The Cockcroft Institute
of Accelerator Science and Technology

UNIVERSITY OF
LIVERPOOL

Collective Effects in Particle Accelerators

Part 3

Potential Well Distortion, the Microwave Instability and Landau Damping

There are six lectures in this course on collective effects in accelerators:

1. Space charge and scattering
2. Wake fields and impedances
3. Potential well distortion and the microwave instability
4. Head-tail instability
5. Coupled bunch instabilities
6. Luminosity and the beam-beam effect

In this lecture, we shall discuss how wake fields in a storage ring can change the longitudinal distribution (potential well distortion), or lead to a beam instability (microwave instability).

By the end of this lecture, you should be able to:

- explain how the Haissinski equation can be used to determine the equilibrium longitudinal distribution of charge in a bunch in a storage ring in the presence of wake fields;
- outline the mechanism leading to the microwave instability;
- explain what is meant by *Landau damping*, and describe how (below a threshold bunch charge) Landau damping can suppress the development of the microwave instability;
- apply simple approximate formulae to estimate the microwave instability threshold in a storage ring.

In an electron storage ring, the combined effects of synchrotron radiation and longitudinal focusing (from the RF cavities) determine the longitudinal distribution of charge within individual bunches.

Longitudinal wake fields can contribute to the change in energy of particles as a bunch moves around a storage ring. If the wake fields are long enough, this can have observable effects.

The effects depend on the strengths of the wake fields. The wake fields may be weak enough that their effects are negligible; or, they may be strong enough to distort the equilibrium shape of the bunch (potential well distortion).

Very strong wake fields can lead to an instability, in which the longitudinal charge distribution fails to reach equilibrium at all.

Example: Potential Well Distortion in the SLC Damping Ring

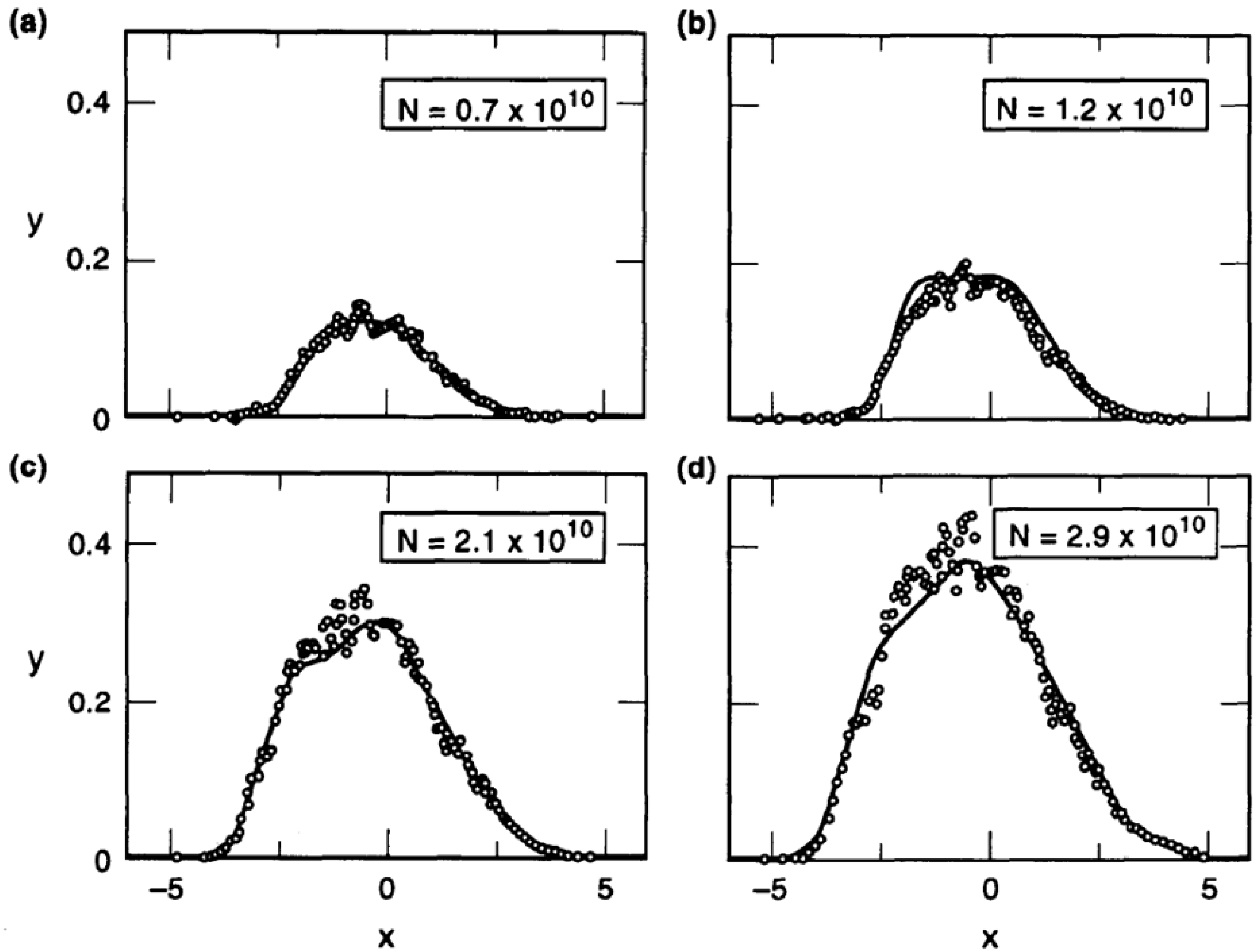


Figure 6.4. Potential-well distortion of bunch shape for various beam intensities for the SLC damping ring. The open circles are the measured results. The horizontal axis is $x = -z/\sigma_{z0}$, where σ_{z0} is the unperturbed rms bunch length. The vertical scale gives $y = 4\pi e\rho(z)/V'_{rf}(0)\sigma_{z0}$. (Courtesy Karl Bane, 1992.)

To understand the impact of wake fields on the charge distribution, we first consider a simpler case: the motion of a single particle in the absence of wake fields.

We shall first review the equations of motion for a single particle performing synchrotron oscillations in a synchrotron storage ring.

We shall then show that the equilibrium bunch length and energy spread are both described by Gaussian distributions.

In an electron storage ring, the equilibrium energy spread is determined by synchrotron radiation. The equilibrium bunch length is related to the energy spread by the parameters of the storage ring.

After reviewing these results for the case without wake fields, we shall consider the impact that longitudinal wake fields will have on the equilibrium distribution.

Averaged over one turn of the storage ring, the rate of change of the longitudinal co-ordinate z (relative to a reference particle at the centre of the bunch) is determined by the momentum compaction factor α_p of the lattice and the energy deviation $\delta = (E - E_0)/E_0$ of the particle:

$$\frac{dz}{ds} = -\alpha_p \delta, \tag{1}$$

where s is the longitudinal position along the closed orbit.

The momentum compaction factor is determined by the optics:

$$\alpha_p = \frac{1}{C_0} \oint \frac{\eta_x}{\rho}\, ds, \tag{2}$$

where C_0 is the circumference of the orbit, η_x is the dispersion function, and ρ is the radius of curvature of the closed orbit.

The rate of change of the energy of the particle (averaged over the lattice) is given by the energy gain from the RF cavities and the energy loss from synchrotron radiation:

$$\frac{d\delta}{ds} = \frac{eV_{\mathrm{RF}}}{E_0 C_0} \sin\left(\phi_s - \frac{\omega_{\mathrm{RF}} z}{c}\right) - \frac{U}{E_0 C_0}, \tag{3}$$

where V_{RF} is the RF voltage; E_0 is the beam energy; ϕ_s is the synchronous phase; ω_{RF} is the RF frequency; and U is the energy lost per turn through synchrotron radiation.

