Collective Effects in Particle Accelerators

Part 2

Wake Fields and Impedances

There are six lectures in this course on collective effects in accelerators:

1. Space charge and scattering
2. Wake fields and impedances
3. Potential well distortion and the microwave instability
4. Head-tail instability
5. Coupled-bunch instabilities
6. Luminosity and the beam-beam effect

A beam of charged particles in an accelerator acts as a source of electromagnetic fields. These fields can act back on the beam via interaction with the vacuum chamber.

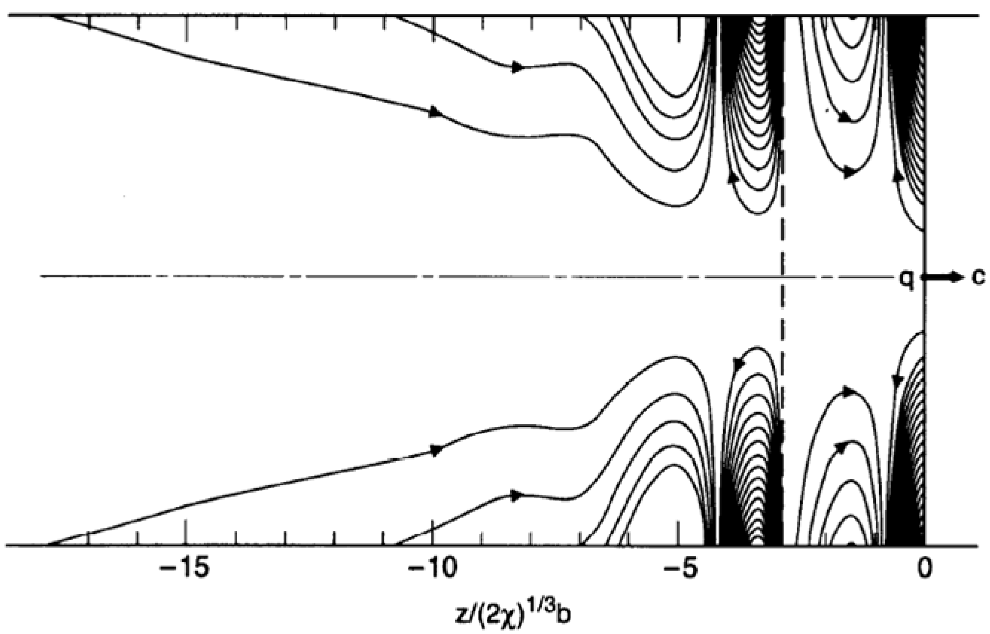
The fields generated by the beam and acting back on it (mediated by the vacuum chamber) are known as *wake fields*.

By the end of this lecture, you should be able to:

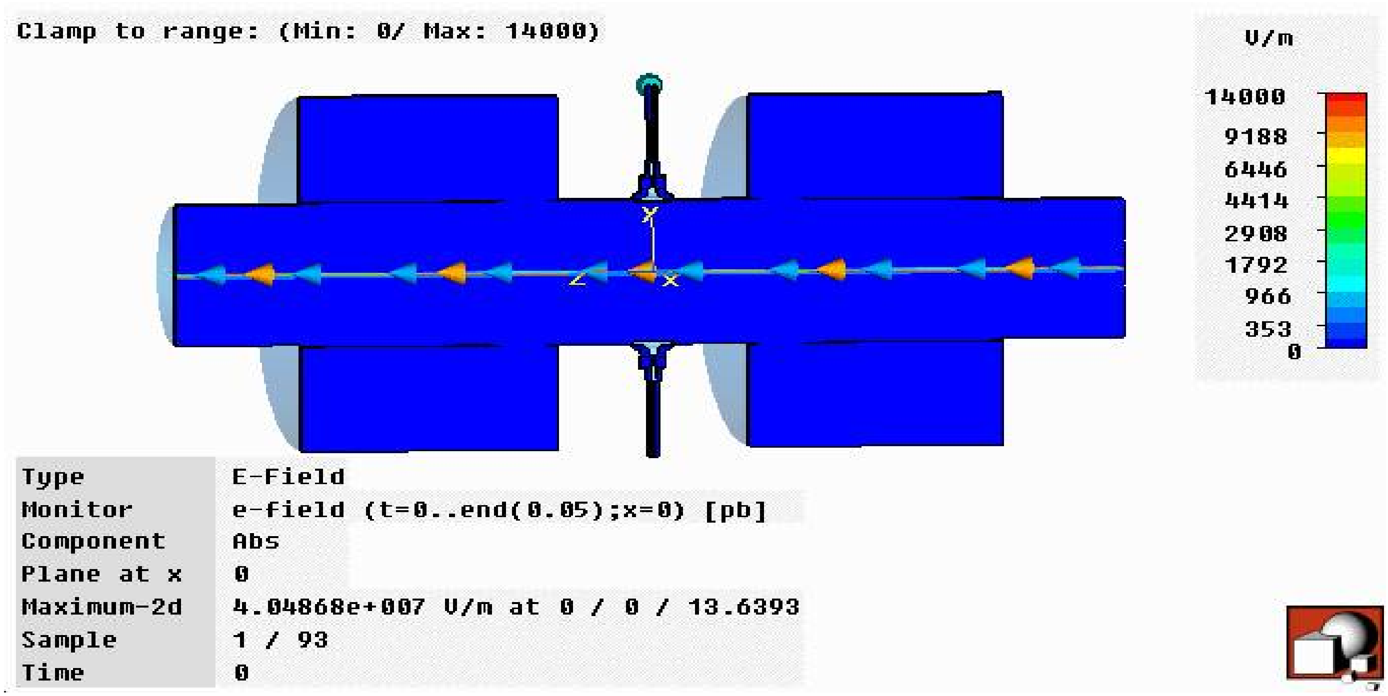
- describe some of the principal features of wake fields;
- describe some of the main effects of wake fields on transverse and longitudinal beam dynamics, including tune shifts and instabilities;
- explain how wake fields may be represented in terms of wake functions and impedances, and how these quantities can be used in the analysis of the effects of wake fields.

The electromagnetic fields around a bunch of charged particles must satisfy Maxwell's equations.

The presence of a vacuum chamber in an accelerator imposes bounday conditions that modify the fields around a bunch of charged particles moving along the accelerator.



Wake fields around a point-like charged particle in a cylindrical beam pipe with resistive walls (K. Bane).



Wake fields in a BPM and bellows (M. Korostelev).

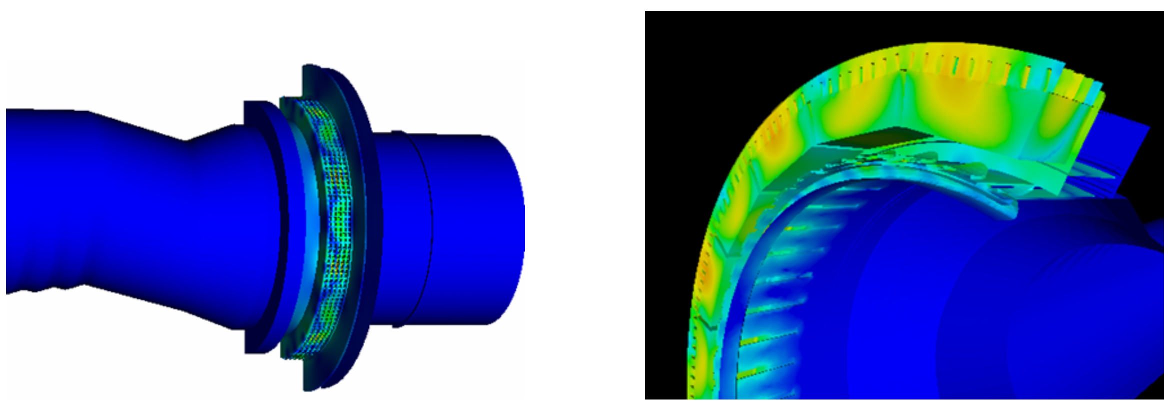
Fields generated by particles passing through a section of an accelerator can act on particles passing through that section at a later time, affecting the dynamics of the later particles.

