would have a found a chromatic perturbation of

$$\Delta K_y = \frac{g}{B\rho}\delta$$

We already know how to compute the effect of a quadrupole field error.
Recall the tune shift from a quadrupole error k(s) in our lattice

$$\Delta\nu = \frac{1}{4\pi}\oint ds\beta(s)k(s)$$

Which means we can write down the tune-shift arising from the chromatic perturbation term,

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)(-1)\left[\frac{2}{\rho^2(s)} + \frac{g(s)}{B\rho}\right]\delta$$

An expression which is linear in the momentum deviation.

Natural Chromaticity

The tune change per unit delta is the linear chromaticity we defined earlier

$$\xi_{x,nat} = -\frac{1}{4\pi}\oint ds\beta_x(s)\left[\frac{2}{\rho^2(s)} + \frac{g(s)}{B\rho(s)}\right]$$

$$\xi_{x,nat} \approx -\frac{1}{4\pi}\oint ds\beta_x(s)\left[\frac{g(s)}{B\rho(s)}\right] \qquad 1/\rho^2(s) \approx 0$$

We call this chromaticity 'natural' as any lattice with quadrupoles generates this chromaticity. Similarly in the vertical plane

$$\xi_{y,nat} = +\frac{1}{4\pi}\oint ds\beta_y(s)\frac{g(s)}{B\rho}$$

As the beta function is biggest in focusing quadrupoles the natural chromaticity is normally negative in both planes.

The linear chromaticity is sometimes written as Q'

$$\Delta Q = Q'\delta$$

For a FODO cell we can show that

$$\xi_{x,nat} = -\frac{\beta_F - \beta_D}{4\pi f}$$

Is chromaticity bad?

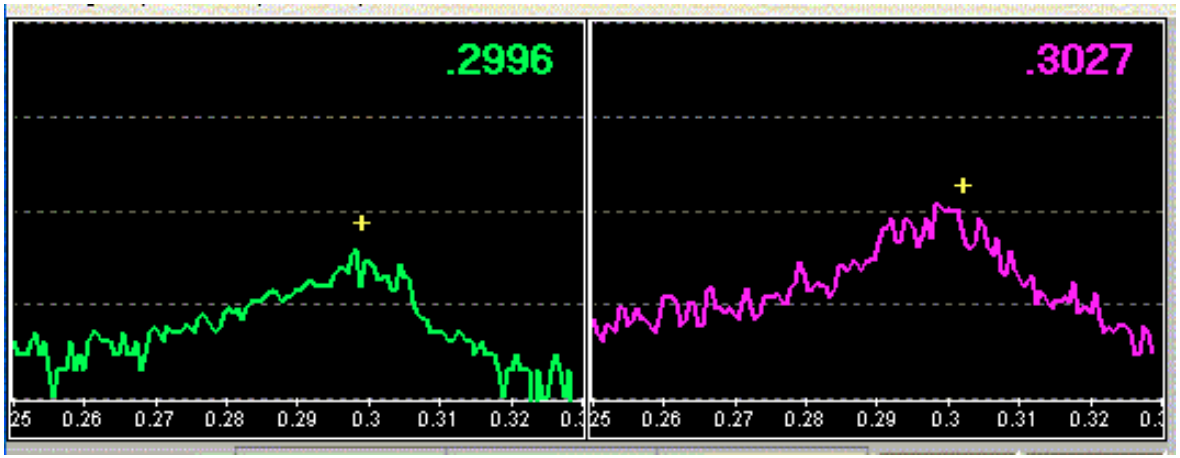
Chromaticity is naturally generated by any focusing lattice. So when we have non-zero k we have chromaticity, and it tends to be negative in both planes.

$$\xi_{x,nat} = -\frac{1}{4\pi}\oint ds\beta_x(s)\frac{g(s)}{B\rho}$$

It tells us how much the tune shifts per a unit shift in the momentum deviation.

Since any beam will an energy/momentum spread, chromaticity tells us the spread of the tune of the beam.

Tune is therefore a blob not a point in tune-space.
This is a plot of tune at HERA, showing the spread of tunes due to uncorrected chromaticity. It crosses many resonances….



