CI-Beam-105

Lattice Design and Computational Dynamics II

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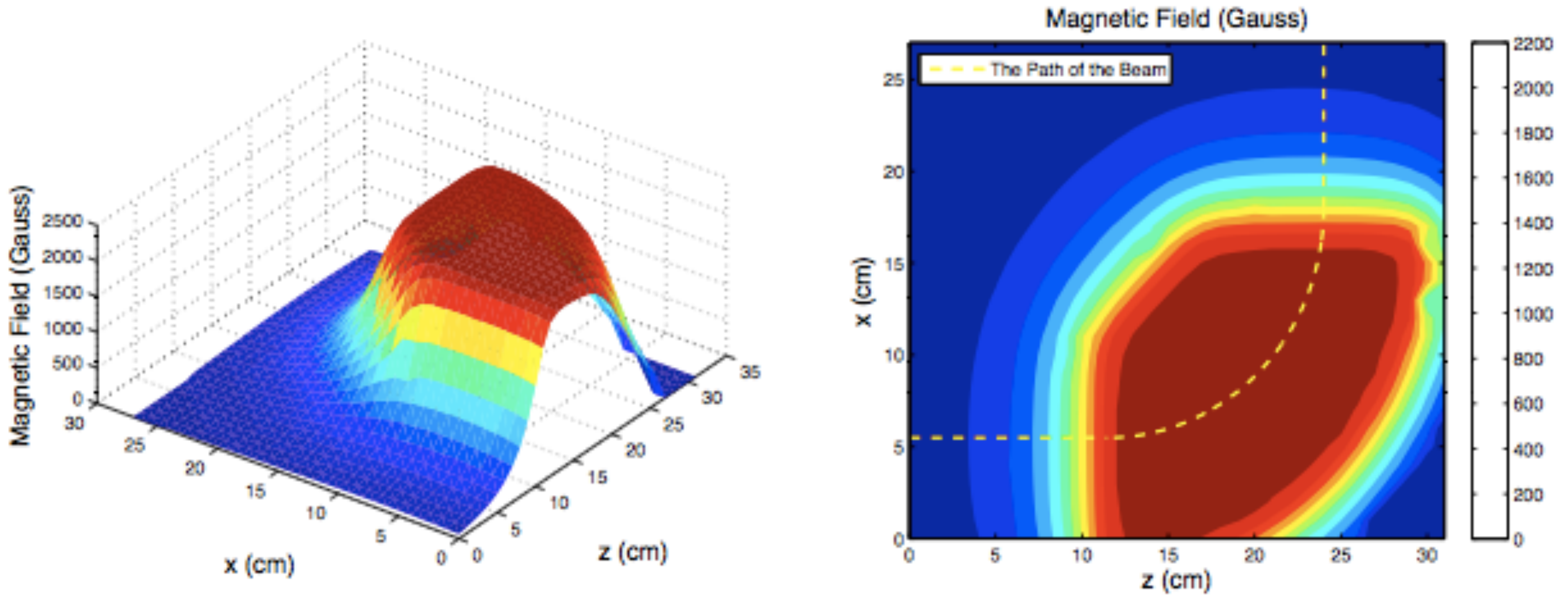
Basic equations on lattice design

Circular orbit



(a) (b)

Magnetic field map of a dipole magnet.

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} \qquad \alpha = \frac{B * dl}{B * \rho}$$

For an entire ring: $$\alpha = \frac{\int Bdl}{B * \rho} = 2\pi \rightarrow \int Bdl = 2\pi * \frac{p}{q}$$

Circular orbit

Dipole magnets are used to keep the particles in a circular orbit in the ring type accelerators. Therefore, fundamentally, the geometry is defined by dipoles in an accelerator.

$$\int B ds = N * B_0 * \ell_{eff} = 2\pi \frac{p}{q}$$

leff -> effective length of the magnet, N -> number of magnets

Example: LHC Dipoles...

$$N = 1232$$

$$l = 15m$$

$$q = +1e$$

$$\int B dl = NlB = 2\pi p/e$$

$$B \approx \frac{2\pi 7000 * 10^9 eV}{e * 1232 * 15m * 3 * 10^8 m/s} = 8.3 Tesla$$