The electromagnetic fields generated by a particle or a bunch of particles moving through a vacuum chamber are referred to as *wake fields*.

In simple cases the fields around a bunch of charge particles can be calculated analytically. However, wake fields are generally calculated numerically, using electromagnetic modelling codes.

The electromagnetic fields are calculated on a mesh as a charge distribution moves through a component with specified geometry and material (electromagnetic) properties.

In this lecture, we shall not discuss the codes or the process used to calculate the wake fields, but focus on the dynamical effects of the wake fields.



Calculation of trapped modes in PEP II bellows (C. Ng).

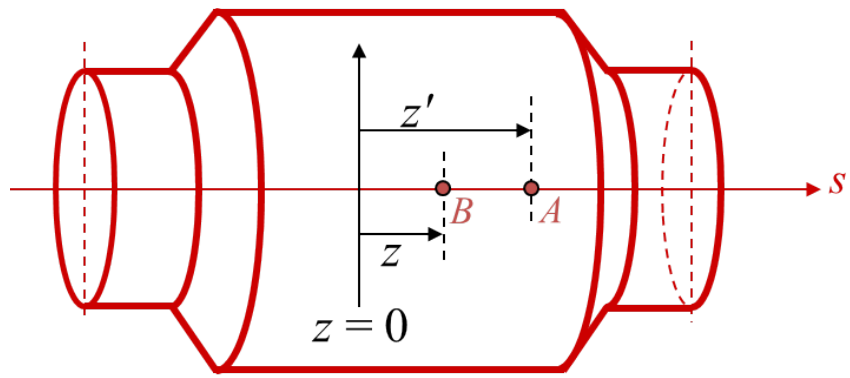
A complete description of a wake field will give the electric and magnetic field components as functions of time and position, and is usually very complicated.

Wake fields depend on the distribution of particles in the beam as well as on the vacuum chamber geometry and material properties.

A simplified description, suitable for analysis of the effect of wake fields on beam dynamics, is provided by *wake functions*.

A *wake function* is associated with a given section of beam pipe, and gives the effect of a leading particle on a following particle when the particles move through that section.

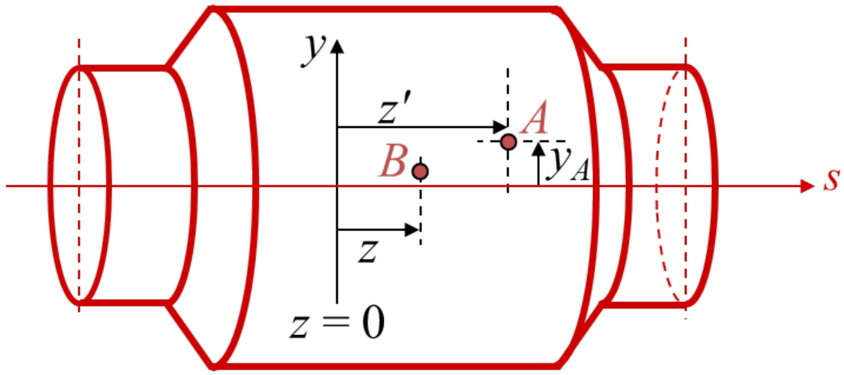
Wake functions are functions of the longitudinal distance between the two particles.



The change in energy of particle B from interaction with the wake field of particle A, when the particles move through a given accelerator component, can be written:

$$\Delta\delta_B = -\frac{q_A q_B}{E_0} W_\parallel(z - z'). \tag{1}$$

- q_A and q_B are the charges of particles A and B, respectively;
- E_0 is the reference energy;
- $W_\parallel(-\Delta z)$ is the *longitudinal wake function* for the given component (SI units: V/C);
- $\Delta z = z' - z$ is the longitudinal separation (assumed constant) between particles A and B as they pass through the component.