The synchronous phase is determined by the condition that a particle arriving at the RF cavities with $z = 0$ gains exactly the right amount of energy from the RF voltage to balance the energy lost in one turn from synchrotron radiation:

$$eV_{\mathrm{RF}} \sin(\phi_s) = U. \tag{4}$$

The synchrotron radiation energy loss per turn, U, depends on the energy of the particle: the more energy a particle has, the more energy it loses in the dipoles from synchrotron radiation.

This effect leads to the damping of synchrotron oscillations.

However, to a first approximation, we can neglect radiation damping, and set $U = U_0$, where U_0 is the energy lost by a particle with the reference energy E_0.

If we assume that z is small, so that the particle crosses the RF cavities close to the synchronous phase, then the equations of motion (1) and (3) become (with some approximations...):

$$\frac{dz}{ds} = -\alpha_p \delta, \tag{5}$$

$$\frac{d\delta}{ds} = -\frac{eV_{\mathrm{RF}}}{E_0 C_0}\frac{\omega_{\mathrm{RF}}}{c}\cos(\phi_s) z. \tag{6}$$

Equations (5) and (6) describe simple harmonic motion in longitudinal phase space (with co-ordinate z and conjugate momentum δ), with angular frequency ω_s given by:

$$\frac{\omega_s^2}{c^2} = -\frac{eV_{\mathrm{RF}}}{E_0 C_0}\frac{\omega_{\mathrm{RF}}}{c}\alpha_p \cos(\phi_s). \tag{7}$$

The equations of motion (5) and (6) can be derived from a Hamiltonian:

$$H = -\frac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\frac{\omega_s^2}{c^2} z^2, \tag{8}$$

by applying Hamilton's equations:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta}, \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z}. \tag{9}$$

The Hamiltonian is a constant of the motion. Therefore, any function of the Hamiltonian will also be a constant of the motion.

An equilibrium distribution of charge within a bunch has the property that it remains unchanged (invariant) from one turn to the next, even though individual particles within the bunch move around the phase space.

Putting together the two statements above, we should be able to write the equlibrium distribution (describing all the particles within a bunch) as a function of the Hamiltonian (which describes the motion of individual particles).

At low bunch charges, where the longitudinal wake fields are negligible, the longitudinal charge distribution in an electron storage ring is usually Gaussian:

$$\Psi(z,\delta) = \Psi_0 \exp\left(-\frac{H}{H_0}\right) = \Psi_0 \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right), \quad (10)$$

where $\Psi(z,\delta)$ is the charge density in longitudinal phase space, with peak value Ψ_0 at $z = \delta = 0$.

H is the Hamiltonian (8), and H_0 is a constant related to the rms energy spread:

$$H_0 = \alpha_p \sigma_\delta^2. \quad (11)$$

Note that the Hamiltonian (8) determines the relationship between the rms bunch length σ_z and the rms energy spread σ_δ:

$$\sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \quad (12)$$

Although this analysis gives the *shape* of the equilibrium distribution, it does not give specific values for σ_δ or σ_z.

In the absence of synchrotron radiation, any value for the energy spread (within a limit set by the RF acceptance) can yield an equilibrium distribution.

But in electron storage rings, the equilibrium energy spread is determined by the balance between radiation damping and quantum excitation.

In the absence of collective effects (i.e. in the limit of low bunch charge) the equilibrium energy spread in an electron storage ring depends on the beam energy and the lattice design: see Appendix A.

Let us now consider the impact of wake fields on the equilibrium charge distribution in an electron storage ring.

The longitudinal wake function $W_{\parallel}(\Delta z)$ for a given section of beam line is defined so that for a particle of charge e following a particle of charge Ne through the beam line, the change in energy of the trailing particle is:

$$\Delta\delta = -\frac{Ne^2}{E_0} W_{\parallel}(\Delta z), \tag{13}$$

where Δz is the longitudinal distance between the two particles.

For a particle within a bunch, we have to sum the contributions from all the "slices" within the bunch ahead of the given particle:

$$\Delta\delta(z) = -\frac{e}{E_0} \int_z^{\infty} \lambda(z') W_{\parallel}(z - z')\, dz', \tag{14}$$

where $\lambda(z)$ is the longitudinal charge density (charge per unit length, C/m) within the bunch.

If the wake function $W_{\|}(\Delta z)$ represents the wake field for the entire storage ring, then the equations of motion (5) and (6) become:

$$\frac{dz}{ds} = -\alpha_p \delta, \tag{15}$$

$$\frac{d\delta}{ds} = \frac{\omega_s^2}{\alpha_p c^2} z - \frac{e}{E_0 C_0} \int_z^\infty \lambda(z') W_{\|}(z - z')\, dz', \tag{16}$$

where ω_s is the synchrotron frequency in the absence of wake fields, given by (7).

These equations of motion can again be derived from a Hamiltonian, which now takes the form:

$$H_{\text{wf}} = -\frac{1}{2}\alpha_p \delta^2 - \frac{\omega_s^2}{2\alpha_p c^2} z^2 + \frac{e}{E_0 C_0} \int_0^z dz' \int_{z'}^\infty dz''\, \lambda(z'') W_{\|}(z' - z''). \tag{17}$$

Using the same reasoning as before, the equilibrium distribution can be assumed to be a function of the Hamiltonian, H_{wf}.

Assuming that the particles again have a Gaussian momentum distribution, the charge density in longitudinal phase space will be given by:

$$\Psi(z,\delta) = \Psi_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right) \exp\left(-\frac{z^2}{2\sigma_z^2} + \frac{e}{\alpha_p \sigma_\delta^2 E_0 C_0} \int_0^z dz' \int_{z'}^{\infty} dz''\, \lambda(z'') W_\parallel(z' - z'')\right), \tag{18}$$

where, as before:

$$\sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{19}$$

Since the longitudinal charge density $\lambda(z)$ is given by:

$$\lambda(z) = \int_{-\infty}^{\infty} \Psi(z,\delta)\, d\delta, \tag{20}$$

we can integrate both sides of (18) with respect to δ, to obtain an integral equation for $\lambda(z)$, known as the *Haissinski equation*:

$$\lambda(z) = \lambda_0 \exp\left(-\frac{z^2}{2\sigma_z^2} + \frac{e}{\alpha_p \sigma_\delta^2 E_0 C_0} \int_0^z dz' \int_{z'}^{\infty} dz''\, \lambda(z'') W_\parallel(z'-z'') \right). \tag{21}$$

The value of the constant λ_0 is determined by the condition that the integral over $\lambda(z)$ is equal to the total charge in the bunch:

$$\int_{-\infty}^{\infty} \lambda(z)\, dz = Ne, \tag{22}$$

The synchrotron radiation effects are not affected by the wake fields, so the rms energy spread σ_δ remains the same as in the case without wake fields (45).

However, the longitudinal distribution $\lambda(z)$ is affected by the presence of wake fields. The equilibrium distribution depends on the lattice parameters, beam energy, wake fields and bunch charge.

Note that the parameter σ_z in equation (21) corresponds to the rms bunch length *in the limit of zero bunch charge* (i.e. σ_z is the rms equilibrium bunch length without wake fields). This is sometimes called the *natural bunch length*.

As an example, we consider potential well distortion in the damping rings for the International Linear Collider (ILC).

The damping rings take the large emittance beams from the electron and positron sources, and produce small emittance beams for acceleration and collision. The bunch length is a key parameter.

