SUMMARY OF TRANSVERSE BEAM DYNAMICS

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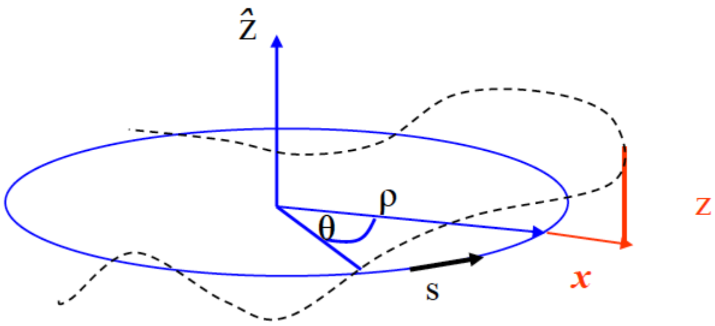
Recall - Hill's Equation and Solutions



$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$
$$z'' - kz = 0$$

Piecewise Solution

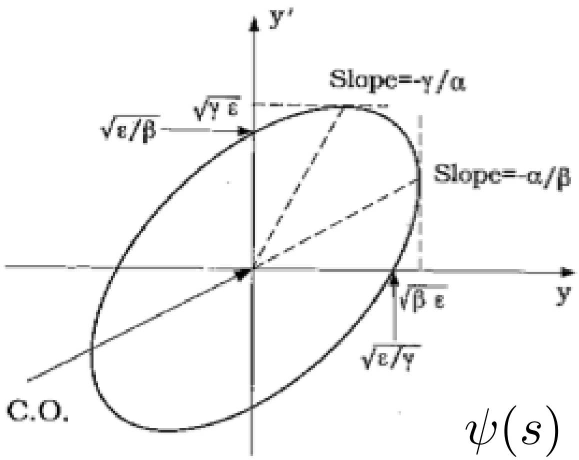
$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Courant-Snyder Solution

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$



$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

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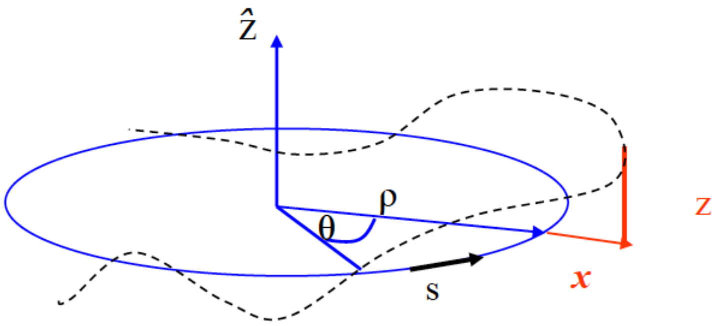
Recall – One-turn map, lattice functions



$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$

$$z'' - kz = 0$$

$$M\left(s + C|s\right) = \prod_i M\left(s_{i+1}|s_i\right)$$

$$M(s+C|s) = \begin{pmatrix} \cos(\Psi) + \alpha\sin(\Psi) & \beta\sin(\Psi) \\ -\gamma\sin(\Psi) & \cos(\Psi) - \alpha\sin(\Psi) \end{pmatrix}$$

One-turn map

Propagating lattice functions

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} m_{11}m_{22} + m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

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Recall – Dispersion, Momentum Compaction

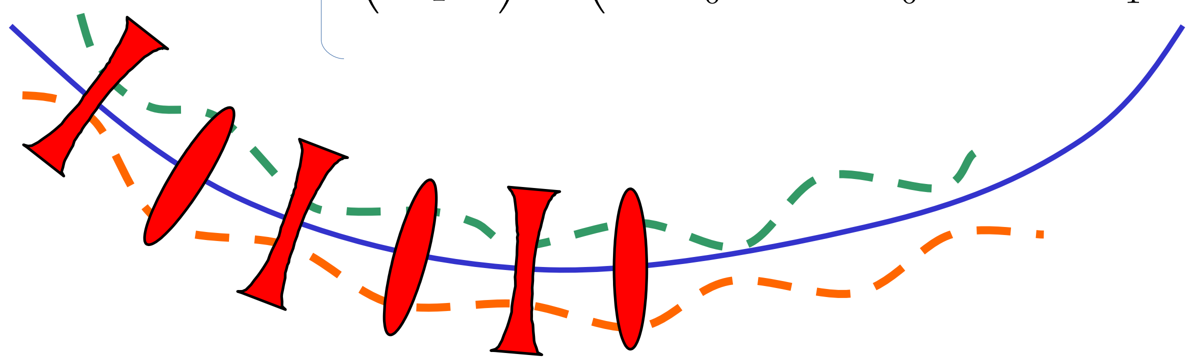
$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Dispersion

$$x(s) = x_h(s) + D(s)\delta$$

E.g. Dipoles

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(s/\rho) & \rho\sin(s/\rho) & \rho(1-\cos(s/\rho)) \\ -\frac{1}{\rho}\sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$



Linear momentum compaction

$$\frac{\Delta C}{C} = \alpha_c \delta = \alpha_c \frac{\Delta p}{p} \Rightarrow \alpha_c = \frac{1}{C}\oint \frac{D(s)}{\rho(s)} ds$$

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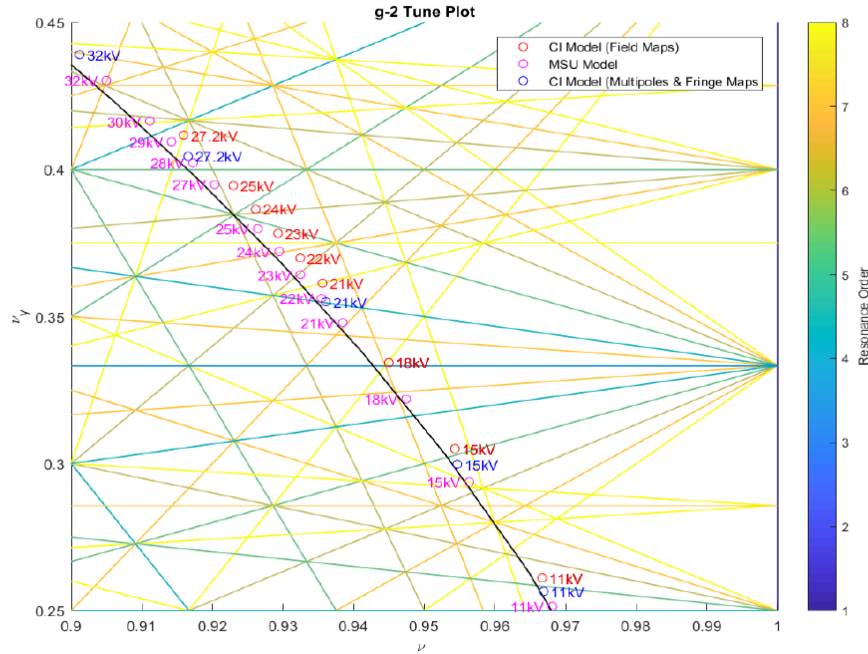
Recall – Tune, Resonances, Chromaticity

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Tune and resonances

$$\nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$



Natural chromaticity

$$\nu_{x,y} = \nu_{x,y}(0) + \xi_{x,y}\delta$$

Reducing chromaticity using sextupoles

$$B_x = S(x_\beta + D\delta)y_\beta = Sx_\beta y_\beta + SD\delta y_\beta$$

$$B_y = \frac{S}{2}(x_\beta^2 - y_\beta^2) + Sx_\beta D\delta + \frac{S}{2}D^2\delta^2$$