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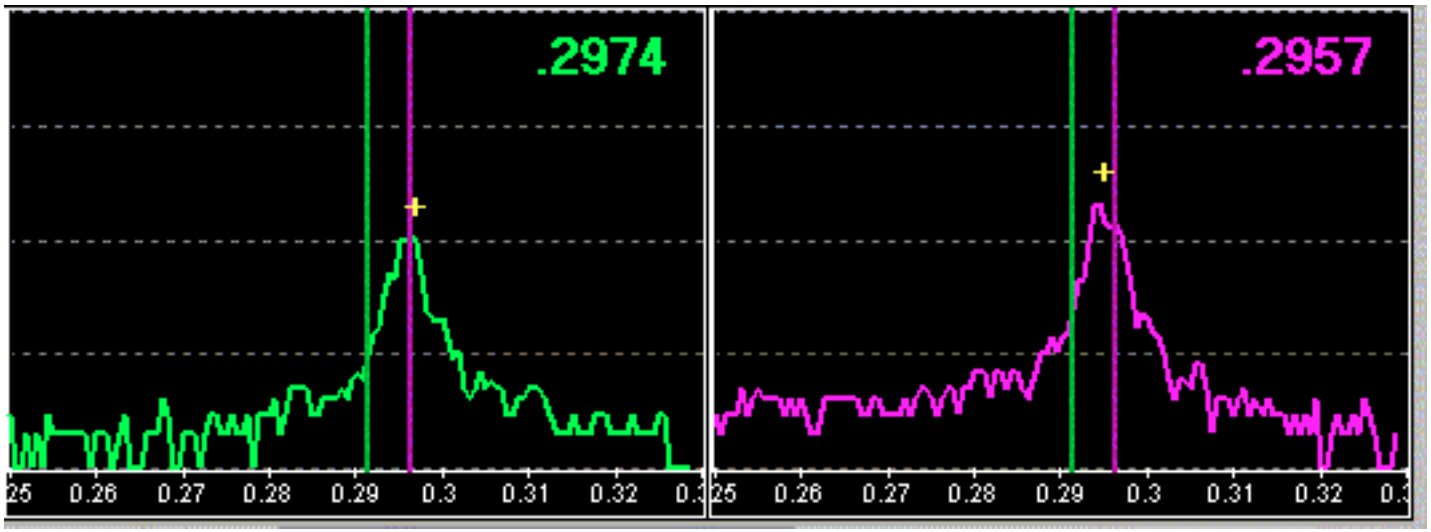
Tune after chromaticity correction



We need a mechanism to correct for chromaticity.

Chromaticity originates when off-momentum particles ‘see’ a different quadrupole field than an on-momentum particle.

So we need a correcting device which has some kind of momentum-dependent focusing…..

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Sextupoles

A sextupole field has field components given by

$$B_x = Sxy \qquad B_y = \frac{S}{2}(x^2 - y^2)$$

where we define the sextupole strength by

$$S = \frac{d^2 B_y}{dx^2}$$

Note the field is quadratic in x and y, and also (for the first time) we see products of x and y in our equations. A sextupole couples the horizontal and vertical beam motion.

Sextupoles

An off-momentum particle passing through the sextupole has displacement

$$x = x_\beta + D\delta \quad y = y_\beta$$

and so the fields seen by the particle are found by substitution

$$B_x = S(x_\beta + D\delta)y_\beta = Sx_\beta y_\beta + SD\delta y_\beta$$
$$B_y = \frac{S}{2}(x_\beta^2 - y_\beta^2) + Sx_\beta D\delta + \frac{S}{2}D^2\delta^2$$

There are many terms here, some helpful and some harmful. The helpful ones for us are

$$B_x = Sy_\beta D\delta \quad B_y = Sx_\beta D\delta$$

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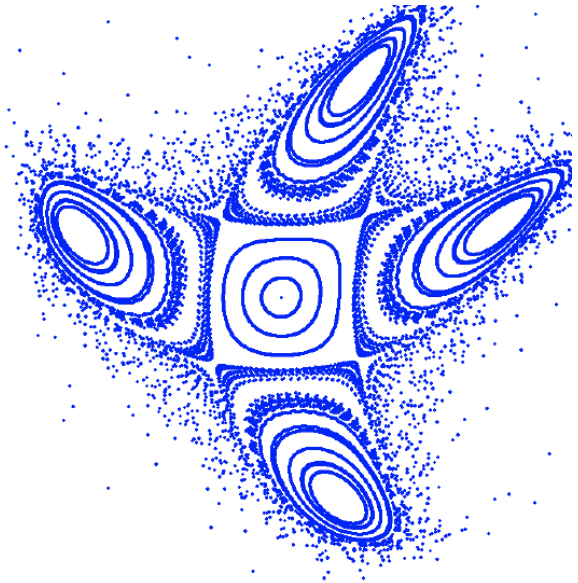
Sextupoles

The dispersion effectively makes the sextupole into a quadrupole with a momentum-dependent focusing gradient

$$B_x = Sy_\beta D\delta \quad B_y = Sx_\beta D\delta$$

This means we can compensate the chromaticity in the ring, and reduce the tune spread, by adjusting the sextupoles.

But it's not all perfect. Some sextupole terms we ignored introduce non-linearities and coupling into our accelerator ring.



It is difficult to represent a sextupole in our linear formalism, and often the best way to understand the impact of sextupole fields is to track particles with matrices, stopping to be more careful every time a sextupole is encountered. This leads to the study of a machine's dynamic aperture (i.e. how large can a particle's deviation from the closed orbit be if we want the particle to survive for many turns.)