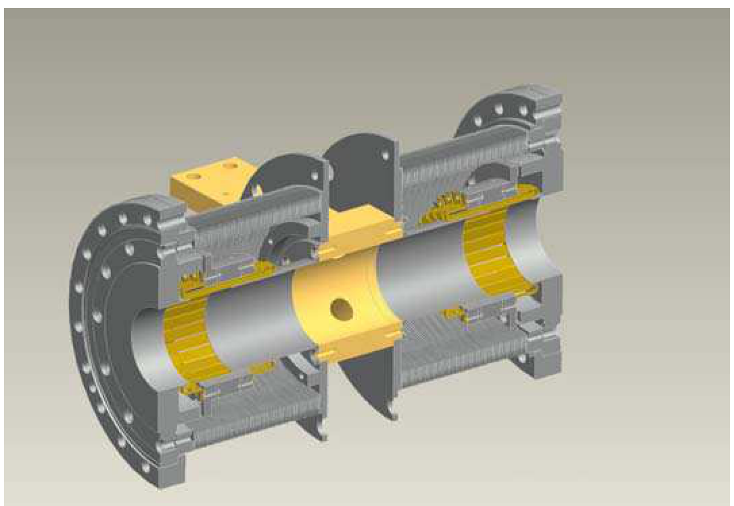
Circumference	6476 m		
Beam energy	5 GeV		
Natural bunch length, σ_z	6 mm		
Particles per bunch	2×10^{10}		
Arc cell phase advance	72°	90°	100°
Momentum compaction, α_p	2.9×10^{-4}	1.6×10^{-4}	1.3×10^{-4}
Natural emittance, $\gamma\epsilon_x$	6.4 μm	4.4 μm	3.9 μm
RF voltage	32.6 MV	20.4 MV	17.1 MV

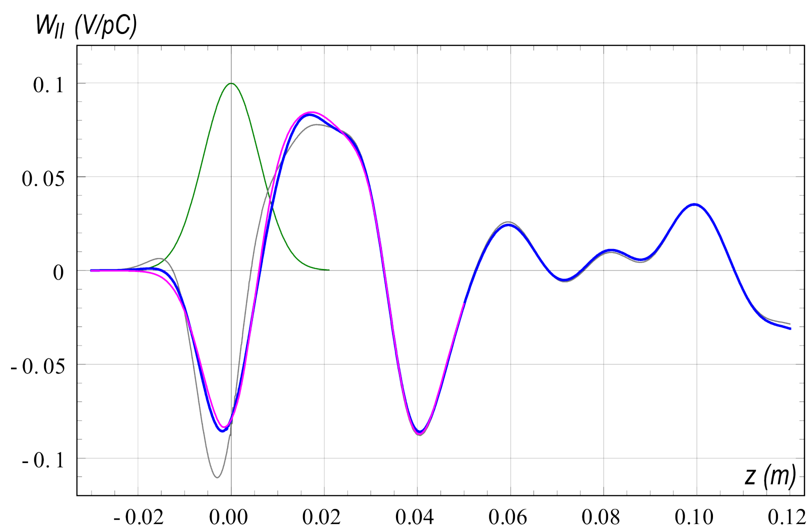
Note that the momentum compaction factor can be adjusted by changing the phase advance in the arc cells. A larger momentum compaction factor raises the threshold for some beam instabilities; but means that a larger RF voltage is needed to achieve the specified bunch length.

The first step is to calculate the wake fields from numerical solutions to Maxwell's equations for a bunch of charged particles passing through a given section of the vacuum chamber.

In the ILC damping rings, the BPMs are expected to make a significant contribution to the longitudinal wake fields.

CST Particle Studio is used to compute the wake potential for a short (few mm) bunch passing through a BPM section: the wake field is the derivative of the wake potential.

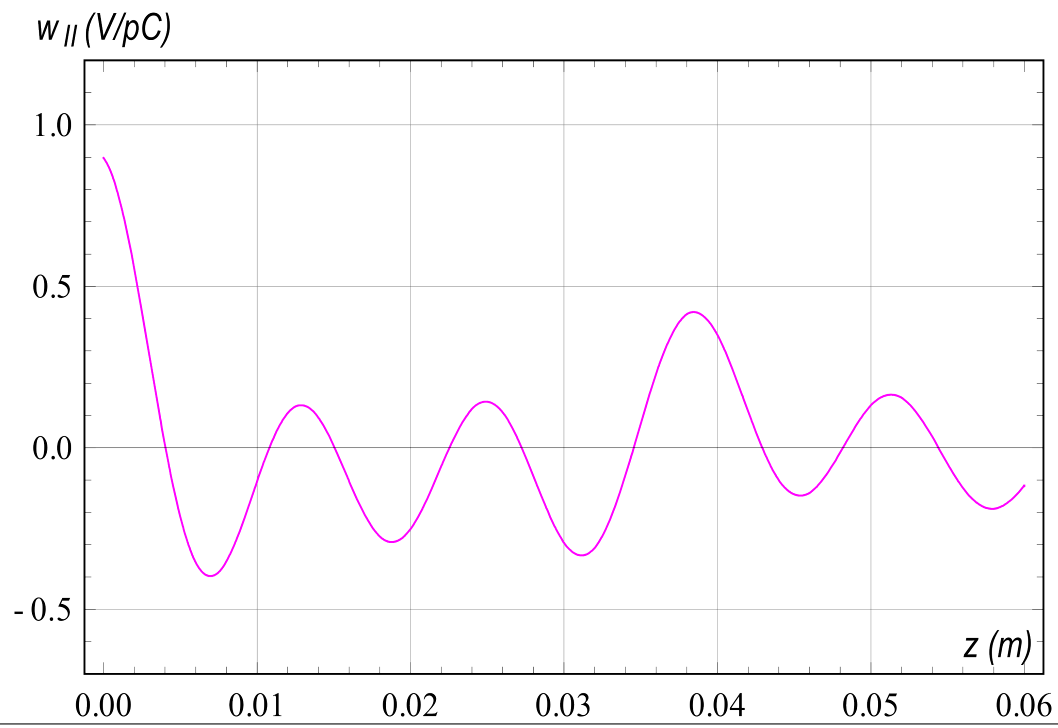




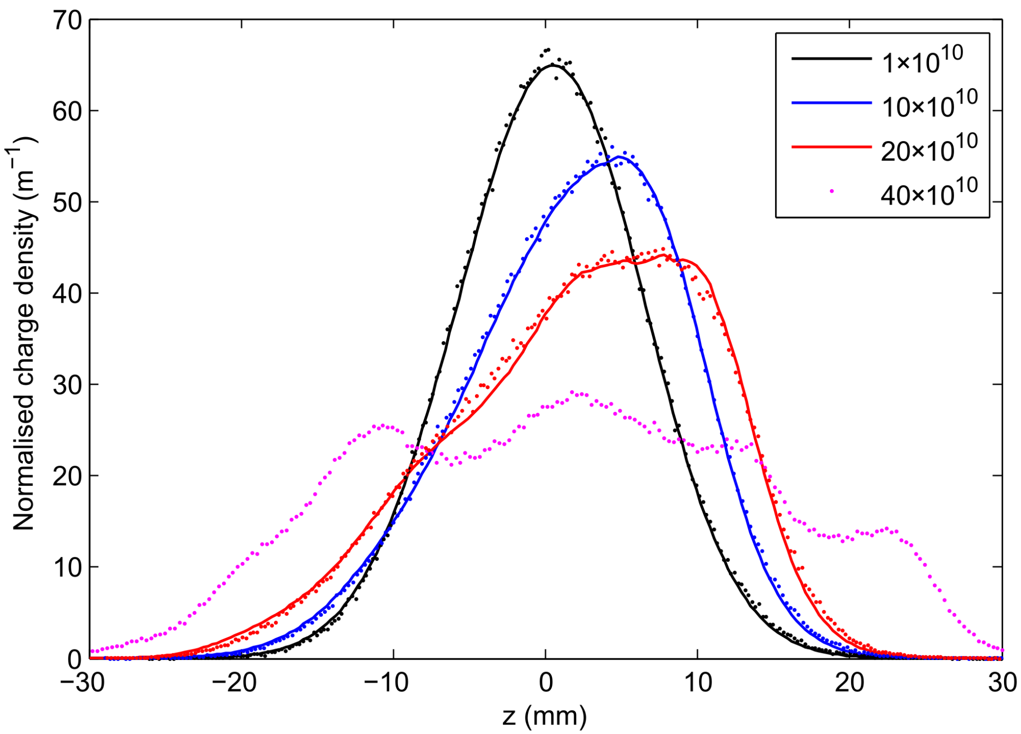
The wake function is the wake field produced by a bunch of charged particles in the limit of zero bunch length.