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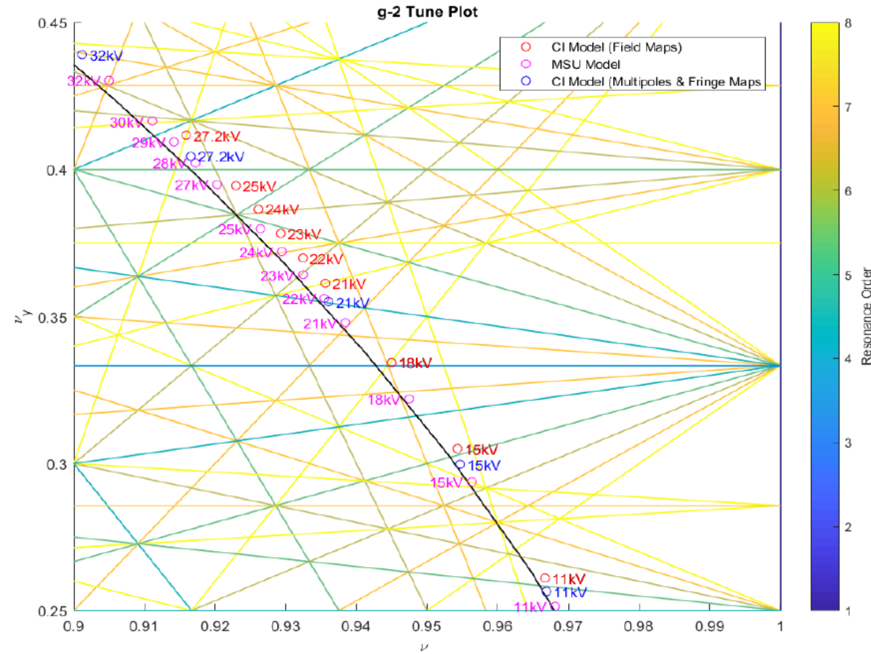

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LONGITUDINAL BEAM DYNAMICS

Longitudinal dynamics

So far we have studied transverse motion (x,x') (y,y'). Now we need to study the remaining motion involving the coordinates in the *longitudinal* direction. This is called **synchrotron motion**.

One way of proceeding would be to define longitudinal lattice functions, by analogy to our studies of transverse beam dynamics, but a different approach is usually taken.

The synchrotron tune is typically much less than the transverse tunes:

$$\nu_{x,y} >> 1 >> \nu_s$$

As the motion is slow, we can ignore the s-dependent effects around the ring, and avoid a longitudinal Courant-Snyder formalism.

In addition, we might expect to use z and z' as the longitudinal coordinates. Instead of z', we typically use the momentum deviation δ or the energy deviation.

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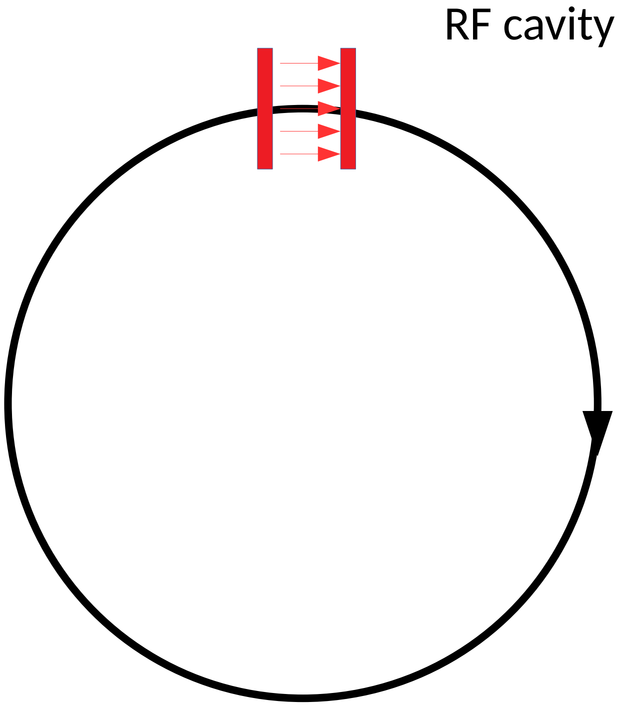
An RF cavity in a ring

Let's add an RF cavity to our storage ring.

This cavity is designed to generate a time-dependent longitudinal electric field to transfer energy to the particle.



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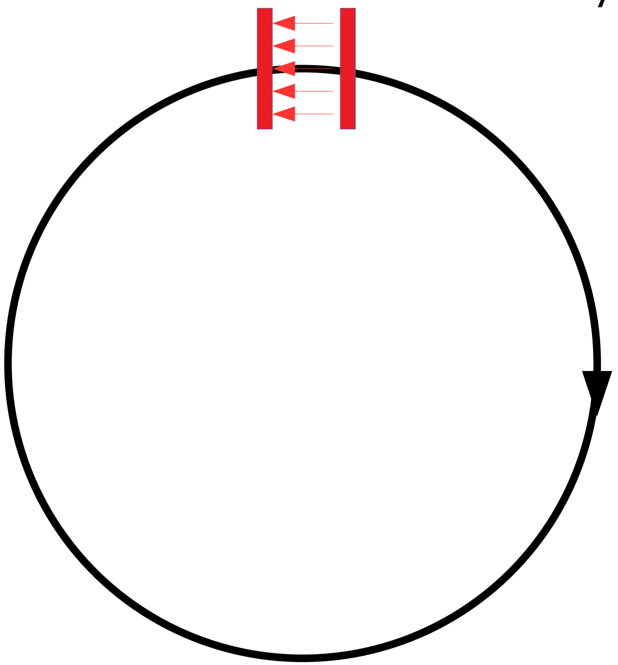
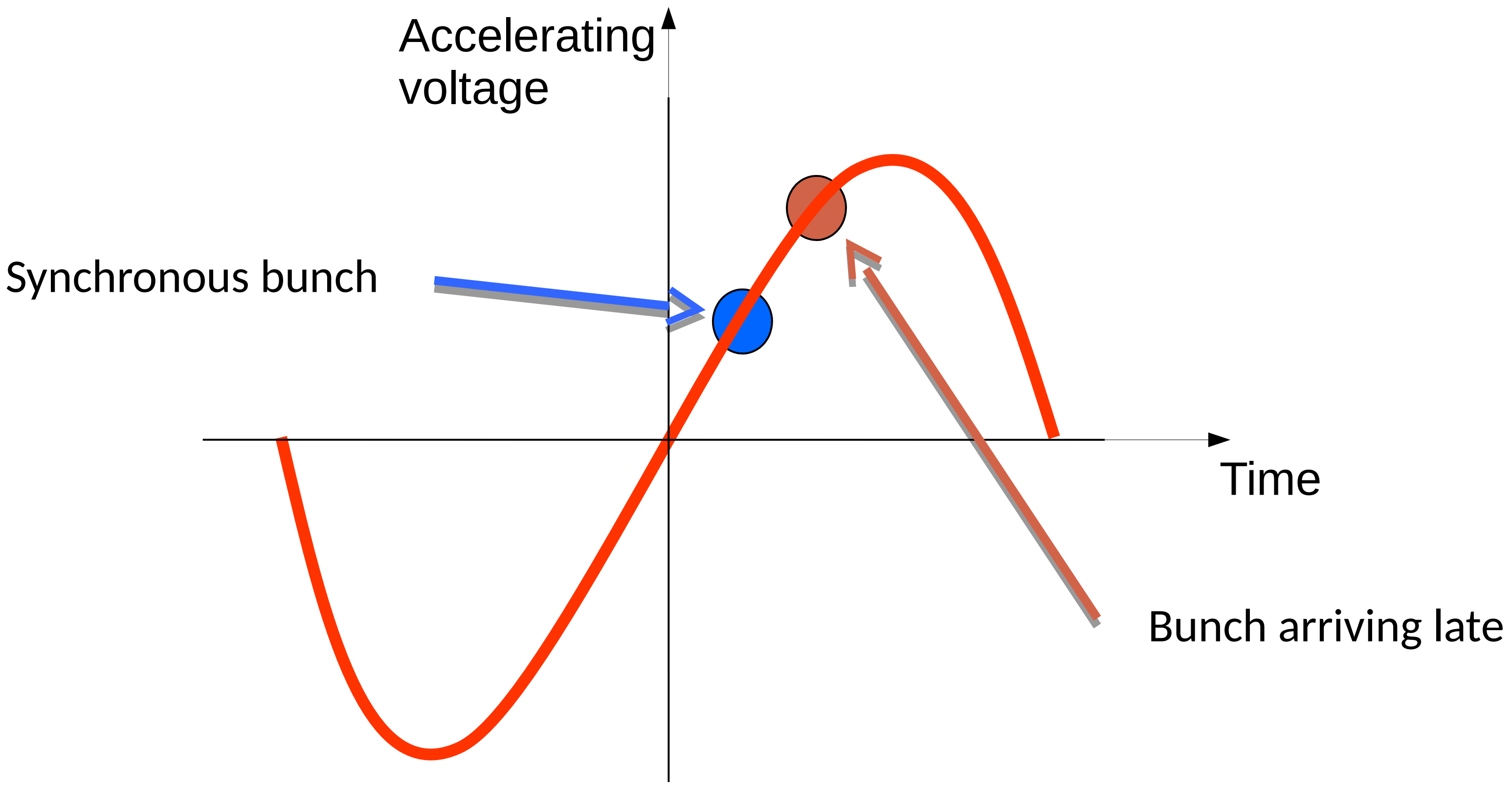


