The Cockcroft Institute
of Accelerator Science and Technology

UNIVERSITY OF LIVERPOOL

Collective Effects in Particle Accelerators

Part 1

Space Charge and Scattering

There are six lectures in this course on collective effects in accelerators:

1. Space charge and scattering
2. Wake fields and impedances
3. Potential well distortion and the microwave instability
4. Head-tail instability
5. Coupled-bunch instabilities
6. Luminosity and the beam-beam effect

By the end of the course, you should be able to:

- describe a range of effects from collective interactions on beam behaviour in high-energy particle accelerators, including space charge, scattering and wake field effects;
- explain how wake fields and impedances can be used to characterise the forces on particles arising from the presence of other particles in an accelerator;
- describe some simple models for single-bunch and coupled-bunch instabilities, and use simple formulae to estimate instability thresholds and growth rates;
- describe the impact of beam-beam interactions in colliders;
- explain how various countermeasures (such as damping mechanisms and feedback systems) can be used to suppress the impact of collective effects.

Many of the general texts on beam dynamics cover collective effects in accelerators, including (for example):

- H. Wiedemann, "Particle Accelerator Physics," Springer (4th Edition, 2015).
- A. Wolski, "Beam Dynamics in High Energy Particle Accelerators," World Scientific (2014).

An authoritative text (out of print, but available on-line):

- A.W. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators," Wiley (1993).

A useful reference:

- A.W. Chao, K.H. Mess, M. Tigner, F. Zimmermann (editors), "Handbook of Accelerator Physics and Engineering," World Scientific (2nd edition, 2013).

In this lecture, we shall discuss the effects arising from electromagnetic fields generated by a beam acting directly on particles within the beam.

The forces in this case are known as *space-charge forces*.

We shall also discuss effects resulting from direct "collisions" between particles within a bunch.

Effects of collisions include *intrabeam scattering*, and the *Touschek effect*.

By the end of this lecture, you should be able to:

- describe some of the main effects of space-charge forces on transverse and longitudinal beam dynamics, including betatron tune shifts, emittance growth, and longitudinal instability;

- explain what is meant by the *perveance* of a beam, and determine, with reference to the envelope equation, whether the transport of a beam with given parameters is dominated by emittance or space-charge effects;

- describe some of the phenomena associated with the effects of collisions between particles in a bunch, including intrabeam scattering and the Touschek effect.

Charged particles in accelerators act as sources of electromagnetic fields.

The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be calculated from Maxwell's equations, with charge density (charge per unit volume) ρ and current density (charge crossing unit area per unit time) $\vec{J}$:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \text{and} \quad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}. \tag{1}$$

Here, we assume there is a perfect vacuum inside the accelerator; ε_0 and μ_0 are (respectively) the vacuum permittivity and permeability.

The electric and magnetic fields must also satisfy:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{2}$$

The fields generated by a beam affect the motion of particles within the accelerator through the Lorentz force, $\vec{F}$:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}), \tag{3}$$

where q is the charge on a particle, and $\vec{v}$ is the velocity of the particle.

To understand the impact of space-charge forces, we need to solve Maxwell's equations for a given distribution of particles, taking into account the boundary conditions imposed by the vacuum chamber, and then apply the Lorentz force to each of the particles in the beam.

This is a challenging problem – especially if we want a "self-consistent" solution (i.e. the distribution of particles is not fixed, but evolves as a result of the space-charge forces).

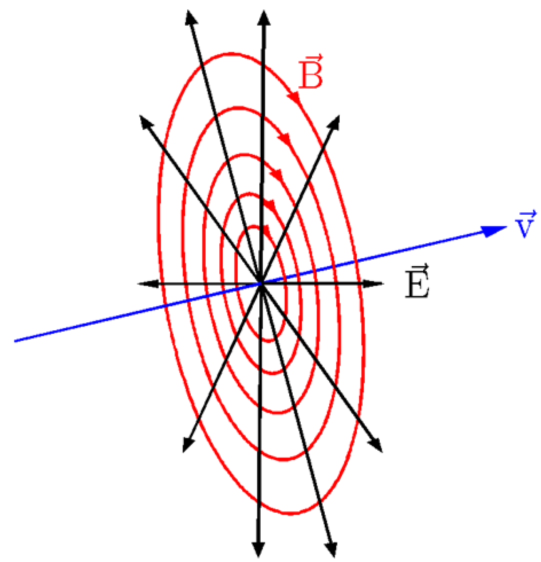
To start to understand the impact of space-charge forces, it is helpful to consider first some simple cases.

One of the simplest cases is that of a single particle moving in a straight line.

Solving Maxwell's equations shows that for an ultra-relativistic charge moving in a straight line, the fields become "flattened" in a plane perpendicular to the direction of motion of the charge.

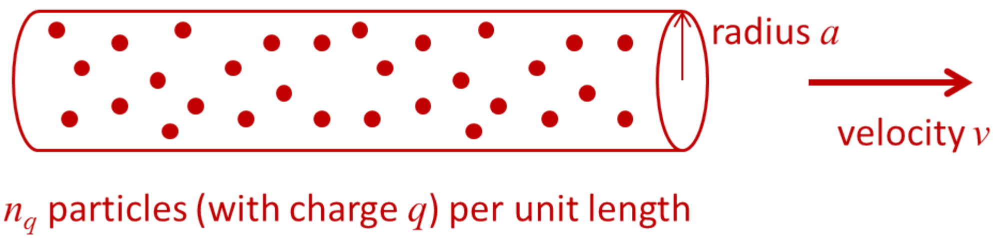
For two charges with the same sign moving parallel to each other:

- the force from the electric field pushes the particles apart;
- the force from the magnetic field pulls the particles together.



