Collective Effects in Particle Accelerators

Part 4

The Head-Tail Instability

There are six lectures in this course on collective effects in accelerators:

1. Space charge and scattering
2. Wake fields and impedances
3. Potential well distortion and the microwave instability
4. Head-tail instability
5. Coupled bunch instabilities
6. Luminosity and the beam-beam effect

In the previous lecture, we discussed the microwave instability as an example of a single-bunch (or coasting beam) longitudinal instability.

Instabilities can appear in the transverse motion as well as the longitudinal motion. In this lecture, we shall discuss the head-tail instability, as an example of an instability involving both transverse and longitudinal motion.

By the end of this lecture, you should be able to:

- outline the mechanism leading to the head-tail instability;
- describe some of the principal phenomena associated with the head-tail instability.

Single-bunch instabilities involving longitudinal and transverse motion can be difficult to describe (and analyse)...

Some of the important aspects of the head-tail instability can be understood in terms of a (relatively) simple model of a bunch as consisting of just two particles, each with charge equal to half the total bunch charge.

We shall base our analysis of the head-tail instability on this model of a bunch consisting of two macroparticles.

At the end of the lecture, we shall briefly outline more sophisticated models, needed to describe the more complex behaviour of bunches displaying transverse instabilities.

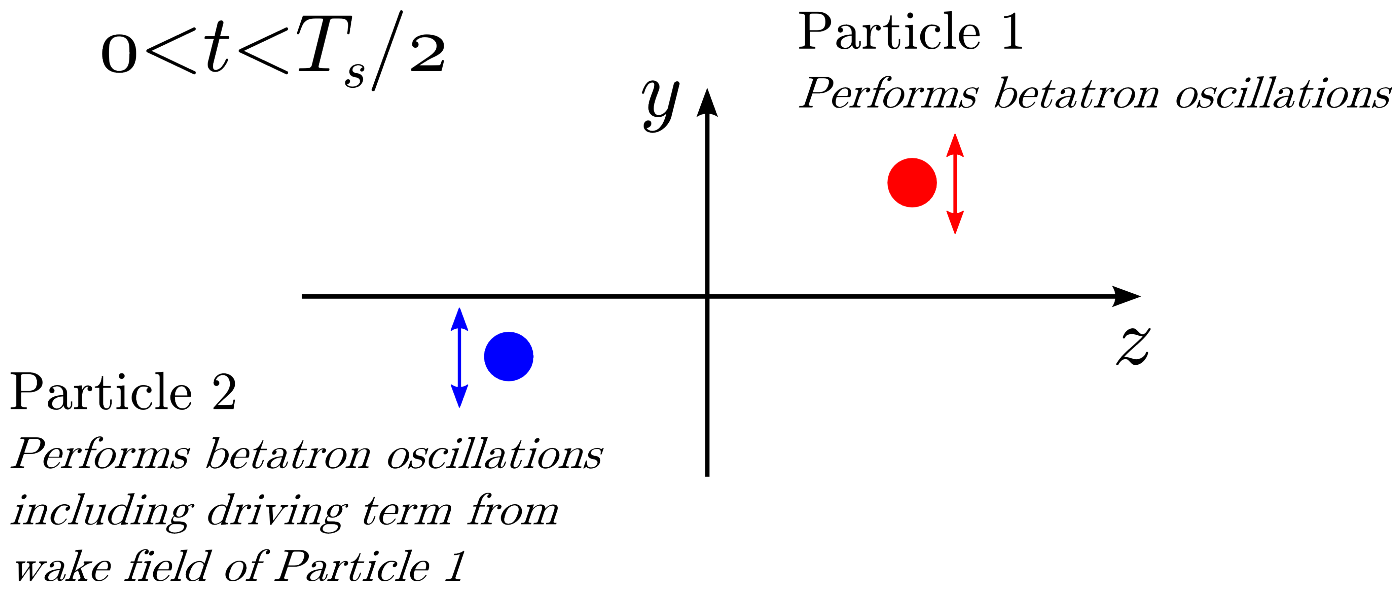
Transverse motion with wake fields: equations of motion