RF waveform and bunches arriving in a RF cavity

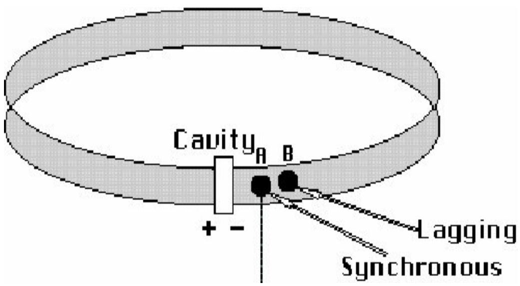


The RF voltage applied to the particle is sinusoidal in time

$$V(t) = V_0 \sin \omega_{RF} t$$

Choose the RF frequency to be an integer multiple of the revolution frequency. This is called synchronism.

Cavity synchronism

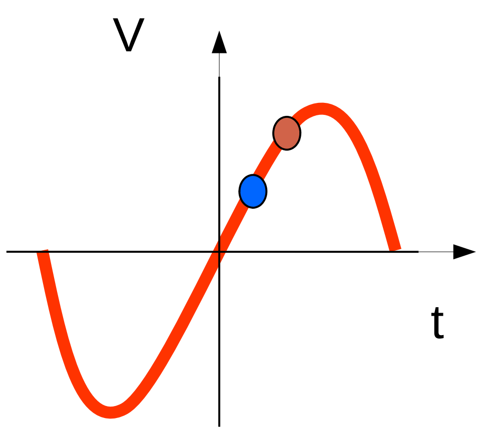


The particle at the centre of the bunch, called the synchronous particle, acquires the right amount of energy such that it experiences the same voltage difference on each turn.

Particles experience a voltage difference across the cavity of

$$V(t) = V_0 \sin(\omega_{RF} t + \phi_0) = V_0 \sin \phi_s(t)$$

In the case of no acceleration, a synchronous particle has ϕ_s= 0, and so it sees the zero of the sine function.



Particles arriving early see $<_s$

Particles arriving late see $>_s$

We'll assume that if 0 <ϕ<π the synchronous particle gains energy on each turn of the machine.

The principle of phase stability

For synchronism to work, the RF frequency must be an integer multiple of the revolution frequency

$$\omega_{RF} = h\omega_0$$

so the beam always sees the correct accelerating field.

h is the '**harmonic number'.**

But what if h is slightly non-integer, e.g. h=200.0000000001 ?

After many turns the beam will be out of phase with the RF system and will no longer be accelerated.

We need the motion to be stable even if there are small errors in the frequencies.

This is **'phase stability**'.

The concept of phase stability was developed by McMillan and Veksler in 1945.

i) Choose your RF frequency. This determines the energy of a synchronous particle.

ii) Particles with slight deviation in longitudinal coordinates will now oscillate (slowly) around this (ideal) synchronous particle.

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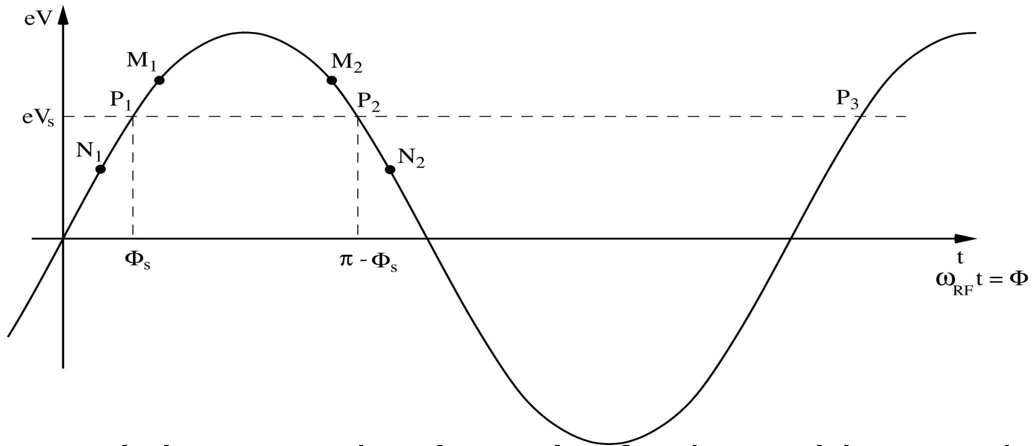
The principle of phase stability (stable phase)

But we have a spread of energy (momenta) in the beam….
Let's consider our ring, for which the synchronism condition is fulfilled for a phase ϕ_s.



eV_s is the energy gain in one accelerating region for the particle to reach the next accelerating region with the same RF phase. This fixes the phases P_1 ,P_2, etc

First consider P_1. Imagine a particle arrives a little later than the synchronous particle. So it sees a slightly later phase of the RF, so sees M_1. This means it gets a larger momentum kick, so has a higher speed and gets around the ring faster. It arrives slightly earlier than it did, and hence moves towards P_1.

Similarly, an early particle will see N_1, get a smaller kick and move towards P_1

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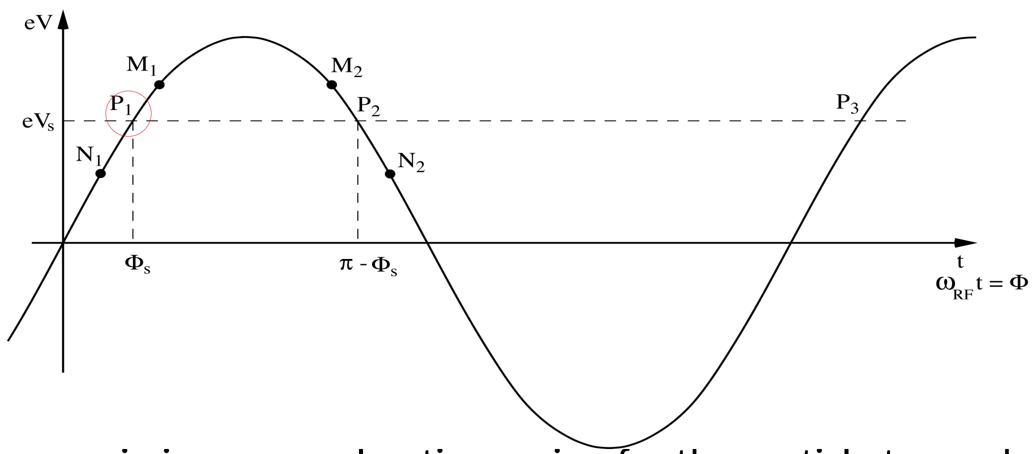

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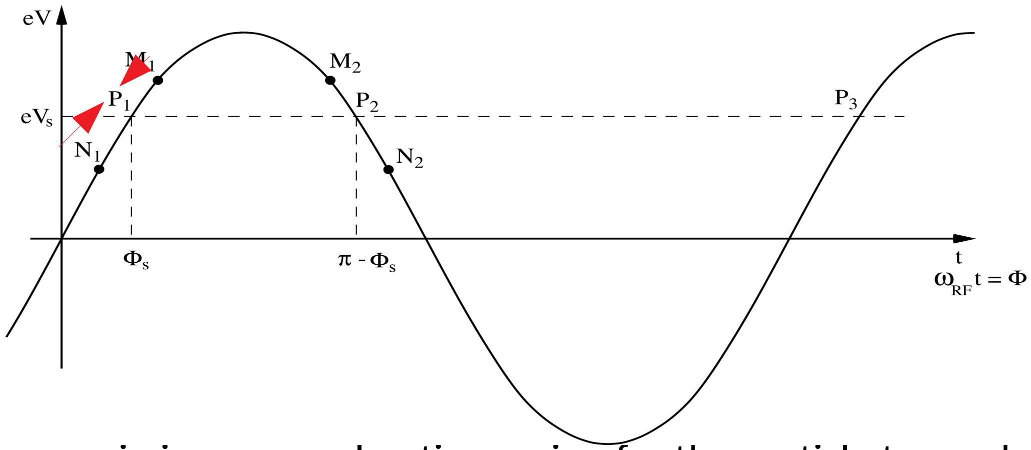