The wake field is the convolution of the charged particle distribution in the bunch with the wake function.

Hence, the wake function is obtained by deconvolving the wake field and the bunch profile.

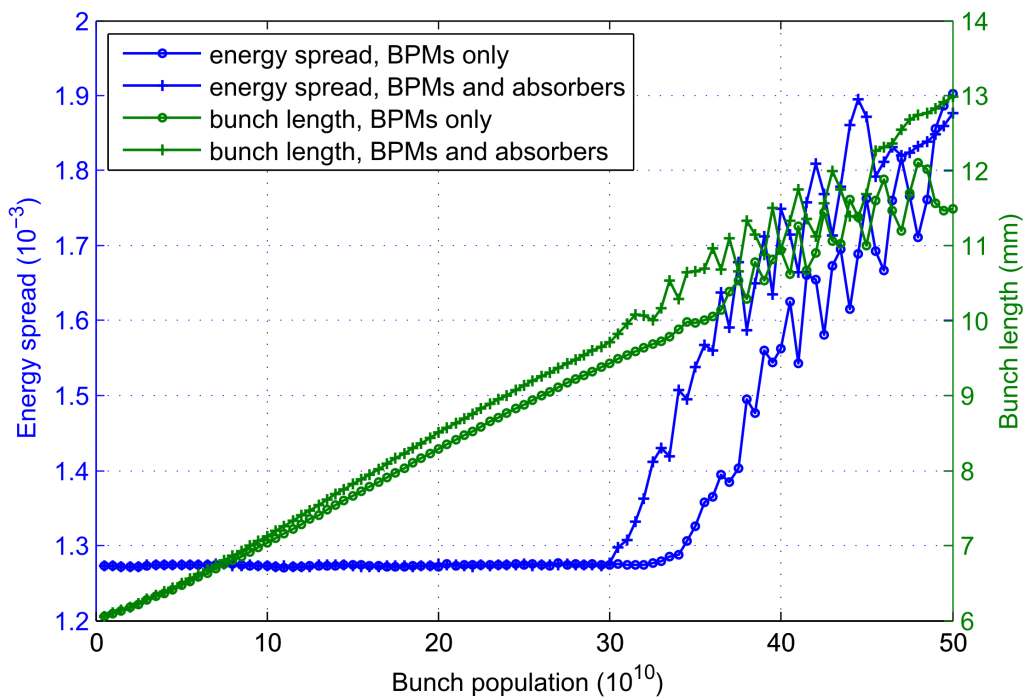


Having obtained the wake function, the equilibrium bunch profile can be found for different bunch charges by solving (numerically) the Haissinski equation (21).



The solutions to the Haissinski equation (solid lines in the above plot) can be compared with the results from particle tracking (dots). At high bunch charges, no equilibrium solution exists: the bunch is unstable.

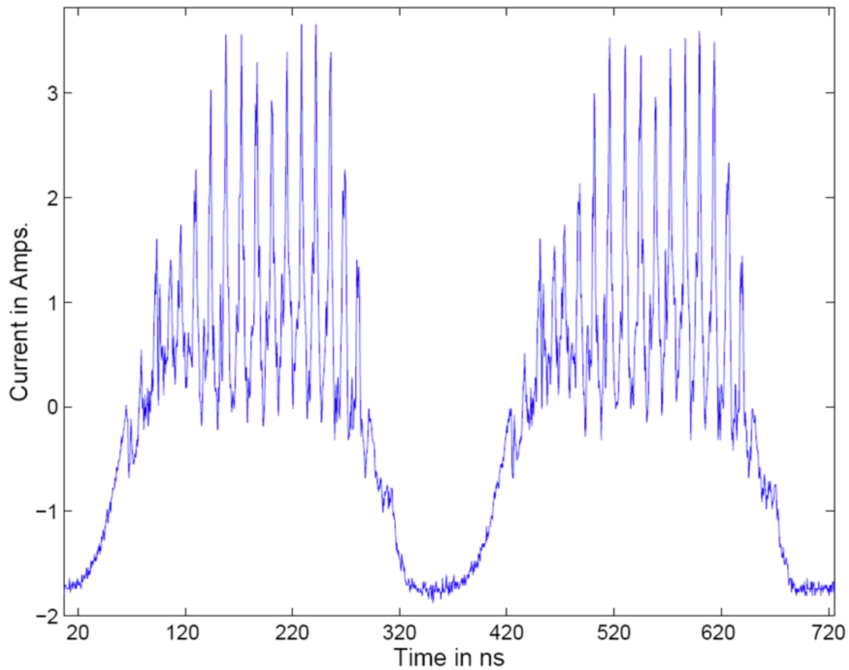
The effect of potential well distortion on the longitudinal phase space distribution can be seen by plotting the rms bunch length and rms energy spread as functions of the bunch population.



Note that the rms energy spread remains (roughly) constant up to a bunch population of (roughly) 30×10^{10} particles: if the population is increased beyond this point, the bunch becomes unstable (fails to reach an equilibrium).

In the remainder of this lecture, we shall discuss how wake fields can lead to instability in the longitudinal phase space distribution of charge in individual bunches in a storage ring.

We shall focus on a type of instability known as the "microwave instability". This is characterised by the appearance of structures within a bunch on a scale small compared to the overall bunch length.



Observation of single-bunch longitudinal instability in the Los Alamos PSR, caused by an inductive impedance.

From C. Beltran, A.A. Browman and R.J. Macek, "Calculations and observations of the longitudinal instability caused by the ferrite inductors at the Los Alamos Proton Storage Ring", Proceedings of the 2003 Particle Accelerator Conference, Portland, Oregon.

To understand the behaviour of a bunch of particles that is not in equilibrium, we need an equation describing the dynamics of the charge distribution within the bunch.

An appropriate description is provided by the *Vlasov equation*, which may be "derived" from Liouville's theorem.

Liouville's theorem tells us that in a storage ring, if we neglect synchrotron radiation and collective effects, the density of charge in phase space is constant. This can be expressed:

$$\frac{d\Psi}{ds} = 0, \tag{23}$$

where Ψ is the charge per unit volume in phase space, and s is the distance along the reference trajectory.

For simplicity, we shall consider the case of a "coasting beam": we neglect synchrotron radiation, and assume that there are no RF cavities in the ring.

Let us consider just the longitudinal phase space, with co-ordinate $\theta = 2\pi s/C_0$ (where C_0 is the ring circumference) and conjugate momentum δ (the energy deviation of a particle).

We shall further assume that the longitudinal motion is decoupled from the transverse (horizontal and vertical) motion.

With these assumptions, Liouville's theorem (23) can be expressed:

$$\frac{\partial \Psi}{\partial s} + \frac{d\theta}{ds}\frac{\partial \Psi}{\partial \theta} + \frac{d\delta}{ds}\frac{\partial \Psi}{\partial \delta} = 0. \tag{24}$$

Equation (24) is the *Vlasov equation*: it is a partial differential equation describing the evolution of charge density in the longitudinal phase space of a coasting beam in a storage ring.

Single-bunch beam instabilities can be described by non-stationary solutions to the Vlasov equation.

However, in general, solving the Vlasov equation is not easy. Usually, accurate solutions can only be obtained by numerical integration.

In some cases, an approach based on perturbation theory can be useful. In this approach, we consider a small change to an assumed equilibrium distribution.

If we find (from the Vlasov equation) that the size of the perturbation grows over time, then this indicates the presence of an instability.