At ultra-relativistic energies ($\gamma \gg 1$) the forces from the electric and magnetic fields almost cancel, though there remains some residual repulsive force.

The simplest "realistic" case in an accelerator is a continuous beam with uniform charge density within a circular cross-section.



A charge at radial distance r from the centre of the beam experiences a net radial force:

$$F_r = \frac{n_q q^2}{2\pi\varepsilon_0} \frac{r}{\gamma^2 a^2}, \tag{4}$$

where there are n_q particles (with charge q) per unit length, $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ is the relativistic (Lorentz) factor, and a is the radius of the beam.

The equation of motion in the horizontal transverse (x) direction for a particle in the beam is:

$$\frac{d^2x}{ds^2} = \frac{K}{a^2}x, \tag{5}$$

where s is the distance along the beamline, and K is the *perveance* of the beam.

The perveance is defined by:

$$K = \frac{2I}{\beta^3\gamma^3 I_c}, \tag{6}$$

where $\beta = v/c$, I is the beam current, and I_c is the *critical current* given by:

$$I_c = \frac{4\pi\varepsilon_0 mc^3}{q}. \tag{7}$$

For electrons, I_c is the Alfvén current $I_A \approx 17.045\,\text{kA}$.

Note that the space-charge force varies linearly with transverse position in the beam: this applies only to the case of a uniform transverse distribution.

For a uniform transverse distribution, space-charge forces act like forces from quadrupole magnets, except that space-charge forces are *simultaneously defocusing* in x and y.

It follows that important properties, in particular the conservation of the emittance, apply to beam transport in the presence of (linear) space-charge forces.

Since we know the equation of motion for individual particles, we can work out the equation of motion for the beam distribution.

The equation of motion for the rms beam size is known as the *envelope equation*:

$$\frac{d^2\sigma_x}{ds^2} + k_1\sigma_x - \frac{\epsilon_x^2}{\sigma_x^3} - \frac{K}{2(\sigma_x + \sigma_y)} = 0, \tag{8}$$

where k_1 is the quadrupole focusing strength, σ_x is the rms horizontal beam size:

$$\sigma_x = \sqrt{\langle x^2 \rangle}, \tag{9}$$

and the emittance ϵ_x is given by:

$$\epsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}. \tag{10}$$

Similar equations apply for the vertical (y) motion.

So far, we have considered only the special case of a beam with uniform charge density within a circular cross-section.

In this case, the horizontal and vertical rms beam sizes evolve according to the envelope equations:

$$\frac{d^2\sigma_x}{ds^2} + k_1\sigma_x - \frac{\epsilon_x^2}{\sigma_x^3} - \frac{K}{2(\sigma_x + \sigma_y)} = 0, \quad (11)$$

$$\frac{d^2\sigma_y}{ds^2} - k_1\sigma_y - \frac{\epsilon_y^2}{\sigma_y^3} - \frac{K}{2(\sigma_x + \sigma_y)} = 0, \quad (12)$$

where the emittances ϵ_x and ϵ_y are constant.

It can be shown that the envelope equations apply to any continuous beam with elliptical symmetry – although in general, the emittances are not conserved.

Each term in the envelope equation has a clear physical origin:

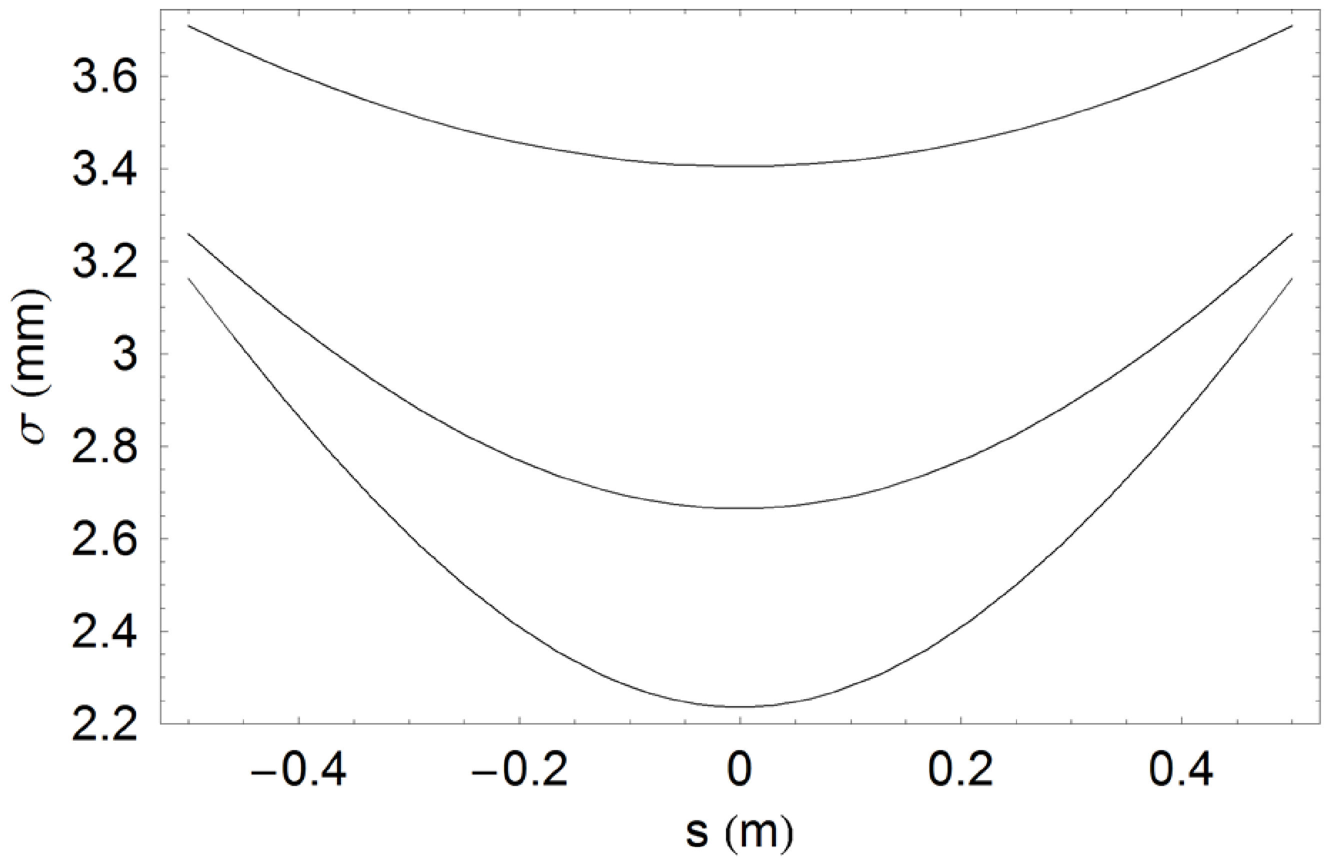
$$\frac{d^2\sigma_x}{ds^2} + k_1\sigma_x - \frac{\epsilon_x^2}{\sigma_x^3} - \frac{K}{2(\sigma_x + \sigma_y)} = 0. \tag{13}$$