The transverse deflection of particle B from interaction with the wake field of particle A, when the particles move through a given accelerator component, can be written:

$$\Delta p_{y,B} = -\frac{q_A q_B}{E_0} y_A W_\perp(z - z'). \tag{2}$$

- q_A and q_B are the charges of particles A and B, respectively;
- E_0 is the reference energy;
- y_A is the transverse displacement of particle A;
- $W_\perp(-\Delta z)$ is the *transverse wake function* for the given component (SI units: V/C/m);
- $\Delta z = z' - z$ is the longitudinal separation (assumed constant) between particles A and B as they pass through the component.

Wake functions provide a simplified representation of the wake fields generated by individual particles in an accelerator.

By integrating over a given distribution of particles, we can estimate the effects of a wake field generated by that distribution.

However, we should remember that wake functions are based on many assumptions and approximations.

For example, for the transverse wake function, we assume that the deflection of a trailing particle has a linear dependence on the transverse displacement of the leading particle from a given axis: this is not necessarily a good approximation.

Wake fields are sometimes classified as "short-range" or "long-range".

In the case of short-range wake fields, we are usually interested in effects *within* individual bunches within an accelerator. For electron storage rings, this typically means length scales less than (of order) 1 mm.

For long-range wake fields, the effects of interest are usually related to the *coherent* motion of individual bunches: that is, we treat each bunch as a single, point-like charge.

In electron storage rings, typical length scales for long-range wake fields can range from several millimetres to many (tens or hundreds) of metres.

Consider the case of a long, straight vacuum chamber with uniform circular cross-section of radius r.

For the regime:

$$-z \gg \sqrt[3]{\frac{r^2}{Z_0 \sigma}}, \tag{3}$$

the longitudinal wake function (for $z < 0$) is given by:

$$W_{\|}(z) = \frac{1}{2\pi r} \sqrt{\frac{Z_0 c}{4\pi} \frac{c}{\sigma}} \frac{L}{\sqrt{-z^3}}, \tag{4}$$

and the transverse wake function (for $z < 0$) is given by:

$$W_{\perp}(z) = -\frac{2}{\pi r^3} \sqrt{\frac{Z_0 c}{4\pi} \frac{c}{\sigma}} \frac{L}{\sqrt{-z}}. \tag{5}$$

L is the length of a given section of the vacuum chamber, σ is the conductivity of the chamber, and Z_0 is the impedance of free space.

Consider, for example, an aluminium vacuum chamber with radius 1 cm.

The conductivity of aluminium is $3.7 \times 10^{7}\,\Omega^{-1}\mathrm{m}^{-1}$.

The expressions for the wake functions shown on the previous slide are valid, in this case, for $-z \gg 20\,\mu$m.

It might be dangerous to use these expressions for single-bunch studies, but they should be safe for studies of multi-bunch effects.

We shall consider the effects of wake fields (on individual bunches, and on beams consisting of multiple bunches) in more detail in later lectures.

For now, we just note that wake field effects can include:

- distortion of (longitudinal and transverse) bunch profiles;
- single-bunch instabilities, where the charge distribution in a bunch fails to reach a stable equilibrium;
- tune shifts, where the betatron or synchrotron frequencies of individual particles (or of coherent motion of entire bunches) vary with beam current;
- multi-bunch instabilities, where individual bunches perform coherent oscillations of increasing amplitude, eventually leading to a loss of beam current.