Transfer matrix for periodic lattice cells

(1) $sin(a+b) = sin(a)*cos(b) + cos(a)*sin(b) \qquad cos(a+b) = cos(a)*cos(b) - sin(a)sin(b)$

(2)
$$x(s) = \sqrt{\epsilon}\sqrt{\beta_s}(cos\psi_s cos\phi - sin\psi_s sin\phi)$$ Solution of Hill's equation.
$$x'(s) = -\frac{\sqrt{\epsilon}}{\beta_s}(\alpha_s cos\psi_s cos\phi - \alpha_s sin\psi_s sin\phi + sin\psi_s cos\phi + cos\psi_s sin\phi)$$

(3) Initially, $x(0) = x_0, \psi(0) = 0 \qquad cos\phi = \frac{x_0}{\sqrt{\epsilon\beta_0}} \qquad sin\phi = -\frac{1}{\epsilon}(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}})$

(4)
$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s)x_0 + (\sqrt{\beta_s\beta_0}sin\psi_s)x_0'$$
$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s)x_0 + \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s)x_0'$$

Transfer Matrix in Terms of Beta Function

(5)
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & (\sqrt{\beta_s\beta_0}sin\psi_s) \\ \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$

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In a periodic lattice Twiss parameters will have the same value as their initial values after a full turn.

$$\beta_s = \beta_{s+L} \quad \alpha_s = \alpha_{s+L} \quad \gamma_s = \gamma_{s+L} \longrightarrow \beta_0 = \beta_s, \alpha_0 = \alpha_s,$$

M(1,1) $\quad cos\psi_{turn} + \alpha_s sin\psi_{turn}$

M(1,2) $\quad \beta_s sin\psi_{turn}$

M(2,1) $\quad -\frac{(1+\alpha_s^2)}{\beta_s} sin\psi_{turn}$

M(2,2) $\quad cos\psi_{turn} - \alpha_s sin\psi_{turn}$

Transfer Matrix for a Periodic Lattice

$$M(s) = \begin{pmatrix} cos\psi_{turn} + \alpha_s sin\psi_{turn} & \beta_s sin\psi_{turn} \\ -\gamma_s sin\psi_{turn} & cos\psi_{turn} - \alpha_s sin\psi_{turn} \end{pmatrix}$$

Transfer matrix for Twiss parameters

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$x_0 = S'x - Sx'$$
$$x'_0 = -C'x + Cx'$$

$$\epsilon = constant \xrightarrow{\text{Liouville Theorem}} \begin{array}{l} \epsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2 \\ \epsilon = \beta_0 x_0'^2 + 2\alpha_s x_0 x'_0 + \gamma_0 x_0^2 \end{array}$$

- **Substitute x0 and x0' in the equation; reorganise in terms of x and x', then compare the coefficients.**

$$\epsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

Transfer matrix for Twiss Parameters

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - sS'C'\alpha_0 + S'^2\gamma_0$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -SS' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

- **Twiss parameters given at any point of the lattice and an appropriate transfer matrix can be used to calculate the values of the parameters at another location at the ring.**
- **Transfer matrix depends on the focusing properties of the lattice.**

In summary…

Transfer Matrix for Periodic Lattice

$$M = \begin{pmatrix} cos\mu + \alpha(s) sin\mu & \beta(s) sin\mu \\ -\gamma(s) sin\mu & cos\mu - \alpha(s) sin\mu \end{pmatrix}$$