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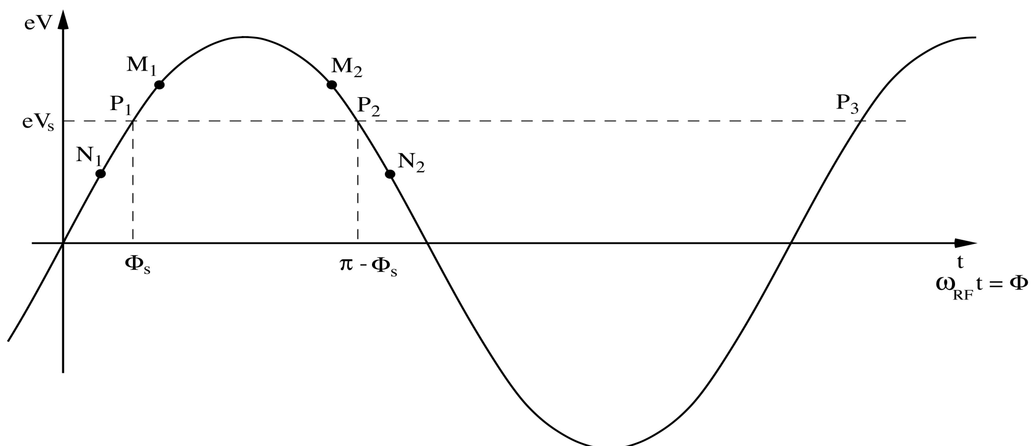

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Next consider P_2. Imagine a particle arrives a little later than the synchronous particle. So it sees a slightly later phase of the RF, so sees N_2. This means it gets a smaller momentum kick, so has a lower speed and gets around the ring slower. It arrives even later than it did, and hence moves away from P_2.

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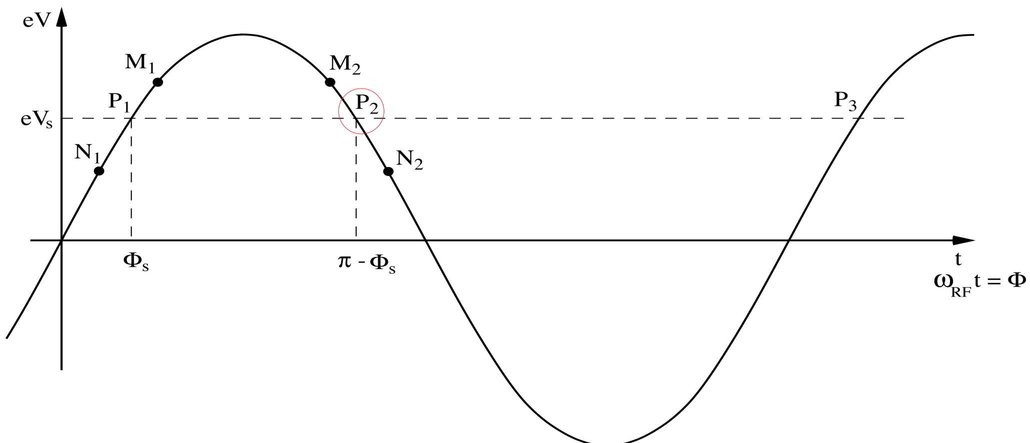

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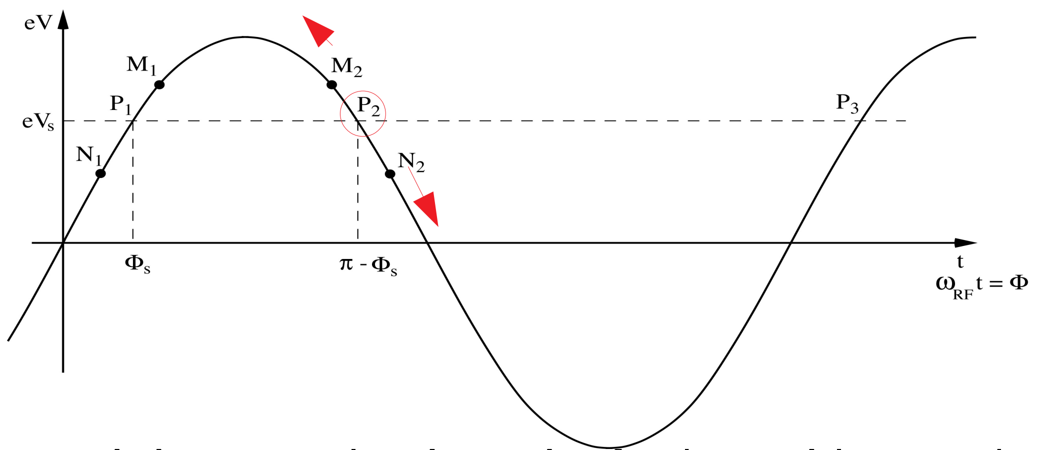

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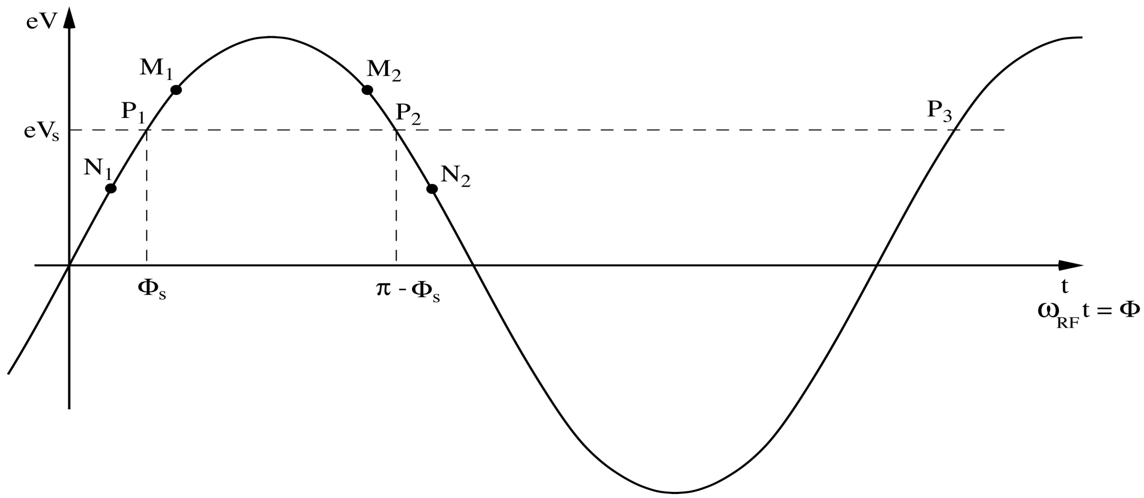

So if an increase in energy is transferred into an increase in speed, M_1 & N_1 will more towards P_1(stable), while M_2 & N_2 will move away from P_2 (unstable).

This is the principle of phase stability.

This suggests that with the right choice of phase, particles will oscillate around the synchronous phase, even though they have a natural spread of momenta.