Beams of particles and emittance

So far we've defined 'emittance' as a property of each particle. In the Courant-Snyder analysis we showed the motion of an individual particle is completely specified by its emittance and initial phase.

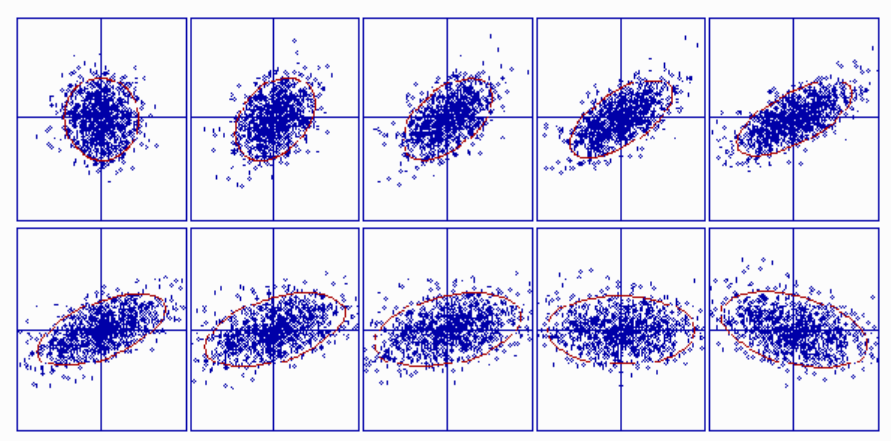
Different particles have different emittances and initial phases but they all have the same Courant-Snyder functions.

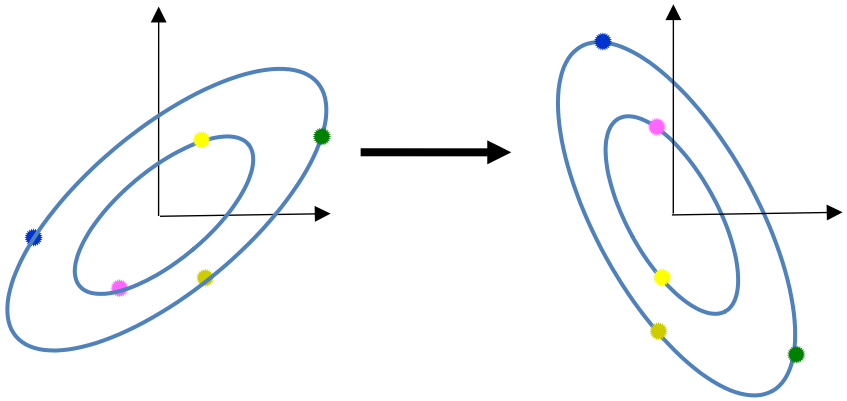
For example, the particle with x=x'=0 will have zero emittance and always stay at x=x'=0. This is the ideal particle.

But, we always have more than one particle in our beam and so need to understand how to characterise a beam of particles, each with their own emittance.

Beams of particles and emittance

We can plot the emittance (also known as the Courant-Snyder invariant for all the particles in a beam





We choose the emittance of one particle to represent the emittance of the entire beam.
For example, we can characterise the beam by the emittance of the particle for which 95% of the beam particles are within the ellipse of this particle.

Another useful definition, when dealing with complicated distributions, is the RMS emittance, which we find by averaging over the beam distribution

$$\epsilon_{\mathrm{rms}} = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$$

Beam moments

For a complex and non-linear beam distribution, we often work with the moments of the beam distribution where ρ here is the particle density not the radius of curvature!

$$< x >= \int dx \int dx' \;\; x \;\; \rho(x, x')$$

$$< x' >= \int dx \int dx' \;\; x' \;\; \rho(x, x')$$

$$< x^2 >= \int dx \int dx' \;\; (x - < x >)^2 \;\; \rho(x, x')$$

$$< x'^2 >= \int dx \int dx' \;\; (x' - < x' >)^2 \;\; \rho(x, x')$$

$$< xx' >= \int dx \int dx' \;\; (x' - < x' >)(x - < x >) \;\; \rho(x, x')$$

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Liouville's theorem

Liouville's theorem : the density of points representing particles in 6-D (x, p) phase space is conserved if all forces are conservative and differentiable.

Radiation and dissipation do not satisfy this requirement, but magnetostatic forces and (Newtonian) gravitational forces do.

There must be no (or very slow) time-dependence in the system.

Note: acceleration keeps (x,p) phase space constant, but reduces (x, x') phase space, so there is no violation of Liouville theorem.

Transfer maps derived from a Hamiltonian have a mathematical property called symplecticity, which is linked to Liouville's theorem. But this is beyond this course…

Sympleciticty in 2D phase space is equivalent to det(M)=1

J. Liouville
(1809-1882)