The steps appropriate for an analysis of the microwave instability are as follows:

1. Assume an initial phase space distribution of the form:

$$\Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi \, e^{i(n\theta - \omega_n t)}, \tag{25}$$

   where $\Psi_0(\delta)$ is an assumed stationary (equilibrium) distribution, and $\Delta\Psi$ is the amplitude of a density modulation with "wavelength" C_0/n and oscillation frequency ω_n.

2. Substitute the distribution into the Vlasov equation, and expand each term to first order in $\Delta\Psi$.

3. Solve the resulting equation for the frequency of oscillation of the perturbation, ω_n.

If there is a solution for ω_n with a positive imaginary part, then the amplitude of the perturbation will grow exponentially: this indicates an instability.

Following these steps (see Appendix B) we obtain an integral equation for the frequency ω_n:

$$1 = -iZ_{\parallel}(\omega_n)\frac{eI_0}{C_0 E_0}\int_{-\infty}^{\infty}\frac{\partial\Psi_0/\partial\delta}{(n\omega - \omega_n)}\,d\delta. \qquad (26)$$

Here, I_0 is the beam current (associated with the phase space distribution Ψ_0), C_0 is the ring circumference, E_0 is the beam energy, $\omega = \omega_0\,(1 - \alpha_p\delta)$ is the particle revolution frequency assuming momentum compaction factor α_p, and $Z_{\parallel}(\omega_n)$ is the longitudinal impedance of the ring.

Equation (26) relates the wavelength of the density modulation (characterised by the "mode number" n) to the frequency of the modulation: it is known as the *dispersion relation*.

In practice, as a result of random fluctuations in the particle density, all modes will be present to some extent. Therefore, if there exists a mode n for which the frequency ω_n has a positive imaginary part, then the beam is likely to be unstable.

Note that the dispersion relation includes the average beam current I_0 and the beam energy E_0: from the form of the dispersion relation, we can see that the stability of the beam depends on the ratio I_0/E_0.

In practice, the smaller this quantity (lower current, or higher energy) the more likely the beam is to be stable.

Note also that the stability of the beam depends on the form of the energy distribution Ψ_0: we shall consider this point in more detail shortly.

Because of the various assumptions and approximations that we have made, the dispersion relation usually gives only a rough indication of beam stability.

Furthermore, because we have retained terms in the Vlasov equation only up to first order in the perturbation $\Delta\Psi$, the dispersion relation can only give an indication of whether the beam is stable or not: it cannot be used to describe the behaviour of the beam if an instability is present.

A more rigorous analysis of beam stability in a storage ring needs to be based on solution of the full Vlasov equation, without (for example) linearising the equation by retaining terms to first order in a specified density perturbation.

Usually, a rigorous analysis will depend upon numerical techniques: however, the analytical methods that we describe in this lecture can still be useful for providing some insight into the physical mechanisms associated with beam instabilities.

As an example, let us consider the case of a "cold" beam, i.e. a beam with zero energy spread.

In this case, the energy spread is described by a Dirac delta function: the energy distribution function $\Psi_0(\delta)$ is zero, except for $\delta = 0$.

Integrating by parts, and using $\omega = \omega_0(1 - \alpha_p \delta)$, gives:

$$\int_{-\infty}^{\infty} \frac{\partial \Psi_0 / \partial \delta}{(n\omega - \omega_n)} \, d\delta = \int_{-\infty}^{\infty} \frac{\Psi_0}{(n\omega - \omega_n)^2} \, n \frac{\partial \omega}{\partial \delta} \, d\delta = -\frac{n\omega_0 \alpha_p}{(n\omega - \omega_n)^2}. \tag{27}$$

We then find from the dispersion relation:

$$(n\omega_0 - \omega_n)^2 = i Z_{\|}(\omega_n) \frac{I_0}{E_0/e} \frac{n\omega_0^2 \alpha_p}{2\pi}, \tag{28}$$

and hence:

$$\frac{\omega_n}{n\omega_0} = 1 \pm \sqrt{i \frac{Z_{\|}(\omega_n)}{n} \frac{I_0}{E_0/e} \frac{\omega_0 \alpha_p}{2\pi}}. \tag{29}$$

From equation (29) we see that (except in the very special case that the impedance has complex phase $3\pi/2$) there is always a solution for ω_n with positive imaginary part.

Hence, a beam with zero energy spread will *always* be unstable in the presence of any longitudinal impedance.

This is because there is no mechanism in this case for suppressing or damping the growth of an instability: any density modulation on the beam will generate longitudinal wake fields, which will act on the beam to modulate the energy of the particles, leading to a variation in revolution frequency and an *enhancement* of the original density modulation.

In practice, there will always be some spread in energy for the particles in a storage ring.

Combined with the (non-zero) momentum compaction of the lattice, the energy spread will lead to a range in revolution frequency for the particles in the beam.

The spread in revolution frequencies means that any density modulation will tend to get "smeared out" over some number of turns, leading to a reduction in the amplitude of the density modulation.

If the rate of reduction in amplitude of the density modulation is sufficient to suppress the growth in amplitude from the impedance, then the beam will be stable.

The suppression of the beam instability arising from the spread in energy of particles in the beam is known as "Landau damping" (from a similar effect described by Landau in plasma physics).

As an example of the effect of Landau damping, let us consider the case of a beam in a storage ring with Gaussian energy spread:

$$\Psi_0 = \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta}. \tag{30}$$

Substituting this into the dispersion relation (26) gives:

$$1 = i\frac{Z_\parallel(\omega_n)}{n}\frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p\sigma_\delta^2}\int_{-\infty}^{\infty}\frac{\zeta\, e^{-\zeta^2/2}}{\zeta+\Delta_n}\,d\zeta, \tag{31}$$

where:

$$\Delta_n = \frac{\omega_n - n\omega_0}{n\omega_0\alpha_p\sigma_\delta}. \tag{32}$$

The dispersion relation in this case, equation (31), appears rather formidable.

However, we are only really interested in whether the beam is stable or not, and this can be determined from the imaginary part of ω_n; and hence, from the imaginary part of Δ_n.

To apply the dispersion relation to determine the stability of the beam, we write the dispersion relation in the form:

$$F(n) = U + iV, \tag{33}$$

where:

$$F(n) = \frac{Z_\parallel(\omega_n)}{n} \frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p \sigma_\delta^2}, \tag{34}$$

and:

$$U + iV = \left(i \int_{-\infty}^{\infty} \frac{\zeta\, e^{-\zeta^2/2}}{\zeta + \Delta_n}\, d\zeta \right)^{-1}. \tag{35}$$

For a known impedance, by making the approximation $Z_{\|}(\omega_n) \approx Z_{\|}(n\omega_0)$, we can plot a curve in the complex plane representing the real and imaginary parts of $F(n)$ over a range of values of n.