- $k_1\sigma_x$ represents the quadrupole focusing;
- ϵ_x^2/σ_x^3 represents the evolution of the beam size arising from the beam emittance (non-zero divergence);
- $K/2(\sigma_x + \sigma_y)$ represents the defocusing effect of the space-charge forces.

If the emittance term is much larger than the space-charge term, then the beam transport is said to be *emittance dominated*.

If the space-charge term is much larger than the emittance term, then the beam transport is said to be *space-charge dominated*.

As an example, we can integrate the envelope equation to calculate the beam size along a drift length of 1 m. We choose an emittance of 10 mm mrad, and set the initial conditions to minimise the maximum beam size for a given perveance.



The defocusing effects of space-charge forces lead to changes in the betatron oscillation frequencies.

For a continuous beam with uniform charge density, all particles experience the same frequency shifts.

For a bunched beam (for example, in a storage ring) the situation is more complicated: the space-charge forces are nonlinear, and depend on the longitudinal position of the particle in the bunch.

Consider the particles in a bunch in a storage ring. Although each particle within the bunch experiences a betatron frequency shift from space-charge forces, if we neglect interactions with the vacuum chamber then the overall space-charge force on the bunch is zero.

This kind of tune shift is sometimes called an *incoherent tune shift*: it cannot be measured by observing the coherent motion of a bunch of particles.

For particles in a bunch with non-uniform density, there is a *tune spread* representing the range of tune shifts for different particles. The vertical tune spread can be estimated using the formula:

$$\Delta\nu_y = -\frac{K}{4\pi}\oint \frac{\beta_y}{\sigma_y(\sigma_x + \sigma_y)}\,ds, \tag{14}$$

where K is the peak perveance in the bunch, β_y is the vertical beta function, and the integral is taken over the entire circumference of the ring.

In addition to direct forces between particles in a bunch, particles experience forces from image charges in the walls of the vacuum chamber.

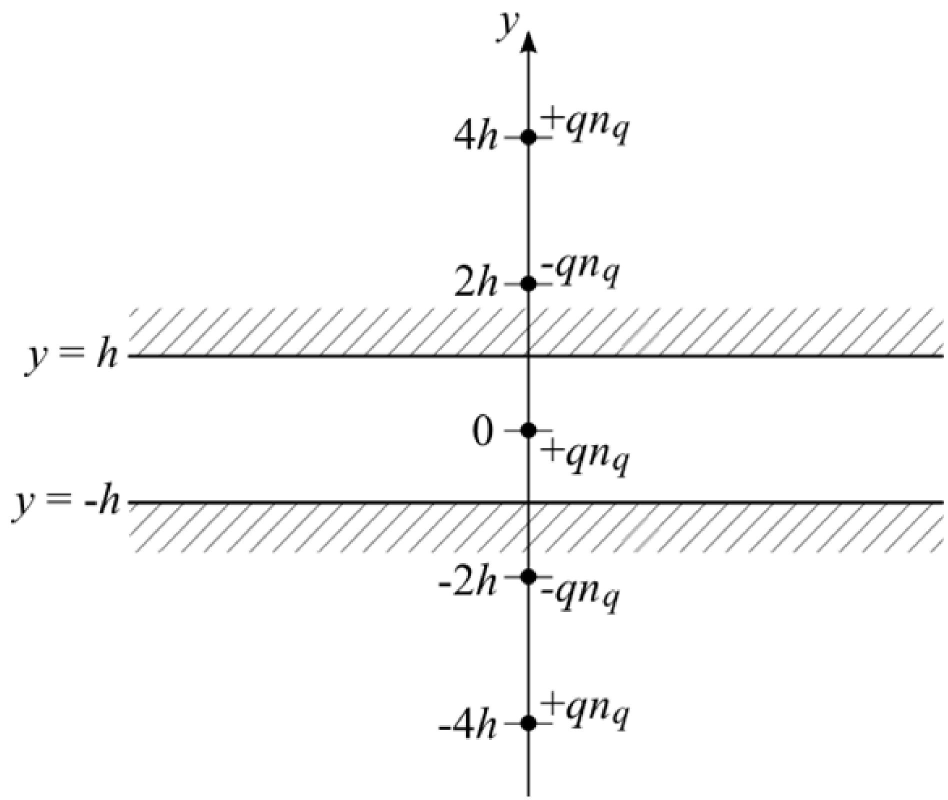
The forces from image charges depend on the geometry of the vacuum chamber, and the position of the bunch within the chamber.

