Moment tracking to improve PIC codes for astrophysical plasmas

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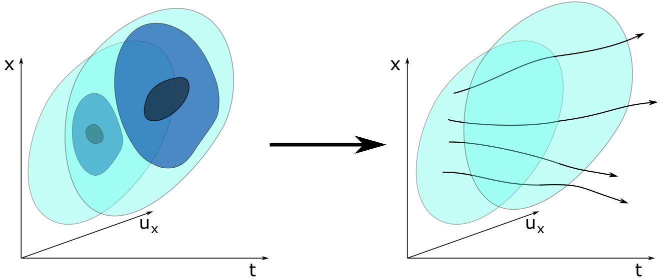
Macroparticles

- Often there are too many electrons to solve the Lorentz force for every particle individually
- Instead, particle-in-cell (PIC) codes group particles together into 'macroparticles'
- This allows simulations to be computationally feasible, but small scale detail is lost



The Vlasov equation

- The Vlasov equation models how particles travel in any system where we can neglect collisions
- In 7 dimensional phase space + time, the Vlasov equation is given by

$$\frac{\partial f}{\partial t} + \sum_{\mu=1}^{3} W^{\mu} \frac{\partial f}{\partial \mathbf{x}^{\mu}} + \sum_{\mu=1}^{3} W^{\mu+3} \frac{\partial f}{\partial \mathbf{u}^{\mu}} = 0$$

where f is the particle distribution, and $\mathbf{u} = \gamma \mathbf{v}$, and

$$W^1 = \frac{\mathbf{u}^1}{\gamma}, \quad W^4 = \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{u}}{\gamma} \times \mathbf{B}\right)^1$$

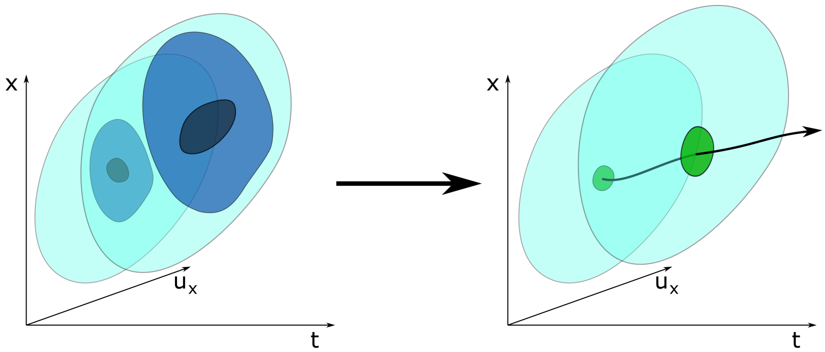
Moments

- As well as tracking particle position and velocity, we can track moments
- The moments are taken around the ideal orbit $(\mathbf{X}, \mathbf{U})$, and are integrated over all of position and velocity space
- The dipole moments V^a give the centre of charge
- The quadrupole moments V^{ab} represent the extent of the beam in phase space
- Higher order moments V^{abc} represent the skew and V^{abcd} represents the kurtosis of the beam

$$V^1 = \int \left(\mathbf{x}^1 - \mathbf{X}^1\right) \, f \, d^3\mathbf{x} \, d^3\mathbf{u}, \quad V^4 = \int \left(\mathbf{u}^1 - \mathbf{U}^1\right) \, f \, d^3\mathbf{x} \, d^3\mathbf{u}$$

Supermacroparticles

- We propose a new method that also transports the moments of the macroparticles
- We call these extended macroparticles ‘Supermacroparticles’



Supermacroparticles

- The amount of information in a supermacroparticle depends on the order of the moments tracked
- A supermacroparticle containing the dipole moments needs twice as much memory to store information
- The increased accuracy of the supermacroparticle approach needs to be balanced against increased computational load

Order of moments tracked	**Information known**	**Number of differential equations to solve**	**Size relative to macro-particles**
Monopole	$\mathbf{X}^\mu,\mathbf{U}^\mu$	6	1
Dipole	$\mathbf{X}^\mu,\mathbf{U}^\mu,\ V^a$	12	2
Quadrupole	$\mathbf{X}^\mu,\mathbf{U}^\mu,\ V^a,\ V^{ab}$	33	5.5
Octopole	$\mathbf{X}^\mu,\mathbf{U}^\mu,\ V^a,\ V^{ab},\ V^{abc}$	89	14.83
Hexadecapole	$\mathbf{X}^\mu,\mathbf{U}^\mu,\ V^a,\ V^{ab},\ V^{abc},\ V^{abcd}$	215	35.83