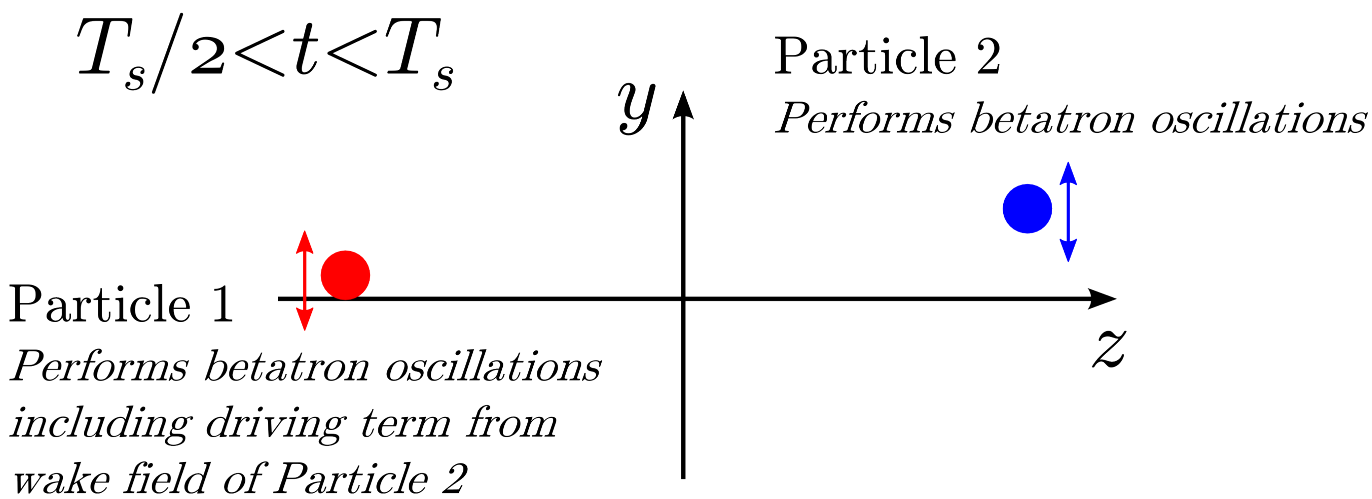
Consider a "bunch" consisting of two particles in a storage ring, with particle 1 ahead of particle 2 by a distance of order of the bunch length σ_z. We assume that the particles are ultra-relativistic.

Both particles will perform betatron oscillations as they travel around the ring. In the absence of wake fields, the betatron frequency is determined solely by the focusing in the lattice.

When wake fields are present, the trailing particle will experience additional forces from the wake field generated by the leading particle.

Synchrotron motion means that the leading and trailing particles interchange roles after half a synchrotron period, T_s.


$0 < t < T_s/2$
y
z
Particle 1
Performs betatron oscillations
Particle 2
Performs betatron oscillations including driving term from wake field of Particle 1


$T_s/2 < t < T_s$
y
z
Particle 2
Performs betatron oscillations
Particle 1
Performs betatron oscillations including driving term from wake field of Particle 2

Consider the case that particle 1 is leading particle 2.

Because the particles are ultra-relativistic, the motion of particle 1 will not be affected by any (short-range) wake fields from particle 2.

The equation of motion for transverse oscillations of particle 1 can be written:

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0, \tag{1}$$

where the dots indicate a (second) derivative with respect to time, and ω_β is the betatron oscillation frequency.

Particle 2 will observe the wake field from particle 1. When the particles pass through a section of the accelerator with wake function $W_\perp(-\Delta z)$, the transverse deflection of particle 2 from the wake field will be:

$$\Delta p_2 = -\frac{e^2 N_0}{2E_0} y_1 W_\perp(-\Delta z), \tag{2}$$

where each particle has total charge $eN_0/2$ and total energy $E_0 N_0/2$. y_1 is the transverse co-ordinate of particle 1, and the particles have longitudinal separation Δz.