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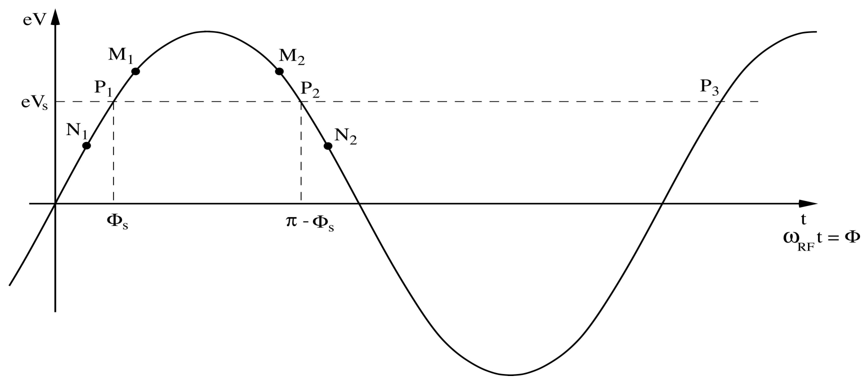
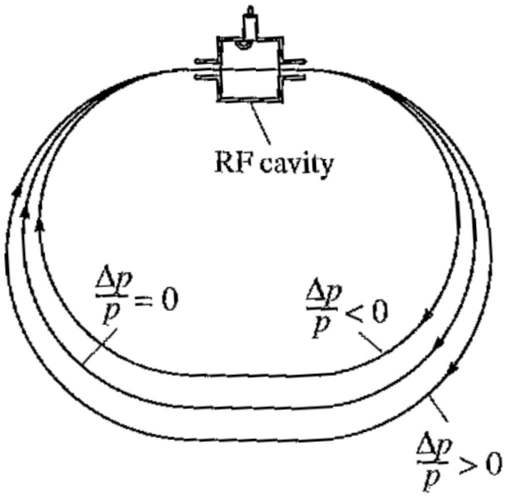
Relativistic considerations



i) For a highly-relativisitic particle an increase in energy translates to an increase in Lorentz factor but only a small increase in speed.
ii) Particles with lower/higher momenta move on an inner/outer dispersive orbit, with a shorter/longer revolution time.



For highly-relativistic particles, higher energy means a longer revolution time, this means P_2 becomes a stable point and P_1 becomes an unstable fixed point. Now M_1 & N_1 will move away from P_1(unstable), while M_2 & N_2 will go towards P_2 (stable).

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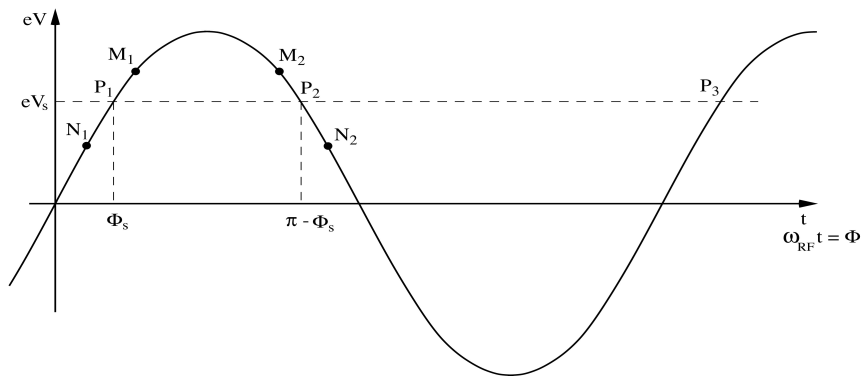
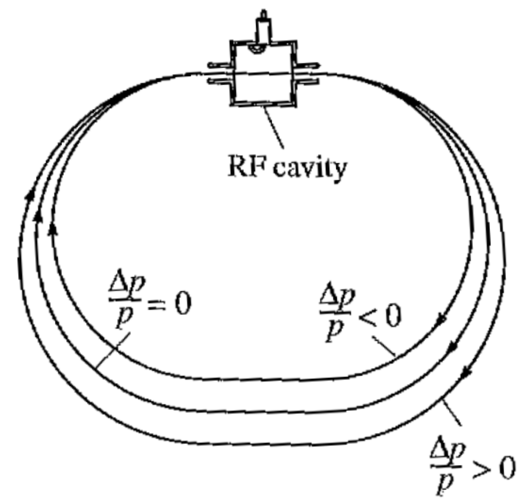


The change of ring behaviour when P_1 and P_2 swap between stable and unstable is called **transition.**



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Phase slip and transition energy

Particles with different momenta travel on different paths. The revolution period depends on the circumference taken by a particle and its speed

$$T = \frac{C}{v}$$

The fractional revolution frequency for a slightly different circumference and velocity is therefore given by

$$\frac{\Delta f}{f} = -\frac{\Delta T}{T_0} = -\frac{\Delta C}{C} + \frac{\Delta v}{v}$$

The particle arrival time is affected both by a longer path around the machine and also by the particle moving faster. We can relate both these contributions to the fractional momentum deviation,

$$\frac{\Delta v}{v_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$\frac{\Delta C}{C_0} = \alpha_c \frac{\Delta p}{p}$$

$$\frac{\Delta f}{f} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta\delta$$

And define the phase slippage factor by

$$\eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

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Transition energy

The quantity γ_t is called the transition gamma and is related to the momentum compaction factor of the lattice:

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

Below the transition energy we have

$$\gamma < \gamma_t \quad \eta < 0$$

and a higher momentum particle has a revolution period shorter than that of the synchronous particle and so makes a single turn back to the cavity in a shorter time. This means our fixed point P_1 is stable and P_2 is unstable.

Above the transition energy we have

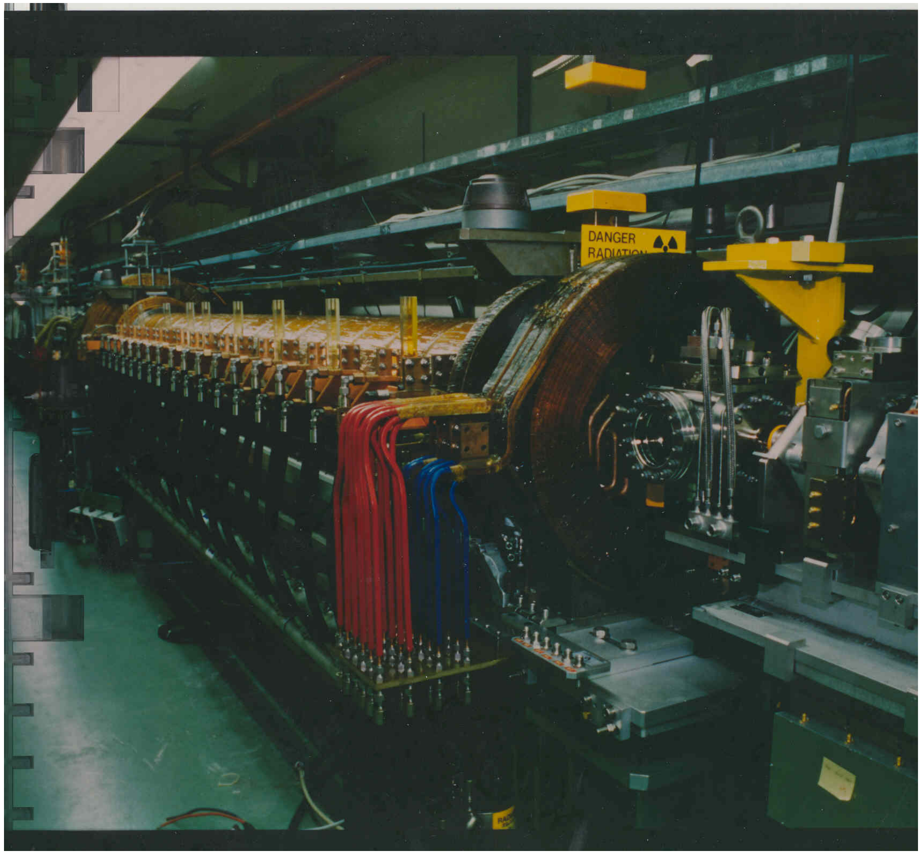
$$\gamma > \gamma_t \quad \eta > 0$$

And the opposite is true. Now higher momentum particles have a revolution period greater than that of the synchronous particle. This means our fixed point P_1 is unstable and P_2 is stable.