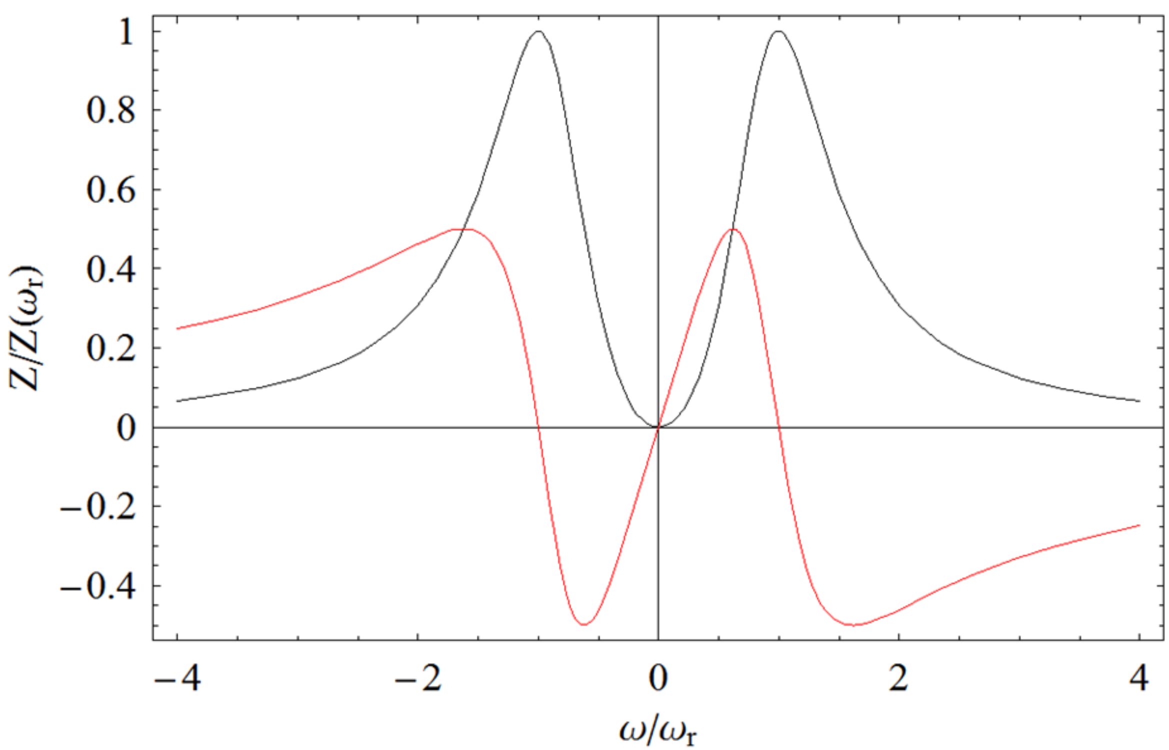
Similarly, we can plot a curve representing $U + iV$ for a range of *real* values of Δ_n. This curve represents a "boundary of stability": on one side of the curve, $\text{Im}(\Delta_n) < 0$, and on the other side $\text{Im}(\Delta_n) > 0$.

If all the points on the curve of $F(n)$ lie in the region where $\text{Im}(\Delta_n) < 0$, the beam is likely to be stable.

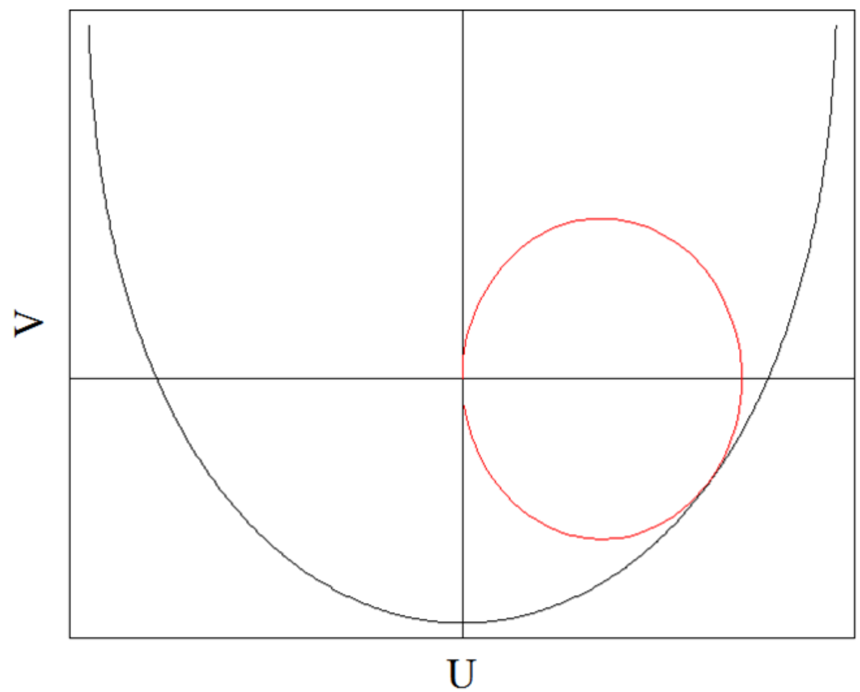
If any points on the curve of $F(n)$ lie in the region where $\text{Im}(\Delta_n) > 0$, the beam is likely to be unstable (for modes corresponding to the values of n for which $F(n)$ is in the unstable region).

As an example, consider the case of a storage ring with a broad-band impedance, with characteristic frequency ω_r:

$$Z_{\parallel}(\omega) = Z_{\parallel}(\omega_r)\frac{1 - i\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + \frac{(\omega^2-\omega_r^2)^2}{\omega_r^2\omega^2}}. \tag{36}$$



We can now plot $F(n)$ for a range of values of n (red curve), and $U + iV$ for a range of real values of Δ_n (black curve).



We have chosen beam and storage ring parameters so that the red curve just touches the black curve: under these conditions, the storage ring is operating at an instability threshold.

The instability threshold corresponds to the condition:

$$I_0 = \frac{\pi^2\sqrt{2\pi}}{3}\alpha_p\sigma_\delta^2\frac{E_0/e}{Z_\parallel(\omega_r)/n}. \tag{37}$$

This represents the maximum current that can be injected into the storage ring while maintaining beam stability.

We see that we can raise the instability threshold by:

- increasing the momentum compaction factor or the energy spread: this increases the rate of Landau damping;
- increasing the beam energy: this increases the beam rigidity;
- reducing the impedance.

It should be remembered that the analysis so far has applied to coasting beams. However, in some circumstances we can also apply the stability criterion to bunched beams.

In the stability diagram shown earlier, the first mode to become unstable has a frequency $\omega \approx 1.2\omega_r$, hence the mode number is:

$$n \approx 1.2\frac{\omega_r}{\omega_0}, \tag{38}$$

where $\omega_0 = 2\pi c/C_0$ is the (angular) revolution frequency.

The broad-band impedance of a storage ring usually has characteristic frequency $\omega_r \gg c/b$, where c is the speed of light and b is the beam pipe radius.

In an electron storage ring, the beam pipe radius is typically of the same order of magnitude as the bunch length σ_z. Hence, the first mode to become unstable is such that:

$$\frac{C_0}{n} \ll \sigma_z. \tag{39}$$

In other words, the length scale of the density modulation resulting from an instability is likely to be short compared to the bunch length.

In this case, if the timescale of the instability is short compared to the synchrotron period, then from point of view of the instability there is little distinction between a coasting beam and a bunched beam.

Under the above conditions, we can apply the same stability criterion to a bunched beam as to a coasting beam.

Finally, we assume (following Boussard) that in applying the stability criterion to a bunched beam, we should replace the average current I_0 by the peak current $\hat{I}$, which for a Gaussian bunch is:

$$\hat{I} = \frac{ecN_0}{\sqrt{2\pi}\sigma_z}, \tag{40}$$

where N_0 is the number of particles in a bunch.

Then for bunched beams in an electron storage ring, the stability criterion can be written (as a limit on the impedance):

$$\frac{Z_\parallel(\omega_r)}{n} < \frac{\pi^2}{6} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}, \tag{41}$$

where $Z_0 \approx 376\,\Omega$ is the impedance of free space, γ is the relativistic (Lorentz) factor corresponding to the reference energy E_0, and $r_e \approx 2.82 \times 10^{-15}\,\mathrm{m}$ is the classical radius of the electron.

A further commonly used approximation is to replace the stability boundary obtained from $\mathrm{Im}(\Delta_n) = 0$ (black curve in the figure, right) with a circle of radius $1/\sqrt{2\pi}$ (red curve).