The deflection can be included as a "driving force" in the equation of motion for particle 2. Taking into account the usual betatron oscillation, the equation of motion is:

$$\ddot{y}_2 + \omega_\beta^2 y_2 = -\frac{c^2 e^2 N_0}{2E_0} y_1 \frac{W_\perp(-\Delta z)}{L}, \tag{3}$$

where L is the length of the accelerator section with wake function $W_\perp(-\Delta z)$.

To solve the equations of motion (1) and (3), we need to make some assumptions and approximations. In particular, we assume that we can make a linear approximation for the wake function, so that:

$$W_{\perp}(-\Delta z) = \frac{\Delta z}{\sigma_z} W_0, \tag{4}$$

where W_0 is a constant and σ_z is the bunch length.

Furthermore, let us assume that W_0 characterises the transverse wake function over the entire circumference of the storage ring, so that the wake function per unit length is $W_0 \, \Delta z / C_0 \sigma_z$, where C_0 is the circumference of the ring.

Finally, we shall assume that the beta function is (approximately) constant, so that the betatron frequency ω_β is also constant.

With these assumptions and approximations, the equations of motion (when particle 1 is ahead of particle 2) are:

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0, \tag{5}$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = -iAy_1 e^{-i\omega_s t}, \tag{6}$$

where the constant A is defined:

$$A = \frac{c^2 e^2 N_0}{2E_0} \frac{W_0}{C_0}. \tag{7}$$

Note that we have taken the synchrotron motion into account by writing:

$$\Delta z = i\sigma_z e^{-i\omega_s t}, \tag{8}$$

where ω_s is the synchrotron frequency. The particles start with zero separation at $t = 0$, and particle 1 remains ahead of particle 2 (so that Δz is positive) for half a synchrotron period.

The next step is to find a solution to the equations of motion (5) and (6). For particle 1, the solution is straightforward:

$$y_1(t) = y_1(0)e^{-i\omega_\beta t}. \tag{9}$$

Particle 2 has a natural oscillation at frequency ω_β, but is also subject to a driving force at frequency $\omega_\beta + \omega_s$. Thus, we write a solution to the equation of motion (6):

$$y_2(t) = B_1 e^{-i\omega_\beta t} + B_2 e^{-i(\omega_\beta + \omega_s)t}, \tag{10}$$

The constants B_1 and B_2 can be determined from the initial condition $B_1 + B_2 = y_2(0)$, and by substituting the solution (10) into the equation of motion (6).

Assuming that $\omega_s \ll \omega_\beta$, we find:

$$B_2 \approx \frac{iAy_1(0)}{2\omega_\beta \omega_s}, \qquad B_1 \approx y_2(0) - \frac{iAy_1(0)}{2\omega_\beta \omega_s}. \tag{11}$$

After some further algebra (an exercise for the student!) we find from the solution to the equations of motion:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s/2} \approx e^{-i\omega_\beta T_s/2} \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}, \tag{12}$$

where $T_s = 2\pi/\omega_\beta$ is the synchrotron period, and the constant a is given by:

$$a = \frac{A}{2\omega_\beta \omega_s}\left(e^{i\omega_\beta T_s/2} - 1\right). \tag{13}$$

After half a synchrotron period, the roles of the particles are reversed, so that particle 2 sees no wake field, but particle 1 experiences the wake field from particle 2.

By symmetry, we can immediately write down, for the transverse co-ordinates of the particles after a full synchrotron period:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s} \approx e^{-i\omega_\beta T_s/2} \begin{pmatrix} 1 & ia \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s/2}. \qquad (14)$$

Combining equations (12) and (14) we obtain:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s} &\approx e^{-i\omega_\beta T_s} \begin{pmatrix} 1 & ia \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}, & (15) \\ &\approx e^{-i\omega_\beta T_s} \begin{pmatrix} 1-a^2 & ia \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}. & (16) \end{aligned}$$

Equation (16) provides us with a "transfer matrix" R for the motion of the two (macro)particles over a full synchrotron period:

$$R = e^{-i\omega_\beta T_s} \begin{pmatrix} 1 - a^2 & ia \\ ia & 1 \end{pmatrix} \tag{17}$$