At the transition energy the machine is isochronous for all momenta and all particle circulate with the same period. I.e. $\eta = 0$

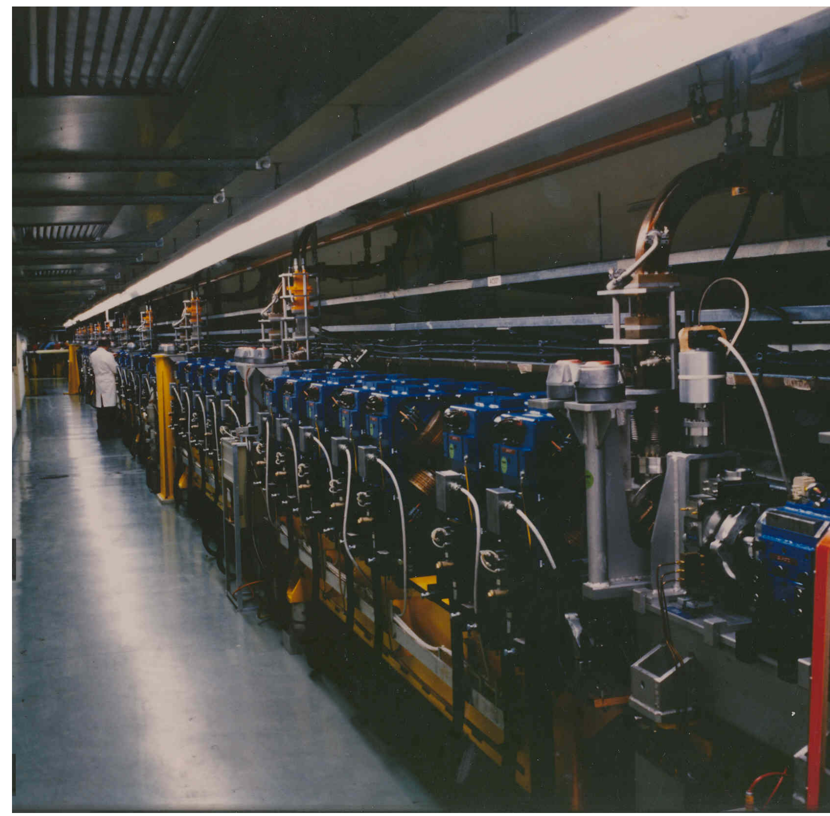
(Some authors define the phase slippage eta the other way around!)

Linacs



Accelerating section, of an electron linac, equipped with solenoids

Accelerating section, of an electron linac, equipped with quadrupoles



(In a linac, only speed changes matter for longitudinal stability as there are no dipoles and hence no momentum compaction)

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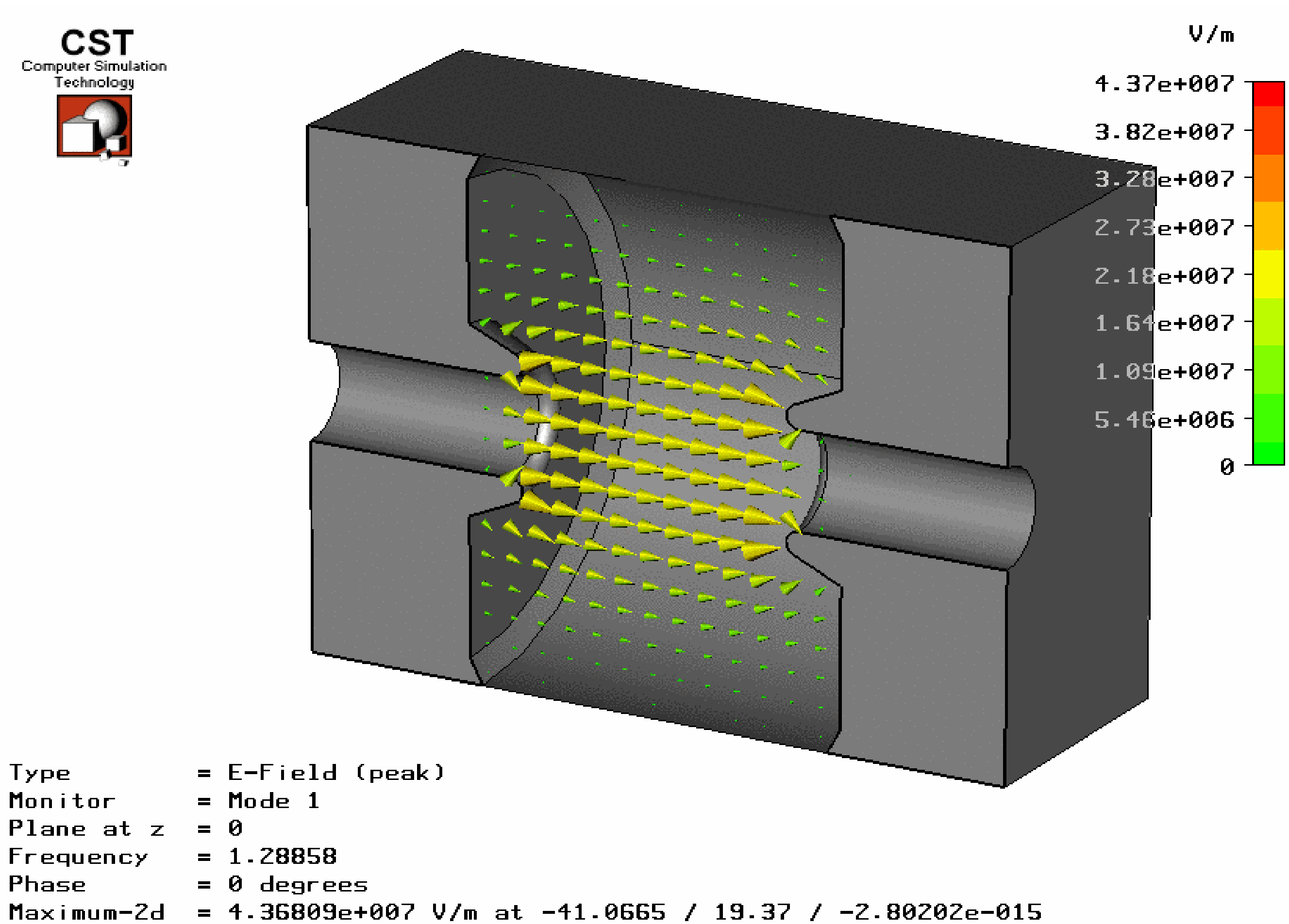
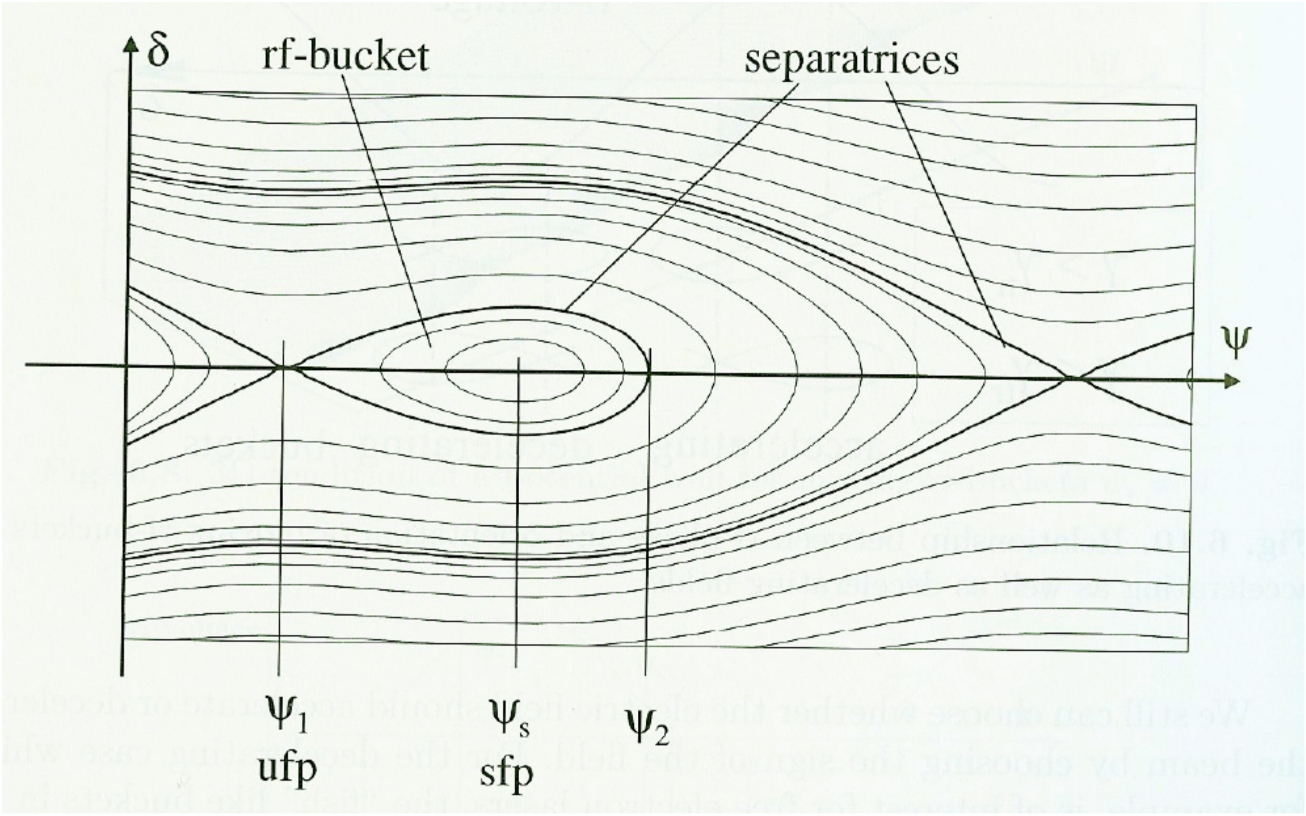
A ‘real’ cavity on-axis field



Courtesy of Graeme Burt

The phase space portrait

Given the parameters of the cavity, we can compute the motion in the longitudinal phase space.