Tune shifts resulting from image charges are known as *Laslett tune shifts*.

In general, there will be coherent Laslett tune shifts as well as incoherent Laslett tune shifts.

Consider a beam centred on the mid-plane between infinite, plane parallel plates.

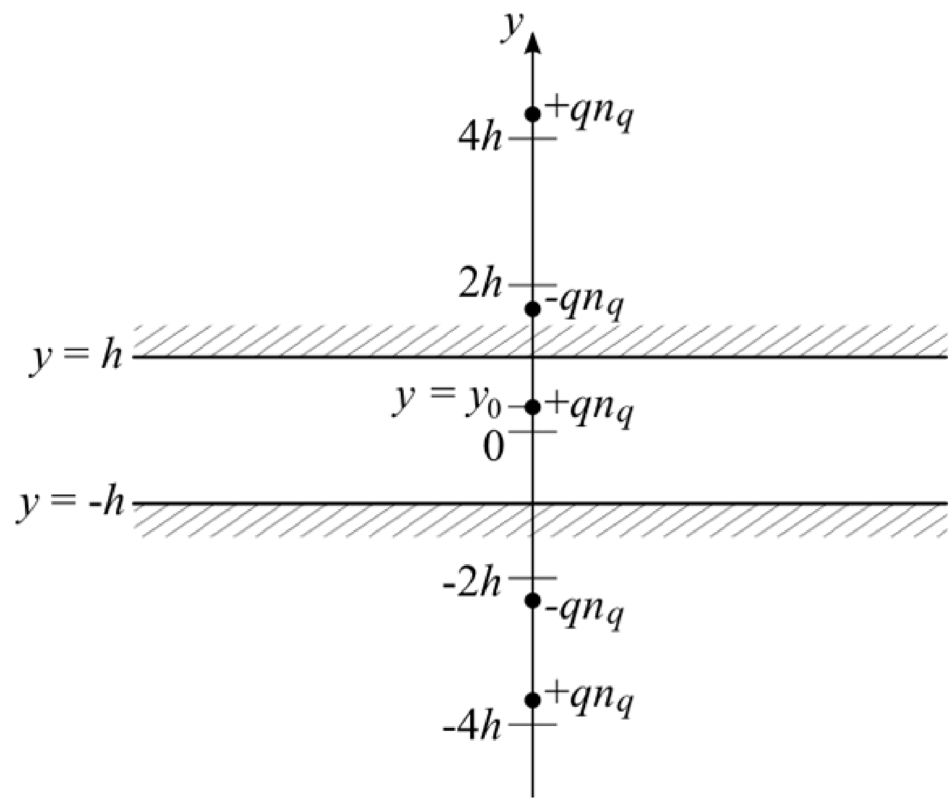
The overall space-charge force on the beam, even including image charges is zero; however, the individual particles will see effects from the image charges.



The result is an incoherent tune shift arising from the image charges.

However, if the beam is displaced from the mid-plane, then the forces from the image charges no longer balance.

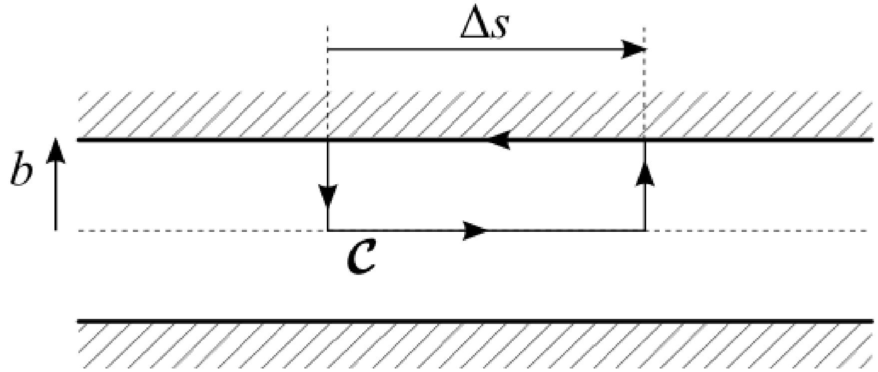
The result is that the image charges cause a coherent tune shift: the frequency of the coherent betatron motion of the entire bunch is changed by the forces from the image charges.



In a continuous beam with uniform charge density (within a given cross-section) the longitudinal fields are zero (by symmetry).

Longitudinal components of the fields occur when there is some variation in the density of particles along the beamline.

The longitudinal components of the fields are related to the transverse components through Maxwell's equations. This provides a method for calculating the longitudinal fields.

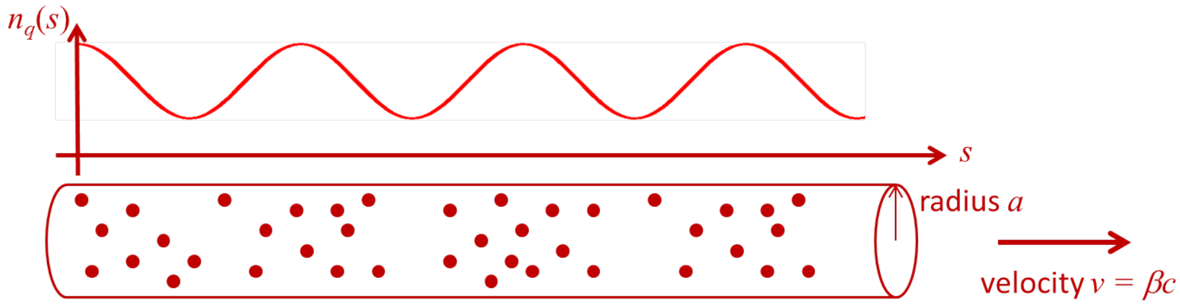


$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{and hence} \quad \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{A}. \qquad (15)$$

In a chamber with perfectly conducting walls, the longitudinal component of the electric field vanishes at the walls.