Since the matrix is symplectic (it is a 2×2 matrix with unit determinant) we can say immediately that the motion of the particles will be stable if the trace of the transfer matrix has magnitude less than 2, that is:

$$\left| 2 - a^2 \right| < 2. \tag{18}$$

Using (13) and assuming the maximum magnitude for $a = |A|/\omega_\beta \omega_s$, the stability condition (18) can be written in terms of the ring and beam parameters:

$$\frac{c^2 e^2 N_0 W_0}{2\omega_\beta \omega_s C_0 E_0} < 2. \tag{19}$$

In terms of the total bunch population N_0, the stability condition is:

$$N_0 < \frac{16\pi^2 \nu_\beta \nu_s E_0}{e^2 W_0 C_0}, \tag{20}$$

where ν_β and ν_s are the betatron and synchrotron tunes, respectively.

Equation (20) expresses a limit on the bunch population in a storage ring, above which the betatron oscillation amplitudes of particles in the bunch will increase exponentially, driven by the transverse wake fields.

The instability that occurs when the bunch population exceeds the limit (20) is known as the "fast head-tail instability".

The presence of transverse wake fields will also lead to a shift in the frequency of betatron oscillations performed by the particles.

We can find the frequency shift from the solution to the equations of motion.

The frequencies of the normal modes are given by the eigenvalues of the full 4×4 transfer matrix, describing the changes in co-ordinates *and the transverse momenta* of the particles over one synchrotron period.

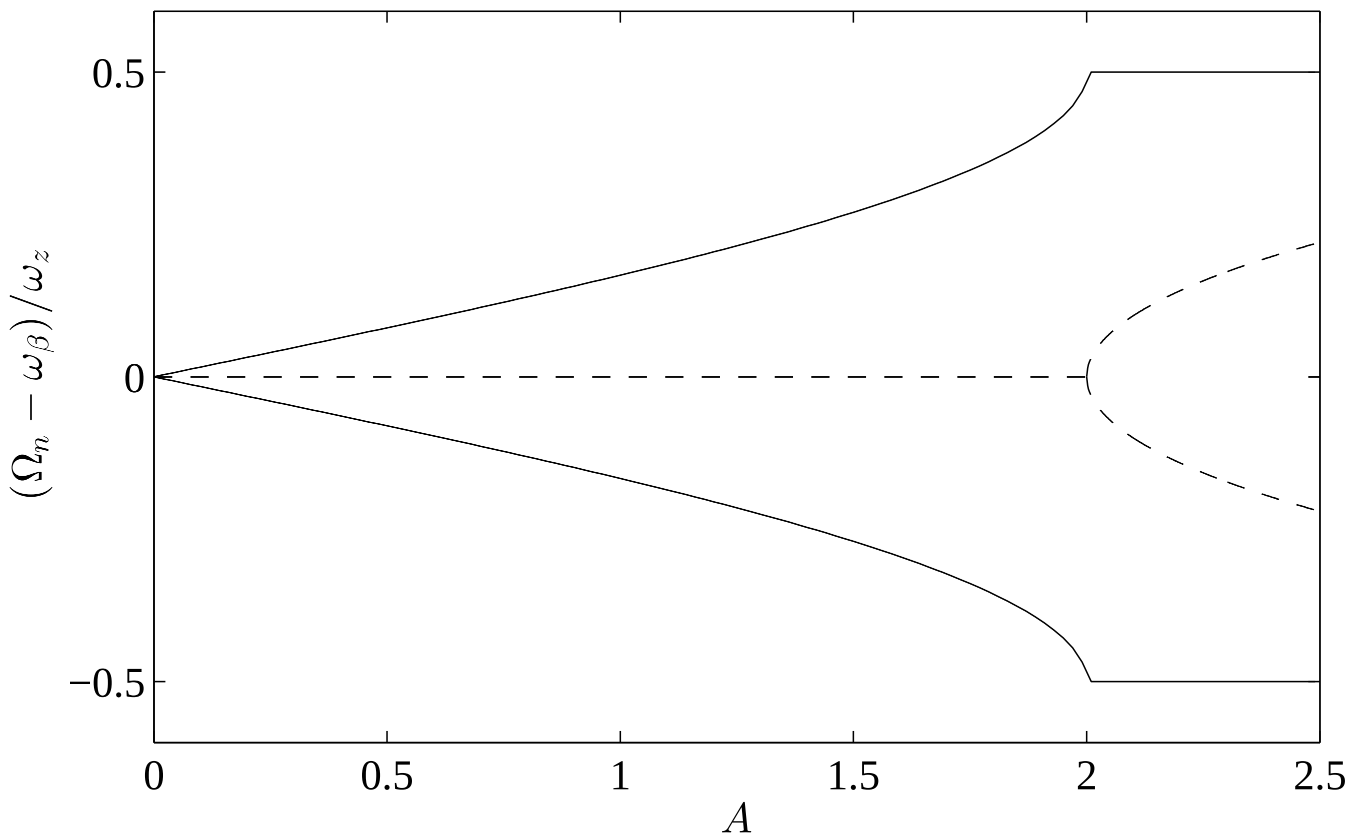
We do not give the details of the calculation here: for further information, see (for example) "Beam Dynamics in High Energy Particle Accelerators," (Wolski, 2014) section 15.4.1.