Stability condition

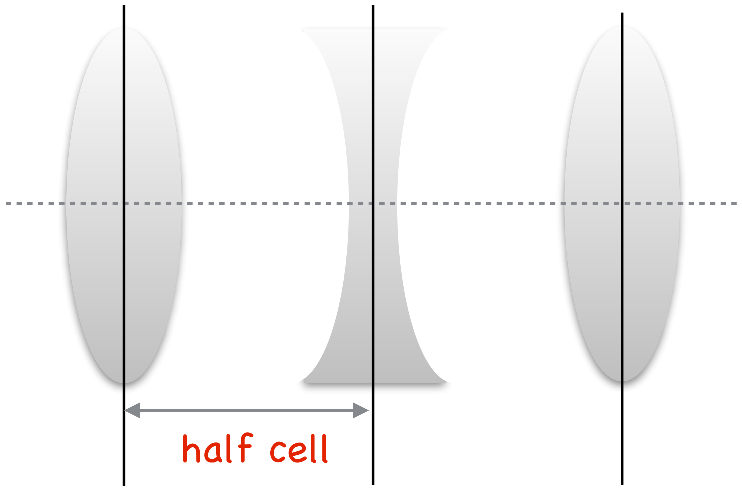
$$Trace(M) < 2$$

Transfer Matrix for Twiss Parameters

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -SS' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

- **Twiss matrix is simpler for periodic lattices.**
- **Twiss parameters α, β, γ, depend on the position, "s", around the ring.**
- **Phase advance, μ, is independent of the position.**

-.-.-.-.- Let's Remember the FODO Cell -.-.-.-.-



$$M_{FoDo} = M_{QF} * M_D * M_{QD} * M_D * M_{QF}$$

$$M_{QF} = \begin{pmatrix} cos\sqrt{|K|}s & \frac{1}{\sqrt{|K|}}sin\sqrt{|K|}s \\ -\sqrt{|K|}sin\sqrt{|K|}s & cos\sqrt{|K|}s \end{pmatrix}$$

$$M_D = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

-.-.-.-.- Let's Remember the FODO Cell -.-.-.-.-

Transfer matrix of a FODO cell under thin lens approximation:

$$M_{FoDo} = \begin{pmatrix} 1-\frac{l^2}{2f^2} & 2l(1-\frac{l}{2f}) \\ -\frac{l}{2f^2}(1+\frac{l}{2f}) & 1-\frac{l^2}{2f^2} \end{pmatrix}$$

Phase advance in terms of FODO cell parameters in a periodic lattice:

Trace of a transfer matrix per turn

$$cos(\mu) = \frac{1}{2}(1-\frac{l_D^2}{2f^2}+1-\frac{l_D^2}{2f^2})$$

$$1-2sin^2(\mu/2) = (1-\frac{l_D^2}{2f^2})$$

$$\boxed{sin(\mu/2) = \frac{L_{Cell}}{4f} = \frac{L_{Cell}}{2\tilde{f}}}$$

f is due to the half length of a quadrupole.

$$\tilde{f} = 2f$$

Drift length is half the cell length.

$$l_D = L_{Cell}/2$$

-.-.-.-.- Let's Remember the FODO Cell -.-.-.-.-

Magnetic length and strength of quadrupoles:

$$K = \pm 0.54102 m^{-2}$$
$$l_q = 0.5m$$
$$l_d = 2.5m$$

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$ Using thick lenses.

Stability of a FODO cell:

$$Trace(M_{FoDo}) = 1.415 \rightarrow < 2$$

Phase advance per cell:

$$M(s) = \begin{pmatrix} cos\psi_{turn} + \alpha_s sin\psi_{turn} & \beta_s sin\psi_{turn} \\ -\gamma_s sin\psi_{turn} & cos\psi_{turn} - \alpha_s sin\psi_{turn} \end{pmatrix}$$

$$cos(\mu) = \frac{1}{2} * Trace(M_{FoDo}) = 0.707$$
$$\mu = arccos(\frac{1}{2} * Trace(M_{FoDo})) = 45^o$$

Alpha and beta functions:

$$\alpha = \frac{M(1,1) - cos(\mu)}{sin(\mu)} = 0 \qquad \beta = \frac{M(1,2)}{sin(\mu)} = 11.611m$$

Summary...

Phase advance per FODO cell
(under thin lens approximation)

$$sin\frac{\mu}{2} = \frac{L_{Cell}}{4f_Q}$$

Stability of a FODO cell

$$f_Q > \frac{L_{Cell}}{4}$$

L_{Cell} , length of the FODO cell
f_Q , focal length of a quadrupole
μ , phase advance per cell

Beta functions in a FODO cell (under thin lens approximation)

Remember...

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & (\sqrt{\beta_s\beta_0} sin\psi_s) \\ \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$

- In a FODO cell, α=0 in the centre of a focusing quadruple.
- Therefore, the beta functions evolve from βmax to βmin along the first half of the cell.