Wiedemann p212

The separatrix starts from a point very close to (but not exactly at) the unstable fixed point, moves away and forms an ‘alpha’ or fish shape around the stable fixed point. The area enclosed is the **bucket** and corresponds to stable motion.

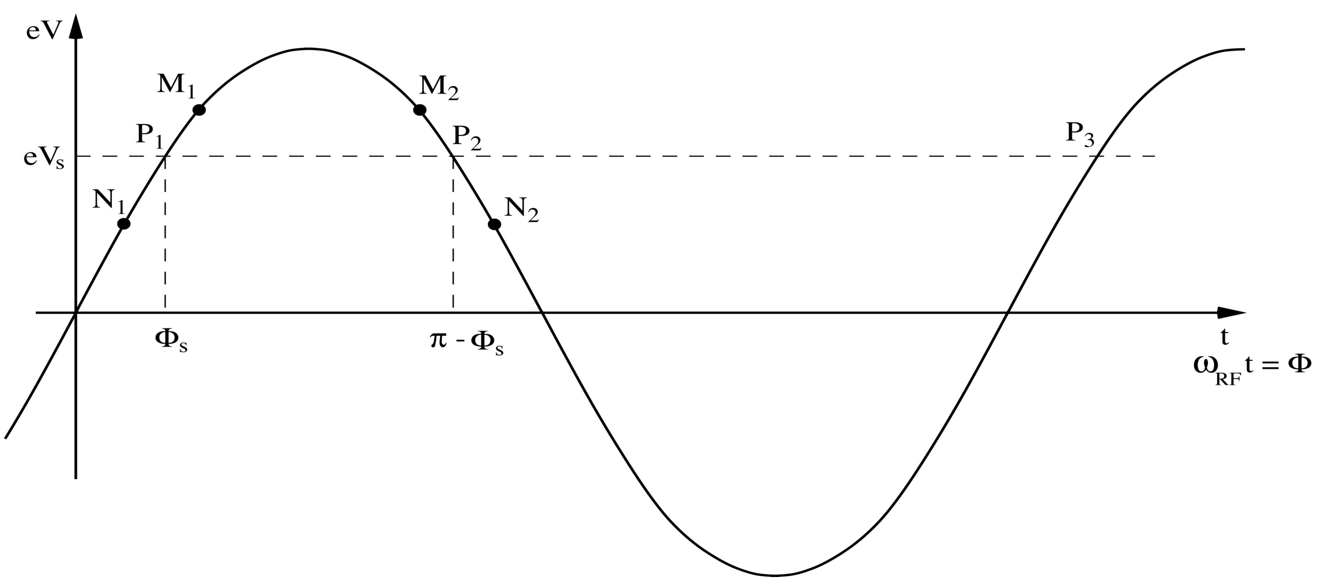
Stable and unstable fixed points (SFP, UFP)

Our "fixed points" are

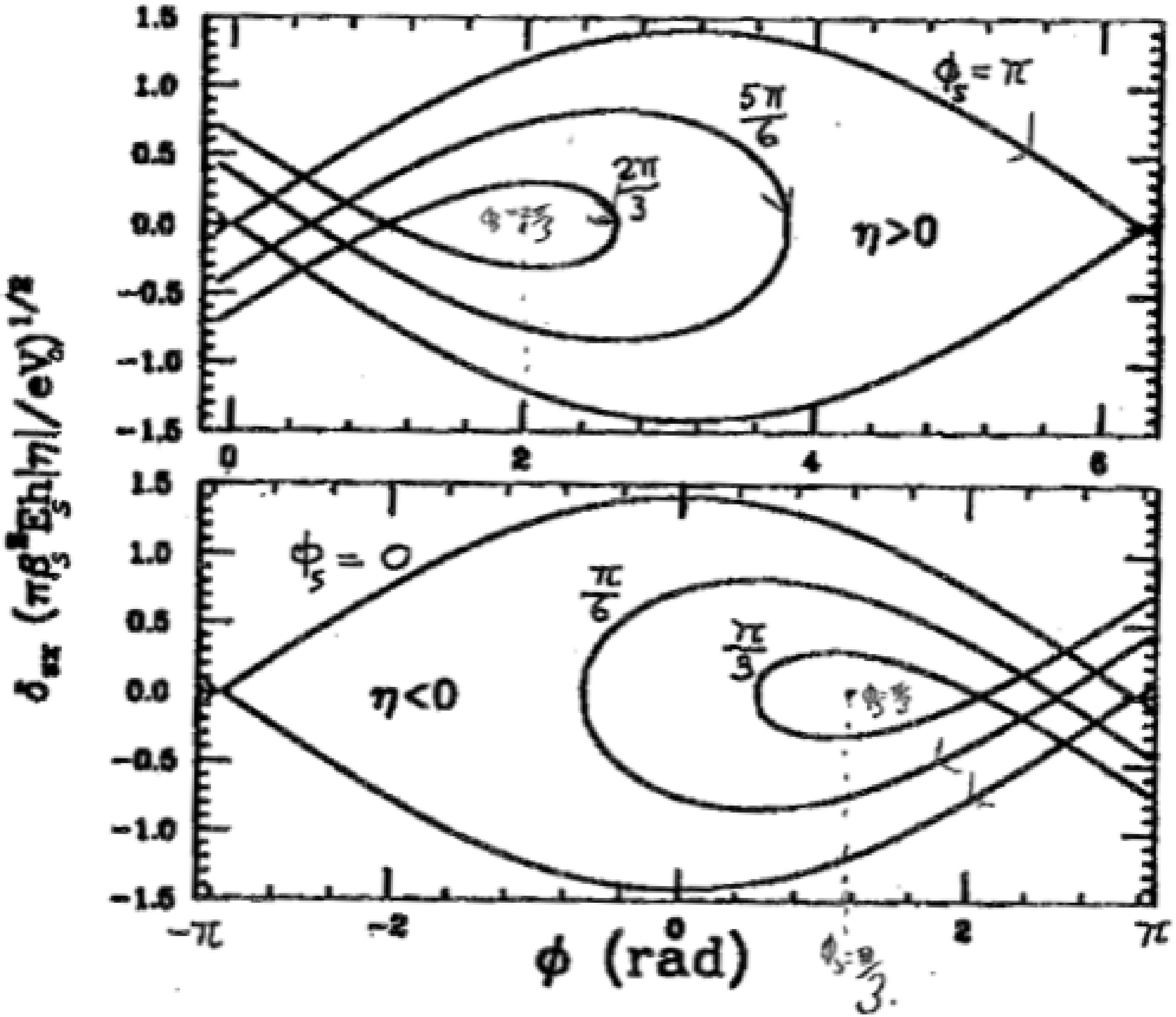
$$\Psi = \Psi_0 \quad \Psi = \pi - \Psi_0$$

If the synchronous phase P_2 in the diagram below is stable, P_2 is a SFP. Hence P_1 is an UFP. This is **above** transition.

If the synchronous phase P_1 in the diagram below is stable, P_1 is a SFP. Hence P_2 is an UFP. This is **below** transition.



The phase space portrait – above and below transition



Each fish-shaped curve corresponds to a different value of the synchronous phase (RF settings).

(Ignore the odd units of the energy axis….
It's expressed in terms of a normalised momenta deviation).