The stability criterion for coasting beams is then:

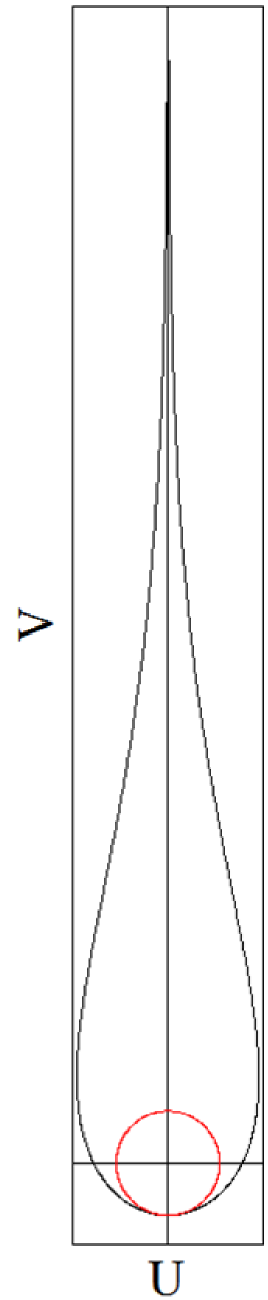
$$\left| \frac{Z_{\|}(\omega_r)}{n} \right| < 2\pi \frac{E_0/e}{I_0} \alpha_p \sigma_\delta^2, \qquad (42)$$

which is known as the *Keil–Schnell criterion*.

For bunched beams, the stability criterion is:

$$\left| \frac{Z_{\|}(\omega_r)}{n} \right| < \sqrt{\frac{\pi}{2}} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}, \qquad (43)$$

which is known as the *Keil–Schnell–Boussard criterion*.

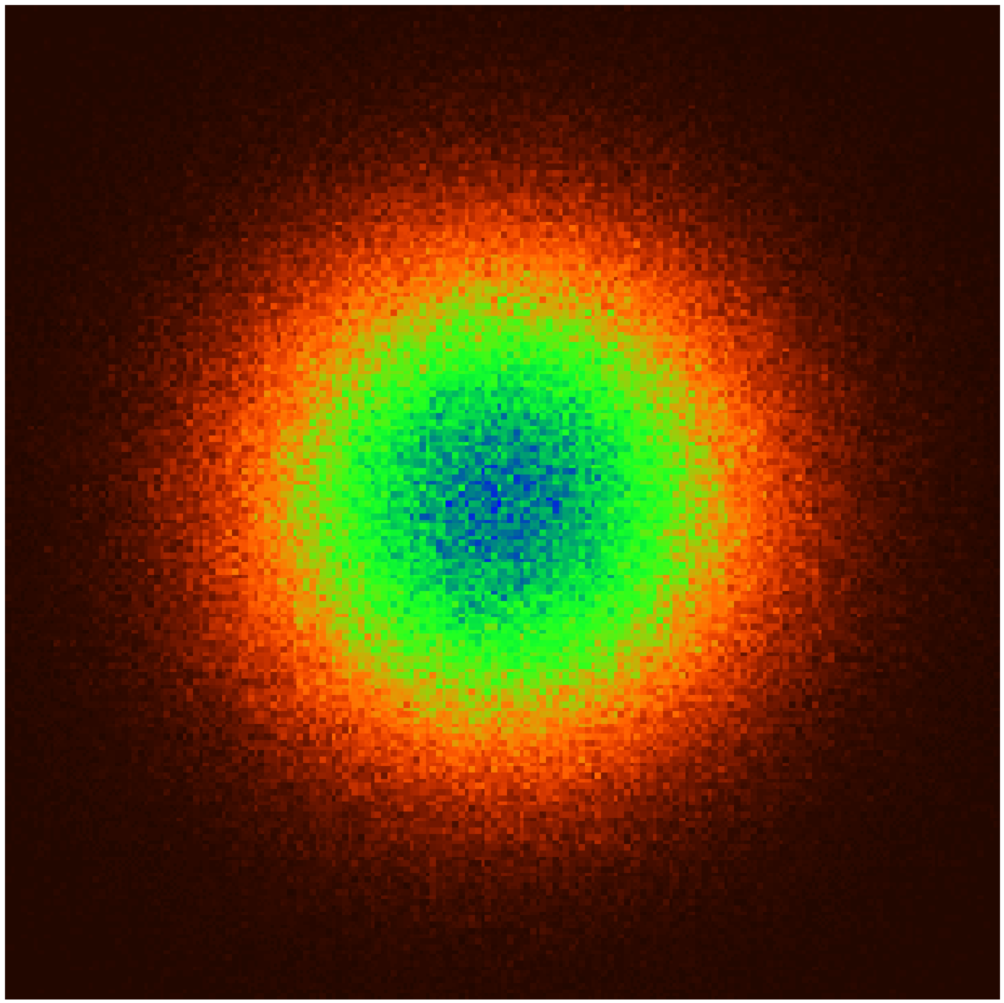


The type of instability that we have described often appears as a modulation in charge density within individual bunches in a storage ring, on a length scale of order of a millimetre.

The charge density modulation can lead to the emission of detectable microwave radiation: the instability is therefore often known as the "microwave instability".

As we have already mentioned, since our analysis is based on linearising the Vlasov equation (i.e. keeping terms only up to first order in the density modulation $\Delta\Psi$), we can only estimate the threshold of the instability: we cannot describe how the beam behaves above threshold.

We can use tracking simulations to study the behaviour of the charge distribution in a bunch above the instability threshold.



Further theoretical analysis, together with numerical modelling and experimental studies, indicate that above the instability threshold there is an increase in beam energy spread following a 1/3 power law:

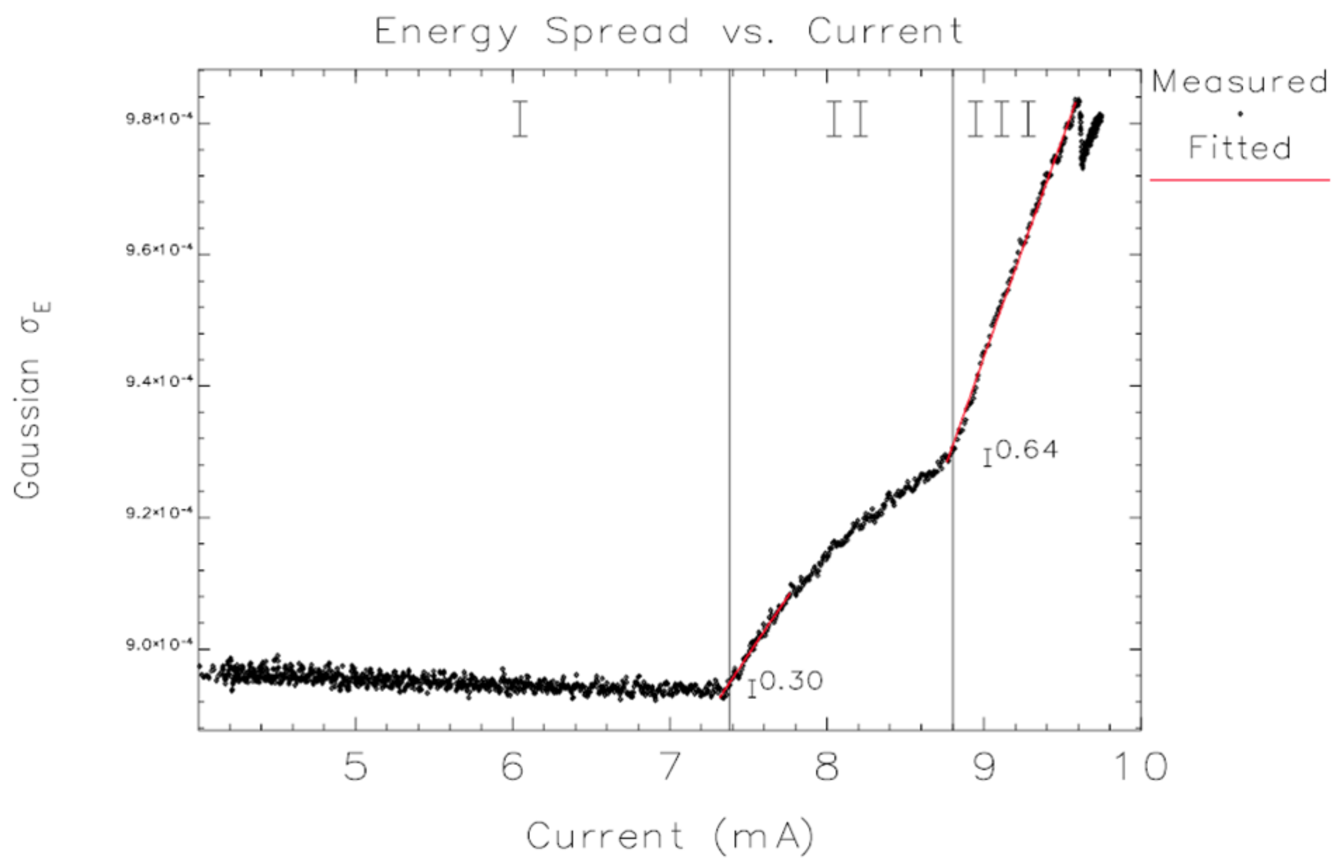
$$\sigma_\delta = \sigma_{\delta,0} + k(N_0 - N_{\mathrm{th}})^{\frac{1}{3}}, \qquad (44)$$