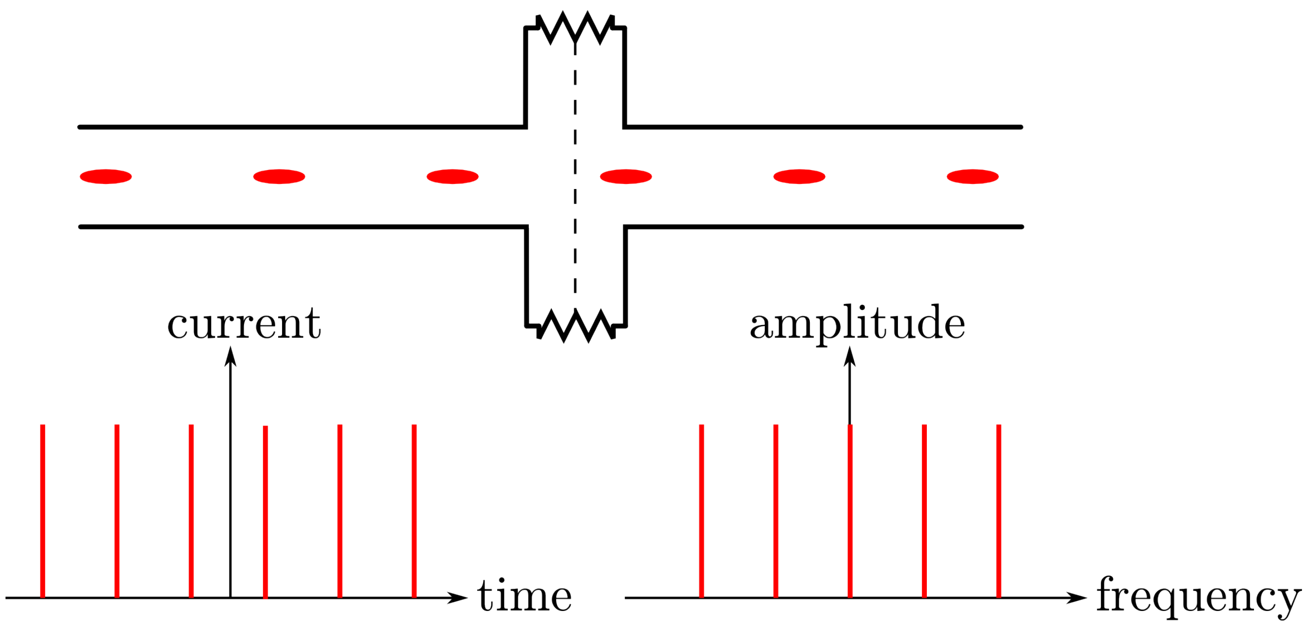
Effects of wake fields: example of a single-bunch instability

Wake field effects can vary widely, and are often complicated to analyse (or even just to describe).

As an example, wake fields can drive single-bunch instabilities...

Structures (such as BPMs and bellows) in accelerator beam pipes often have *resonances*: electromagnetic fields oscillating at the resonant frequency of a given structure can persist for long periods of time.

If the current spectrum of a beam passing through a given structure has a significant amplitude at the resonant frequency of that structure, then the fields within the structure can be driven to large amplitudes, leading to strong wake fields.



The resonant response of a section of beam pipe can be characterised by an *impedance*.

The impedance of a section of beam pipe is the Fourier transform of the wake function; that is, the impedance is a representation of the wake function in the frequency domain (rather than the time domain).

If there is a significant overlap (as a function of frequency) between the impedance of a section of beam pipe and the current spectrum, then there is the possibility of a large impact on the beam from wake fields in that section of beam pipe.

Mathematically, the *longitudinal impedance* is defined:

$$Z_{\parallel}(\omega) = \int_{-\infty}^{\infty} W_{\parallel}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}. \tag{6}$$

The longitudinal wake function can be expressed as the inverse Fourier transform of the longitudinal impedance:

$$W_{\parallel}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z_{\parallel}(\omega) e^{i\frac{\omega z}{c}} \, d\omega. \tag{7}$$

The *transverse impedance* is defined:

$$Z_{\perp}(\omega) = i \int_{-\infty}^{\infty} W_{\perp}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}. \tag{8}$$

The transverse wake function is then expressed in terms of the transverse impedance:

$$W_{\perp}(z) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} Z_{\perp}(\omega) e^{i\frac{\omega z}{c}} \, d\omega. \tag{9}$$

The wake function describes the energy change (longitudinal wake) or transverse kick (transverse wake) resulting from the wake field of one particle acting on another, when the two particles pass through a section of an accelerator.