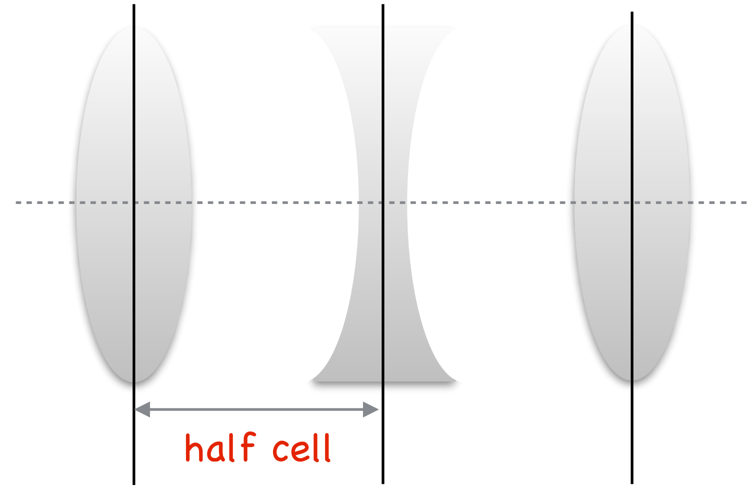
$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}} cos\mu/2 & (\sqrt{\check{\beta}\hat{\beta}} sin\mu/2) \\ -\frac{1}{\sqrt{\hat{\beta}\check{\beta}}} sin\mu/2) & \sqrt{\frac{\hat{\beta}}{\check{\beta}}} cos\mu/2 \end{pmatrix}$$

back...

Beta functions in a FODO cell (under thin lens approximation)

Let's move from the first focusing magnet to defocusing magnet in a FODO cell…

$$M_{FoDo} = \begin{pmatrix} 1-\frac{l^2}{2f^2} & 2l(1-\frac{l}{2f}) \\ -\frac{l}{2f^2}(1+\frac{l}{2f}) & 1-\frac{l^2}{2f^2} \end{pmatrix}$$



From QF to QD

$$M = \begin{pmatrix} 1-\frac{l_d}{2f} & l_d \\ -\frac{l_d}{4f^2} & 1+\frac{l_d}{2f} \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}} cos\mu/2 & (\sqrt{\check{\beta}\hat{\beta}} sin\mu/2) \\ -\frac{1}{\sqrt{\hat{\beta}\check{\beta}}} sin\mu/2) & \sqrt{\frac{\hat{\beta}}{\check{\beta}}} cos\mu/2 \end{pmatrix}$$

$$\frac{S'}{C} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1+\frac{l_d}{2f}}{1-\frac{l_d}{2f}} = \frac{1+sin\mu/2}{1-sin\mu/2} \qquad \frac{S}{C'} = \hat{\beta}\check{\beta} = 4f^2 = \frac{l_d^2}{sin^2\mu/2}$$

$$\boxed{\hat{B} = \frac{(1+sin\frac{\mu}{2})L_{Cell}}{sin\mu} \qquad \check{B} = \frac{(1-sin\frac{\mu}{2})L_{Cell}}{sin\mu}}$$

In addition...

Dispersion for a FODO cell:

$$\hat{D} = \frac{l^2}{\rho} * \frac{1 + \frac{1}{2} sin \frac{\mu}{2}}{sin^2 \frac{\mu}{2}}$$

$$\check{D} = \frac{l^2}{\rho} * \frac{1 - \frac{1}{2} sin \frac{\mu}{2}}{sin^2 \frac{\mu}{2}}$$

Low dispersion:
- weak dipoles
- large angle
- shorter cells

Chromaticity for a FODO cell:

$$Q'_{total} = -\frac{1}{4\pi} \oint (K(s) - mD(s))\beta(s) ds$$

Low chromaticity:
- weak focusing
- small β

Summary...

- An arc on a ring type accelerator (storage ring etc.) generally consists of magnetic elements which repeats periodically, such as: FODO lattice.
- Quadrupole values in an arc can give an initial idea of the beam parameters in that arc.