where $\sigma_{\delta,0}$ is the natural energy spread (in the limit of zero bunch charge), N_0 is the bunch population, and N_{th} is the bunch population at the instability threshold.

The increase in energy spread leads to an increase in the bunch length.

The microwave instability is also sometimes known as "turbulent bunch lengthening".

Characteristics of the microwave instability



Y.-C. Chae et al, "Measurement of the longitudinal microwave instability in the APS storage ring", Proceedings of the 2001 Particle Accelerator Conference, Chicago (2001).

Summary: Potential well distortion

In an electron storage ring in the absence of wake fields, the beam generally has a Gaussian distribution in longitudinal phase space.

Wake fields can drive beam instabilities; but at low currents (below instability threshold) the beam distribution can still reach an equilibrium.

Below instability threshold, longitudinal wake fields have little impact on the energy spread, but the longitudinal charge profile within a bunch can be changed: this effect is known as *potential well distortion*.

The equilibrium charge profile in the presence of longitudinal wake fields is described by the *Haissinski equation*.

Summary: Microwave instability

At high bunch currents, short-range wake fields drive beam instabilities where the charge within the bunch fails to reach an equilibrium.

The dynamics of the charge distribution in longitudinal phase space for a single bunch is described by the Vlasov equation.

With some approximations and assumptions, it is possible to find a solution to the Vlasov equation that relates the frequency of a small modulation on the charge density to the wavelength of the modulation: the equation describing this relationship is known as the *dispersion relation*.

The stability of a modulation of given wavelength is determined by the imaginary part of the oscillation frequency of the modulation, which can be found from the dispersion relation.

A "cold" beam (with zero energy spread) is always unstable in the presence of longitudinal wake fields.

When the energy spread is non-zero, the effects of momentum compaction lead to particles moving round the ring at different rates, depending on their energy deviation.

As a result, with non-zero energy spread (and non-zero momentum compaction) any modulation in charge density tends to get "smeared out", suppressing the development of a beam instability: this process is known as *Landau damping*.

At sufficiently high bunch currents, Landau damping becomes insufficient to prevent the development of beam instability.

A common single-bunch instability occurs when density modulations on a length scale of order 1 mm (or less) grow rapidly in amplitude: this instability is known as the *microwave instability*.

A rough estimate for the threshold for the microwave instability can be found by applying the *Keil–Schnell–Boussard criterion*.

Analysis of the synchrotron radiation effects gives:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{I_2 + I_4}, \tag{45}$$

where γ is the relativistic factor, and C_q is a constant:

$$C_q = \frac{55\hbar}{32\sqrt{3} m_e c} \approx 3.832 \times 10^{-13}\,\text{m}. \tag{46}$$

The synchrotron radiation integrals I_2, I_3 and I_4 are given by:

$$I_2 = \oint \frac{1}{\rho^2} ds, \qquad I_3 = \oint \frac{1}{|\rho|^3} ds, \qquad I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds, \tag{47}$$

where ρ is the local radius of curvature of the closed orbit; η_x is the dispersion function; and:

$$k_1 = \frac{1}{B\rho} \frac{\partial B_y}{\partial x} \tag{48}$$

is the quadrupole field gradient scaled by the beam rigidity $B\rho$.

To complete an explicit form for the Vlasov equation, we need to write expressions for $d\theta/ds$ and $d\delta/ds$.

The rate at which a particle moves around the ring depends on the momentum compaction factor of the ring α_p and the energy deviation of the particle δ:

$$\frac{d\theta}{ds} = \omega_0(1 - \alpha_p \delta), \tag{49}$$

where (for ultrarelativistic particles) $\omega_0 = 2\pi c/C_0$ is the angular revolution frequency for a particle with the reference energy.

Recall that the change in energy of a particle resulting from the longitudinal wake fields is given by the convolution of the current spectrum with the longitudinal impedance.

For simplicity, let us assume that the charge distribution around the ring is described by a sinusoidal modulation, superposed on a uniform distribution.

In other words, the beam current observed at a point $\theta = 2\pi s/C_0$ in the ring is given by:

$$I(\theta, t) = I_0 + \Delta I\, e^{i(n\theta - \omega_n t)}. \tag{50}$$

In this case, the beam current spectrum has only a dc component, and a component at frequency ω_n.

Assuming that the dc impedance is zero, the change in energy of a particle in one revolution of the ring is then:

$$\Delta E = -e\, \Delta I\, Z_{\parallel}(\omega_n)\, e^{i(n\theta - \omega_n t)}. \tag{51}$$

If the beam distribution in phase space is normalised so that:

$$\int_{-\infty}^{\infty} \Psi_0 \, d\delta = 1, \tag{52}$$

then the amplitude of the current modulation is:

$$\Delta I = I_0 \int_{-\infty}^{\infty} \Delta\Psi \, d\delta. \tag{53}$$

The change in energy of a particle in one revolution of the ring is then:

$$\Delta E = -Z_{\parallel}(\omega_n) \, e \, I_0 \int_{-\infty}^{\infty} \Delta\Psi \, d\delta \, e^{i(n\theta - \omega_n t)}. \tag{54}$$

Hence, the rate of change of the energy deviation is:

$$\frac{d\delta}{ds} = \frac{\Delta E}{C_0 E_0} = -Z_{\parallel}(\omega_n) \frac{e I_0}{C_0 E_0} \int_{-\infty}^{\infty} \Delta\Psi \, d\delta \, e^{i(n\theta - \omega_n t)}. \tag{55}$$

We can now substitute the expressions for $d\theta/ds$ (49) and $d\delta/ds$ (55) into the Vlasov equation (24), together with the assumed solution (25).

Expanding to first order in the perturbation $\Delta\Psi$ we obtain:

$$\Delta\Psi = -iZ_{\parallel}(\omega_n)\,\frac{eI_0}{C_0E_0}\int_{-\infty}^{\infty}\Delta\Psi\,d\delta\,\frac{\partial\Psi_0/\partial\delta}{(n\omega-\omega_n)}, \tag{56}$$

where $\omega = \omega_0(1-\alpha_p\delta)$.

Finally, we integrate both sides of this equation over δ, to obtain:

$$1 = -iZ_{\parallel}(\omega_n)\frac{eI_0}{C_0E_0}\int_{-\infty}^{\infty}\frac{\partial\Psi_0/\partial\delta}{(n\omega-\omega_n)}\,d\delta. \tag{57}$$