If the wake fields are weak, then the normal mode frequencies are purely real: the particles perform betatron oscillations with constant amplitude.

If the wake fields are increased (e.g. by increasing the bunch charge) the betatron frequencies acquire non-zero imaginary parts.

At this point the oscillation amplitude of one mode will be damped, but the amplitude of the other mode will grow (exponentially)...

Transverse motion with wake fields: stability condition



Real and imaginary parts (solid and dashed lines, respectively) of normal mode frequency shifts (in units of the synchrotron frequency), as a function of wake field strength.

It is important to remember that in deriving the stability condition (20), we made a number of assumptions and approximations. In particular, our analysis has been based on a rather crude model of a bunch consisting of just two macroparticles.

We should not expect equation (20) to give anything more than a rough indication of the limit of beam stability, in the presence of transverse wake fields.

We have also ignored some important effects, including (for example) chromaticity.

If chromaticity is taken into account, then the behaviour of the system is modified: the instability in this case is known as the "head-tail instability" (rather than the "fast head-tail instability").

The head-tail instability has no threshold, but occurs for any non-zero value of the chromaticity.

However, the growth rates are slow enough that (especially for small, positive chromaticity) the instability can generally be suppressed by natural damping effects such as radiation and decoherence.

A more sophisticated model of transverse single-bunch instabilities can be developed using the Vlasov equation.

The Vlasov equation describes the evolution of a charge distribution, rather than the motion of individual (macro)particles.

Recall that our analysis of the microwave instability (a longitudinal single-bunch instability) in Part 3 of this lecture course was developed from the Vlasov equation.

The analysis of transverse single-bunch instabilities is similar in some ways to the analysis of the microwave instability, but starts from the Vlasov equation extended to include transverse as well as longitudinal degrees of freedom.