Since we know the transverse components of the electric and magnetic fields around the beam, we can calculate the longitudinal field along any line parallel to the axis of the beam.

The longitudinal electric field is conveniently expressed in terms of the Fourier components $\tilde{n}_q(\omega)$ of the particle density.



The particle density $n_q(s,t)$ is related to the components $\tilde{n}_q(\omega)$:

$$n_q(s,t) = \int_{-\infty}^{\infty} \tilde{n}_q(\omega) e^{i(ks-\omega t)} d\omega \quad \text{where} \quad \frac{\omega}{k} = \beta c. \tag{16}$$

The longitudinal electric field in a chamber with circular cross-section of radius b is:

$$E_s(r=0,s,t) = -i\frac{qg}{4\pi\varepsilon_0\beta\gamma^2 c}\int_{-\infty}^{\infty} \omega\tilde{n}_q(\omega) e^{i(ks-\omega t)} d\omega, \tag{17}$$

where the geometry factor g is given by:

$$g = 1 + 2\ln\left(\frac{b}{a}\right). \tag{18}$$

The main effect of the longitudinal space-charge fields is to change the energy of the particles.

The change in energy of a particle over a section of beamline from s_1 to s_2 is found by integrating the force on the particle over that distance:

$$\Delta\delta(z) = \frac{q}{E_0} \int_{s_1}^{s_2} E_s\left(s, t = \frac{s - z}{\beta c}\right) \, ds, \tag{19}$$

where E_0 is the reference energy, and the energy deviation δ of a particle with energy E is given by:

$$\delta = \frac{E - E_0}{E_0}. \tag{20}$$

Combining the expressions for the longitudinal electric field and the energy change, we find that in a storage ring, the energy change over one revolution can be written:

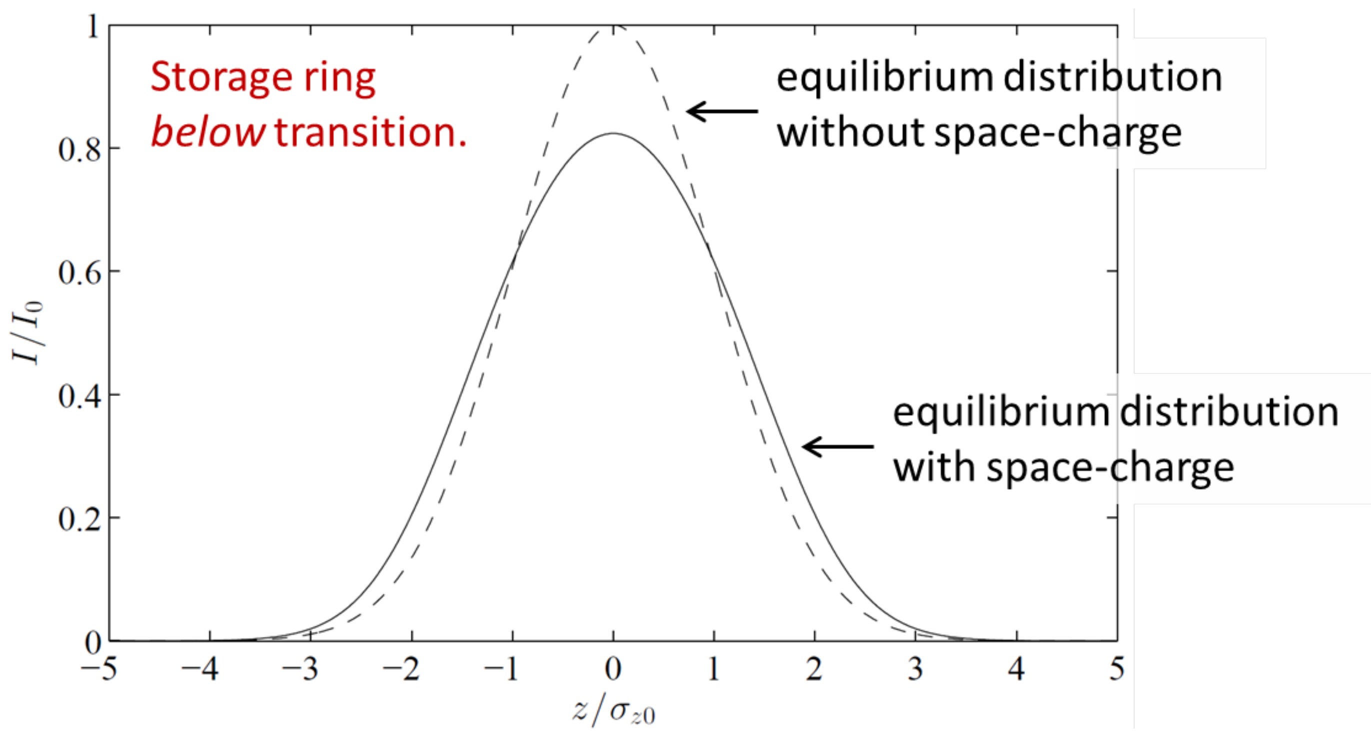
$$\Delta\delta(z) = \frac{q}{E_0} \int_{-\infty}^{\infty} \tilde{I}(\omega) Z_{\parallel \text{sc}}(\omega) e^{i\omega z/c} \, d\omega, \tag{21}$$

where $\tilde{I}(\omega)$ is the Fourier spectrum of the beam current observed at a given location in the ring, and $Z_{\parallel \text{sc}}(\omega)$ is the *longitudinal space-charge impedance* given by:

$$Z_{\parallel \text{sc}}(\omega) = i \frac{g Z_0}{2\beta\gamma^2} \frac{\omega}{\omega_0}. \tag{22}$$

g is the geometry factor (given by (18) for a beam pipe with circular cross-section), and ω_0 is the ring revolution frequency.