The physical interpretation of the impedance is perhaps not immediately obvious.

By considering the beam spectrum (i.e. the frequency components present in the beam current measured at some point in an accelerator) we can understand the significance of the impedance...

Recall that the longitudinal wake function is defined so that the change in energy of a particle with charge e following a "particle" with charge $N_A e$ through an accelerator section with longitudinal wake function $W_\parallel(z)$ is:

$$\Delta\delta(z) = -\frac{N_A e^2}{E_0} W_\parallel(z - z'). \tag{10}$$

For the case of a wake field generated by a charge distribution (charge per unit length) $\lambda(z')$ the energy change becomes:

$$\Delta\delta(z) = -\frac{e^2}{E_0} \int \lambda(z') W_\parallel(z - z')\, dz'. \tag{11}$$

In terms of the beam spectrum $\tilde{\lambda}(\omega)$, the charge distribution is:

$$\lambda(z') = \frac{1}{2\pi} \int \tilde{\lambda}(\omega) e^{i\frac{\omega z'}{c}}\, d\omega. \tag{12}$$

Substituting this into the equation for the energy change, and making a change of variables (replacing z' by $z - z'$) gives:

$$\Delta\delta(z) = \frac{e^2}{2\pi E_0} \iint \tilde{\lambda}(\omega) e^{i\frac{\omega z}{c}} W_\parallel(z') e^{-i\frac{\omega z'}{c}}\, dz'\, d\omega. \tag{13}$$

In terms of the longitudinal impedance, the energy change $\Delta\delta(z)$ can then be written:

$$\Delta\delta(z) = \frac{e^2 c}{2\pi E_0} \int \tilde{\lambda}(\omega) Z_{\|}(\omega) e^{i\frac{\omega z}{c}} \, d\omega, \tag{14}$$

and hence:

$$\int \Delta\delta(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{e^2 c}{E_0} \tilde{\lambda}(\omega) Z_{\|}(\omega). \tag{15}$$

Since, for a particle with energy E, the energy deviation is $\delta = (E - E_0)/E_0$, equation (15) can be written:

$$\int \frac{\Delta E(z)}{e} e^{-i\frac{\omega z}{c}} \frac{dz}{c} = ec\tilde{\lambda}(\omega) Z_{\|}(\omega). \tag{16}$$

The left hand side of equation (16) is the Fourier transform of a voltage, where the voltage represents the energy change per unit charge for a particle moving through a section of accelerator with impedance $Z_{\|}(\omega)$.

The right hand side of equation (16) is simply the product of the current spectrum and the impedance.

With appropriate definitions of the "voltage spectrum" $\tilde{V}(\omega)$ and the current spectrum $\tilde{I}(\omega) = ec\tilde{\lambda}(\omega)$, we can write:

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\|}(\omega). \tag{17}$$

The impedance plays the (usual) role of relating, in the frequency domain, the voltage seen by the beam to the beam current.

Equation (17) emphasises the fact that the impact of an impedance depends on the "overlap" in the frequency domain between the impedance and the beam current spectrum.

Wake fields are the electromagnetic fields generated by charged particles moving through an accelerator beam pipe.

With some assumptions and approximations, the effects of wake fields on trailing particles in a given section of beam pipe can be described by *wake functions*.

A transverse wake function $W_{\perp}(-\Delta z)$ gives the transverse deflection (in a given section of beam pipe) of one particle trailing behind another particle by distance Δz.

A longitudinal wake function $W_{\parallel}(-\Delta z)$ gives the change in energy (in a given section of beam pipe) of one particle trailing behind another particle by distance Δz.

The impedance of a given section of beam pipe is the Fourier transform of the wake function.

In the longitudinal plane, the frequency spectrum of the voltage seen by the beam (resulting in an energy change) is given by the product of the beam current spectrum with the impedance:

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\|}(\omega). \tag{18}$$

Significant effects on the beam behaviour from wake fields can occur when there is a large overlap between the beam current spectrum and the impedance.