The differential equations for the moments

- The theory to track the moments is defined through geometric distributions[1]
- The differential equations for the dipole are given by

$$\dot{V}^a = \sum_{b=1}^{6} V^b \partial_b W^a - \sum_{b=1}^{6}\sum_{c=1}^{6} \frac{1}{2} V^{bc} \partial_b \partial_c W^a$$

- The differential equations for the quadrupole are given by

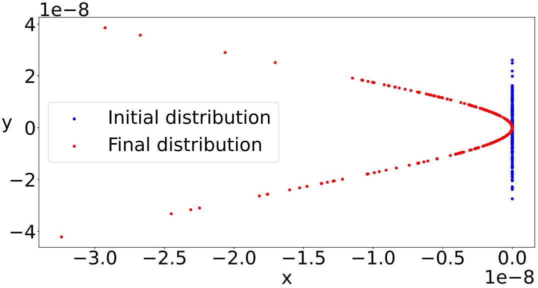
$$\dot{V}^{ab} = \sum_{c=1}^{6} V^{ac} \partial_c W^b + \sum_{c=1}^{6} V^{bc} \partial_c W^a$$

[1] J. Gratus and T. Banaszek. *Proc. Roy. Soc. A*, 474(2213):20170652, May 31, 2018.

The differential equations for the moments

- Quadrupole moments can generate dipole moments
- We can predict the movement in the centre of charge as an elliptical bunch passes through a sextupole magnet

$$\frac{dV^4}{dt} = \frac{q}{m}\frac{SU_z}{2\gamma}V^{22} - \frac{q}{m}\frac{SU_z}{2\gamma}V^{11}, \quad \frac{dV^5}{dt} = \frac{q}{m}\frac{SU_z}{\gamma}V^{12}$$



- The octopole moment will effect the dipole and quadrupole moments

Coordinate transformations

- Other methods can find these differential equations[2] by differentiating

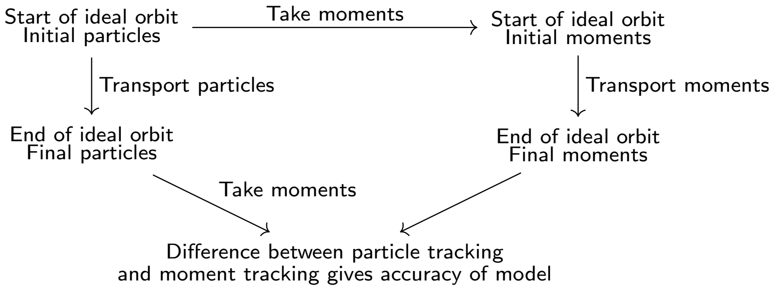
$$V^1 = \int \left(\mathbf{x}^1 - \mathbf{X}^1\right) f \, d^3\mathbf{x} d^3\mathbf{u}$$

- Our method can find the spacetime coordinate transformations for the moments
- This is important for astrophysical plasmas, where there is a choice of coordinate system
- It is also important for particle accelerator scenarios

[2] A. Dymnikov and E. Perelshtein. *Nucl. Instrum. Methods*, 148(3):567–571, Feb. 1978.

Testing the code

- To assess the accuracy of the model, is is compared to particle tracking
- Collective effects are not considered



Future work

- Implement and test the moment tracking code to model the accretion disk of a black hole
- Find a method to use the moments to deposit the charge and current onto the grid
- Find differential equations for the internal structure of the moments

Conclusion

- We propose a new model for PIC codes, transporting the moments of the macroparticles
- The model tracks moments up to second order, but this can be extended to arbitrary order
- Including more moments needs to be balanced against increased computation time and memory usage
- By using our method, we can also find the coordinate transformations for the moments
- We are working on rigorously testing this model in both accelerator and astrophysical scenarios

Thank you for listening
Any questions?