In a synchrotron storage ring, the longitudinal space-charge impedance leads to a change in the synchrotron frequency and a change in the equilibrium longitudinal distribution.



In our discussion of space-charge, we considered the impact of fields calculated with the approximation of a smooth charge distribution.

In reality, bunches consist of large numberse of "point-like" particles.

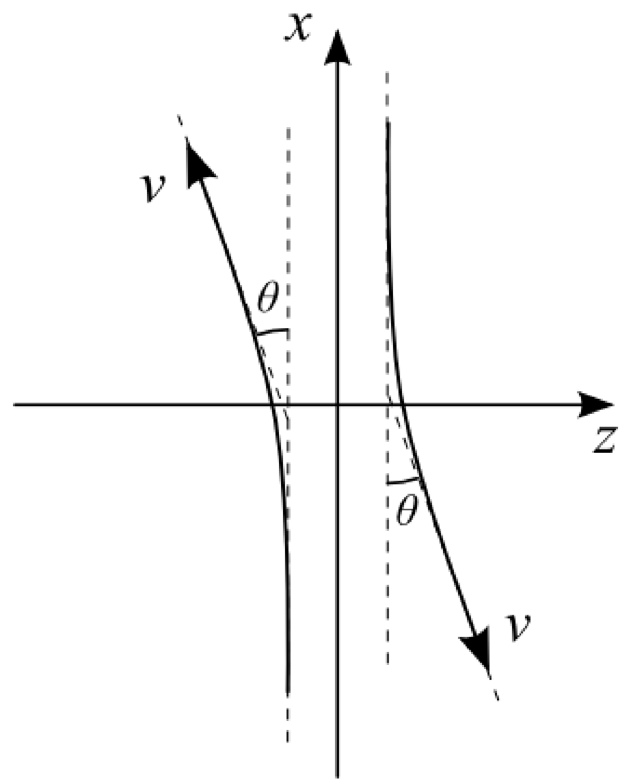
The fields close to a single particle can be very much larger than the fields calculated in the approximation of a smooth charge distribution.

Occasionally, two particles within a bunch can approach each other closely enough for the local peaks in the field strengths to have observable effects.

A "collision" between two electrons (or positrons) is described by *Møller scattering*.

In the rest frame of a bunch (of electrons), the differential cross-section for Møller scattering gives the distribution of the deflection angle.

When transformed into the lab frame, even a small deflection angle can lead to a large change in energy deviation.



If the energy deviation of a particle is outside the energy acceptance of the storage ring, the particle will be lost from the beam: this is the *Touschek effect*.

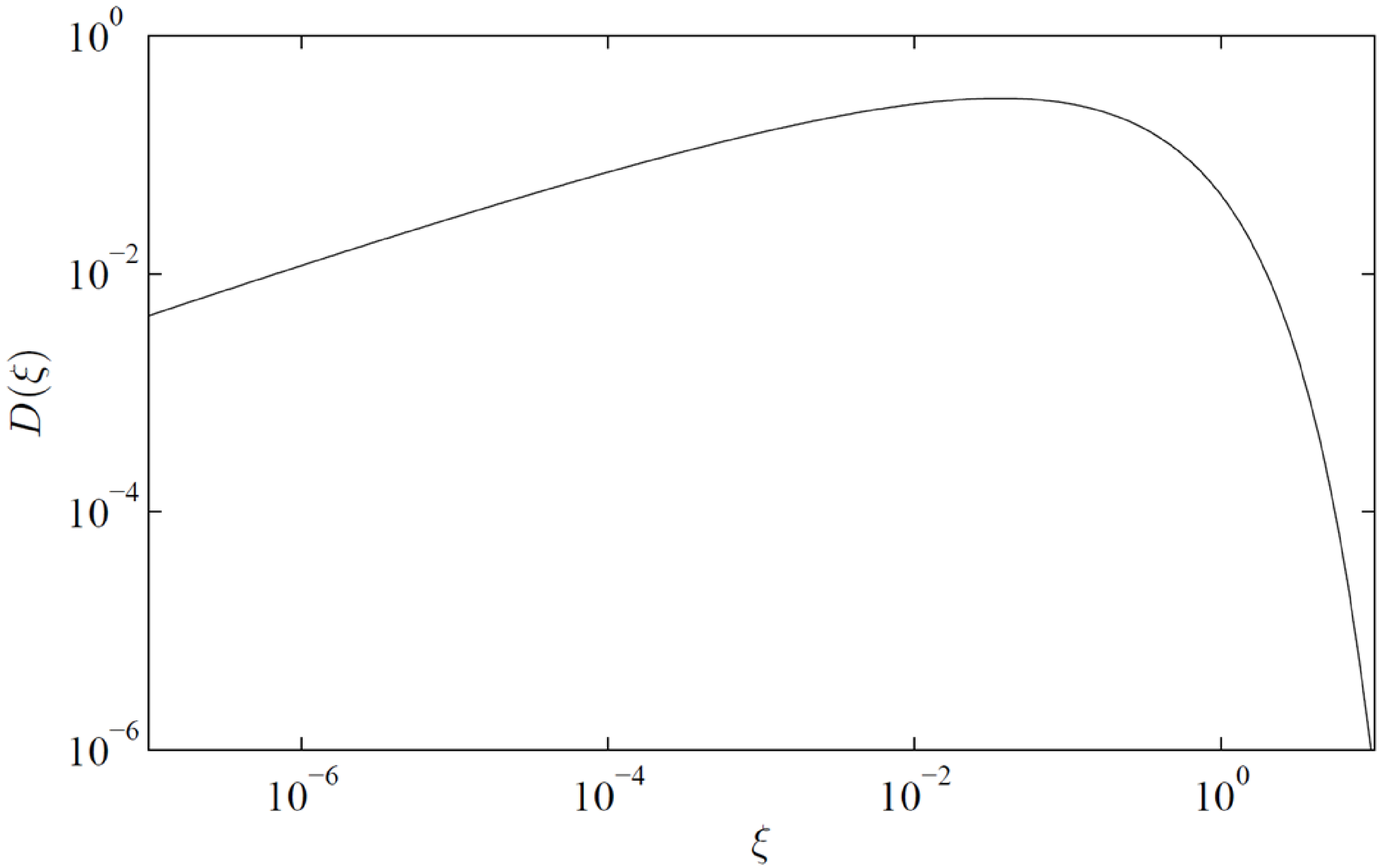
The rate at which particles are lost from a beam can be calculated from the density of particles in the beam and the energy acceptance of the storage ring.