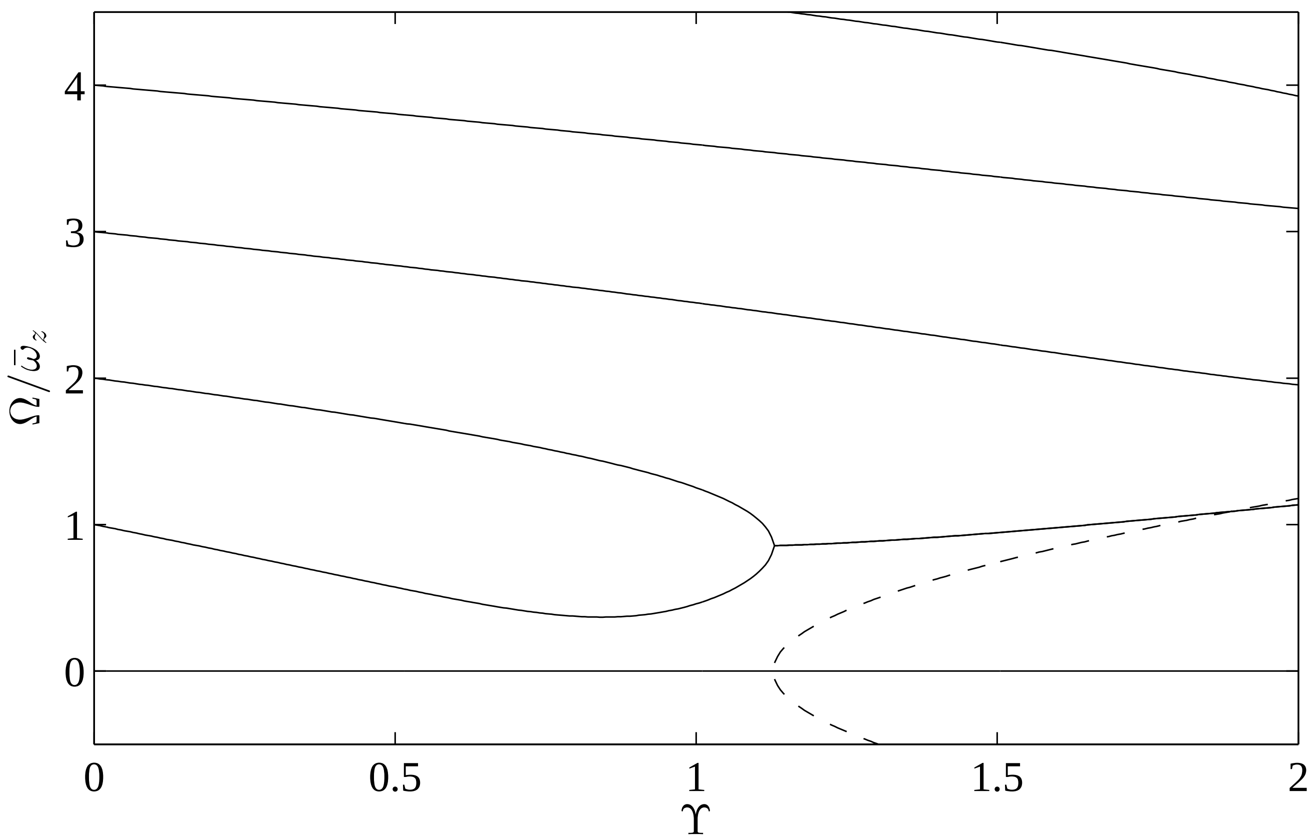
We do not go into details here of the analysis of transverse instabilities using the Vlasov equation, but simply mention the main result.

It is found that perturbations to a given distribution have different oscillation frequencies, depending on the given distribution and the mode of the perturbation.

Wake fields shift the mode oscillation frequencies: if the wake fields are strong enough, the frequencies of two different modes can become equal.

When this happens, the frequencies acquire non-zero imaginary parts, indicating the exponential growth of the perturbation, i.e. the onset of an instability, known as the “transverse mode-coupling instability”.

Transverse mode-coupling instability



Mode frequencies in transverse and longitudinal degrees of freedom, as a function of the wake field strengths.

For further details, see (for example):

- A.W. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators," Wiley (1993).

- A. Wolski, "Beam Dynamics in High Energy Particle Accelerators," Imperial College Press (2014).

A simple model of a transverse single-bunch instability can be developed using a model of the bunch as two macroparticles.

With some assumptions, and ignoring chromaticity, it is found that there is a threshold (in terms of the bunch charge, or the wake field strength) above which the betatron oscillations of the particles grow exponentially. The instability in this case is known as the *fast head-tail instability*.

With non-zero chromaticity, there is no threshold, but the beam appears always to be unstable. However, in practice it is often possible to suppress the instability by operating with a small positive chromaticity.

A more sophisticated model of transverse single-bunch instabilities, including the *transverse mode-coupling instability* can be developed using the Vlasov equation to describe the evolution of a charge distribution in transverse and longitudinal degrees of freedom.