In the simplest case (neglecting dispersion and coupling, and assuming non-relativistic motion in the bunch rest frame) the loss rate from the Touschek effect is given by:

$$\frac{dN_b}{dt} = -\frac{N_b^2 c r_0^2}{8\pi\gamma^2\sigma_x\sigma_y\sigma_z}\left(\frac{\beta}{\delta_{\mathrm{max}}}\right)^3 D(\xi), \tag{23}$$

where N_b is the bunch population, r_0 is the classical radius of the electron, $\beta = v/c$, δ_{max} is the energy acceptance of the storage ring, and:

$$\xi = \frac{\delta_{\mathrm{max}}^2\beta_x}{\beta^2\gamma^2\varepsilon_x}, \qquad D(\xi) = \xi^{\frac{3}{2}}\int_\xi^\infty \frac{e^{-u}}{u^2}\left(\frac{u}{\xi} - 1 - \frac{1}{2}\ln\left(\frac{u}{\xi}\right)\right) du. \tag{24}$$



10^0
10^{-2}
10^{-4}
10^{-6}
$D(\xi)$
10^{-6}
10^{-4}
10^{-2}
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ξ

The loss rate depends on the square of the number of particles in the bunch.

Therefore, the bunch population does not (strictly speaking) decay exponentially; nevertheless, we define the *Touschek lifetime*:

$$\frac{1}{\tau_{\mathrm{T}}} = -\frac{1}{N_b}\frac{dN_b}{dt} = \frac{N_b c r_0^2}{8\pi\gamma^2\sigma_x\sigma_y\sigma_z}\left(\frac{\beta}{\delta_{\mathrm{max}}}\right)^3 D(\xi). \qquad (25)$$

Other effects (in particular, gas scattering) can lead to the loss of particles from a beam in a storage ring.

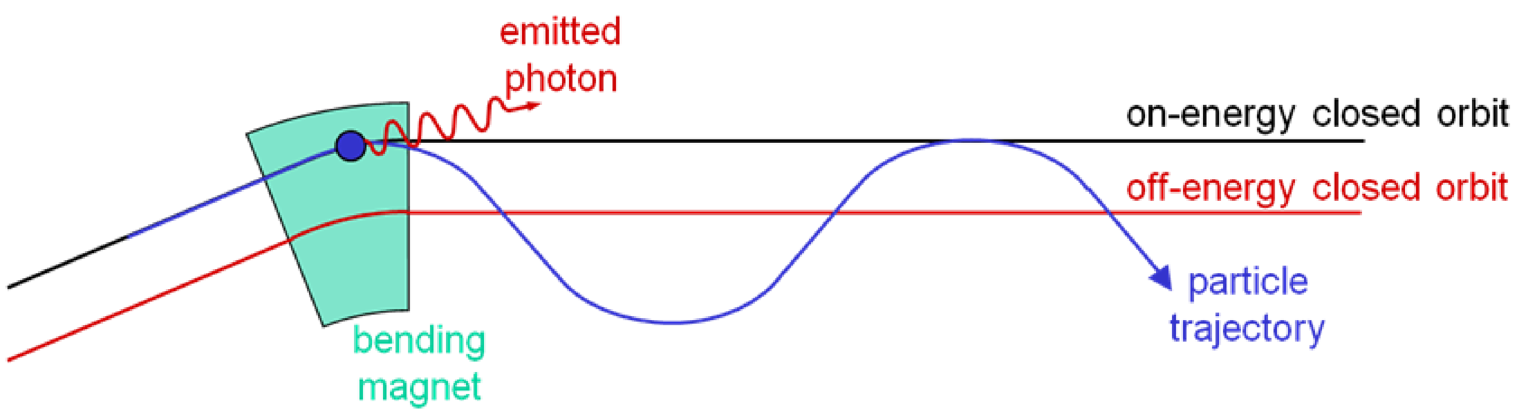
But in low-emittance electron rings (e.g. in third-generation synchrotron light sources) the Touschek effect is usually the dominant lifetime limitation.

The Touschek lifetime has a strong dependence on the energy acceptance: usually the goal is to achieve $\delta_{\mathrm{max}} > 4\%$ (roughly).

Even if a Møller scattering event does not result in the loss of the particles involved, there can be a transfer of momentum from the transverse to the direction to the longitudinal.

If the scattering takes place at a location with non-zero dispersion, then the momentum transfer can lead to an increase in the *transverse* emittance.

Intrabeam scattering (IBS) leads to emittance increase by a mechanism very similar to quantum excitation by synchrotron radiation; but IBS occurs wherever the dispersion is non-zero, not just in dipoles.



There are two widely-used formalisms for the analysis of intrabeam scattering:

- Piwinski (Proc. 9th Int. Conf. High Energy Accel., 1974);
- Bjorken and Mtingwa (Part. Accel. 13, 115–143, 1983).

Both formalisms lead to expressions for the growth rates of the transvserse and longitudinal emittances.

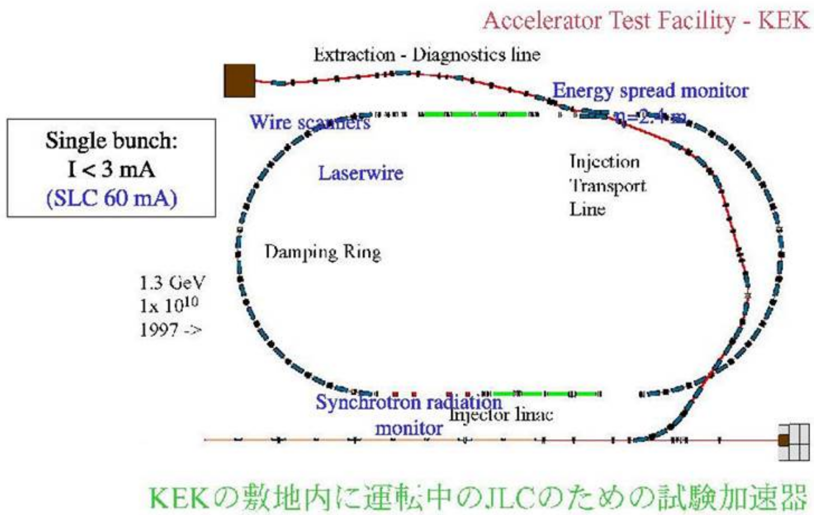
The full expressions are rather complicated; but simplified formulae (e.g. due to Bane) can be derived using various approximations.

In an electron (or positron) storage ring, the IBS growth rates are usually slow compared to the emittance damping rates from synchrotron radiation.

Increases in equilibrium emittance from IBS are usually only observable in proton storage rings.

IBS effects can be observed in electron or positron rings with:

- high bunch population ($> 10^9$ particles);
- low or moderate energy (roughly $1 GeV or less$);
- short bunch lengths and very low emittance (vertical emittance less than a few pm).



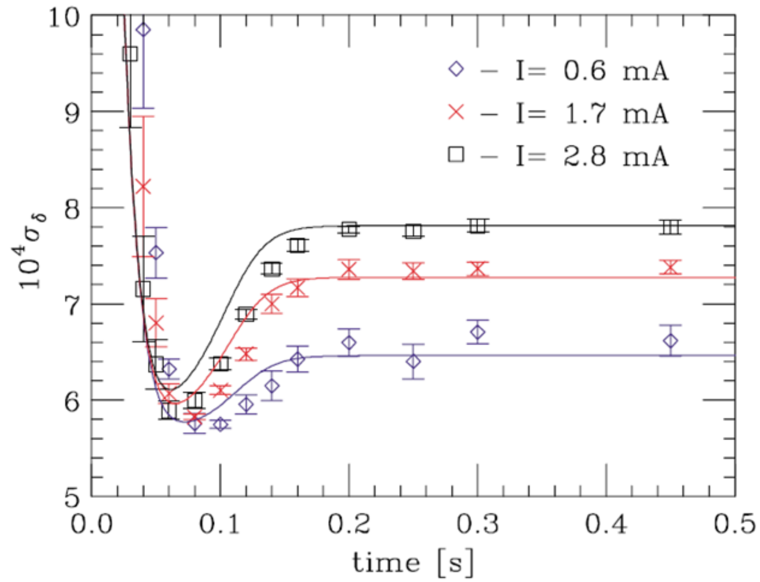


FIG. 4. (Color) Measured energy spread as function of time after injection, for three different currents (the plotting symbols). The curves give BM simulations assuming an x-y coupling of 0.006 and no potential well distortion. This plot is reproduced from Ref. [4].

The energy spread damps more rapidly than the transverse emittances after beam is injected into the ATF storage ring; as the transverse emittances approach small (equilibrium) values, the energy spread increases as a result of IBS.

K.L.F. Bane et al, "Intrabeam scattering analysis of measurements at KEK's Accelerator Test Facility damping ring," PRST-AB 5, 084403 (2002).

Summary: space-charge

Space-charge forces have transverse defocusing effects.

The size of the space-charge forces can be characterised by the *perveance* (a function of the beam current and energy).

The evolution of the transverse beam size, including focusing, emittance and space-charge, is described by the *envelope equations*.

For a uniform transverse distribution, the space-charge forces are linear, and the emittances are conserved.

The *Laslett tune shifts* describe the effects of image charges in the walls of the vacuum chamber.

Longitudinal variations in particle density lead to changes in particle energy, which can be described by an *impedance*.

Summary: scattering

Collisions between particles within a bunch lead to observable (scattering) effects.

The rate of particle collisions depends on the particle density and the (Møller) cross-section.

Collisions can lead to a transfer of momentum from the transverse motion to the longitudinal motion.

If the change in momentum of a particle is large enough that the momentum is outside the momentum acceptance of the storage ring, the particle will be lost from the beam: this is the *Touschek effect*.

For smaller changes in momentum, there can be an increase in the beam emittance: this is *intrabeam scattering*.