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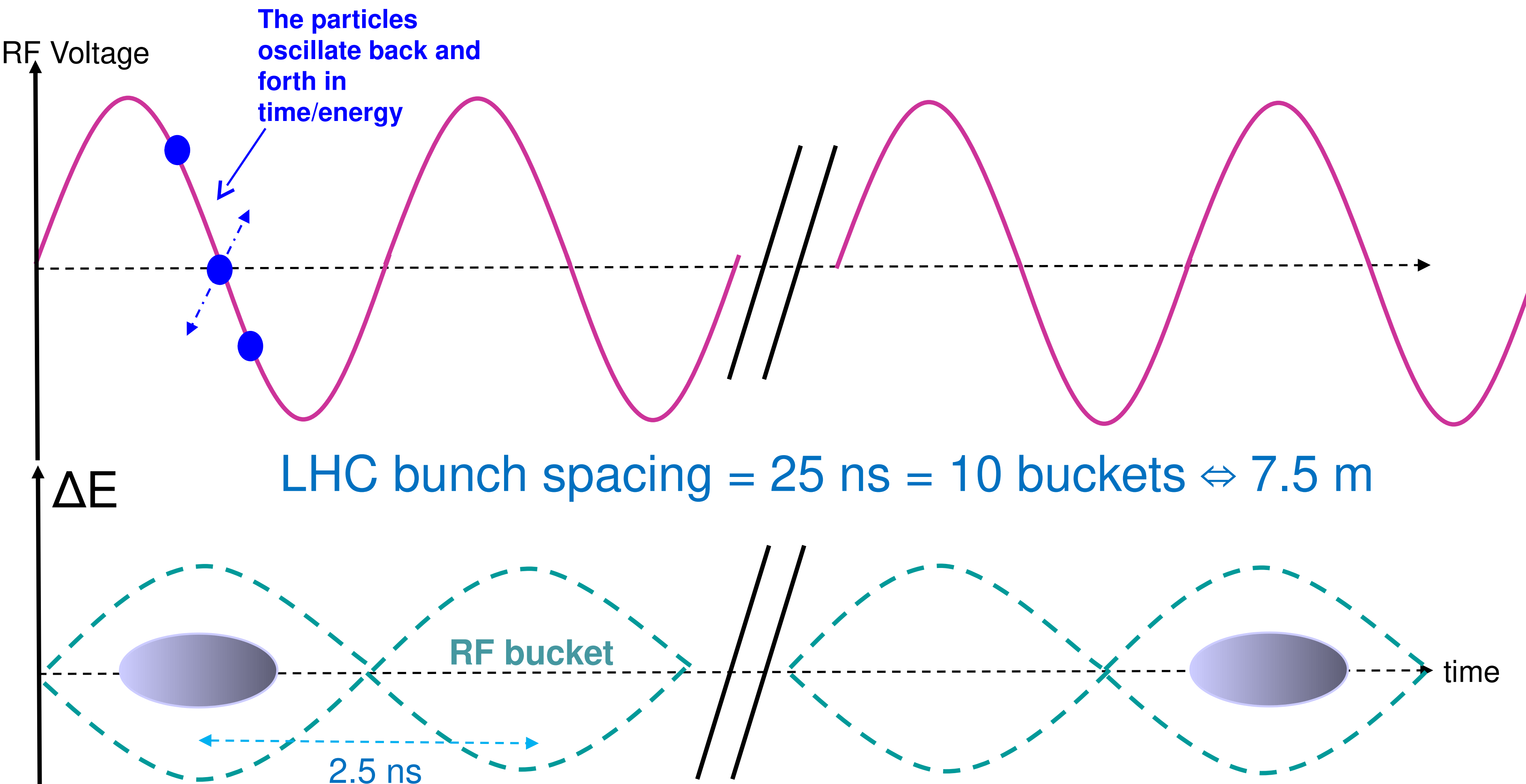
Buckets and bunches – LHC example



$$\omega_{RF} = h\omega_0$$

LHC harmonic number is 35,640
RF frequency is 400 MHz
Every 10^{th} bucket is filled
Bunch spacing is 7.5m

Longitudinal dynamics (variables)

The RF acceleration process relates the energy gained by a particle to the RF phase experienced by the same particle. Since there is a well-defined synchronous particle which always experiences the same rf phase ϕ_0, and gains the nominal energy E_0, it is sufficient to follow other particles with respect to that particle. Possible choice of "reduced variables" are

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\psi = \psi - \psi_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$

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Longitudinal motion

The periodic longitudinal motion around the nominal phase is known as **synchrotron motion**.

Let's study this motion in a more quantitative way by looking at energy balance for one complete turn of the machine.

The nominal quantities are denoted by '0' subscripts, U is the peak voltage and W denotes the energy loss per revolution of the particle.

The nominal particle energy balance is

$$E_0 = eU_0 \sin \Psi_0 - W_0$$

An arbitrary particle which has a slight momentum deviation and a slight phase deviation from the nominal particle has energy balance

$$E = eU_0 \sin(\Psi_0 + \Delta\Psi) - W$$

We write the energy loss per turn for the arbitrary particle as a linear function in the energy deviation

$$W = W_0 + \frac{dW}{dE}\Delta E$$

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Longitudinal motion (smooth approximation)

If we take the difference between the one-turn energy balance of the nominal and arbitrary particle, we get an expression for the energy difference between the two

$$\Delta E = E - E_0 = eU_0 \left[\sin(\Psi_0 + \Delta\Psi) - \sin\Psi_0\right] - \frac{dW}{dE}\Delta E$$

One synchrotron oscillation lasts many turns of the machine. So to get the rate of change of the energy deviation we can simply divide by the revolution time T_0

$$\Delta\dot{E} = \frac{\Delta E}{T_0} = \frac{eU_0}{T_0}\left[\sin(\Psi_0 + \Delta\Psi) - \sin\Psi_0\right] - \frac{dW}{dE}\frac{\Delta E}{T_0}$$

This gives a differential equation for the energy deviation in the **smooth approximation**.

Now we want a differential equation for the phase difference.

Getting an equation for the phase difference

We already have a useful starting expression a few slides ago, when we derived

$$\frac{\Delta T}{T_0} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

in our discussion of the transition gamma.

As the accelerating RF frequency has period T_{RF}

$$\Delta\Psi = 2\pi\frac{\Delta T}{T_{\mathrm{RF}}} = \omega_{\mathrm{RF}}\Delta T$$

Recall the RF frequency should be an integer multiple of the revolution frequency. The harmonic number q (or h) is

$$q = \frac{\omega_{\mathrm{RF}}}{\omega_{\mathrm{rev}}}$$

Therefore the more general expression for the phase deviation is given by

$$\Delta\Psi = \underbrace{q\omega_{\mathrm{rev}}}_{\omega_{\mathrm{RF}}}\Delta T = 2\pi q\frac{\Delta T}{T_0} = 2\pi q\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

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Phase differential equation

Let’s swap momentum deviation for energy deviation (convention) using

$$\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

giving an expression for the phase deviation

$$\Delta\Psi = \frac{2\pi q}{\beta^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E}$$

‘Differentiating’ this with respect to time as before gives us

$$\Delta\dot{\Psi} = \frac{\Delta\Psi}{T_0} = \frac{2\pi q}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E}$$

Together with the first order equation for the energy deviation

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We can either solve the equations numerically, or combine them to make a second order ODE.

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Longitudinal motion

Let's make a second order ODE for small momentum deviations i.e.

$$\Delta\Psi \ll \Psi_0$$

Expanding the sine function using standard identities we get

$$\sin(\Psi_0 + \Delta\Psi) - \sin\Psi_0 = \sin\Psi_0 \cos\Delta\Psi + \cos\Psi_0 \sin\Delta\Psi - \sin\Psi_0$$

Which is approximately

$$\sin(\Psi_0 + \Delta\Psi) \sim \Delta\Psi \cos\Psi_0$$

and so our energy equation becomes

$$\Delta\dot{E} = \frac{eU_0}{T_0}\Delta\Psi\cos\Psi_0 - \frac{dW}{dE}\frac{\Delta E}{T_0}$$

which we can differentiate to get

$$\Delta\ddot{E} = \frac{eU_0}{T_0}\Delta\dot{\Psi}\cos\Psi_0 - \frac{dW}{dE}\frac{\Delta\dot{E}}{T_0}$$

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$$\left.\begin{aligned}\Delta\ddot{E} &= \frac{eU_0}{T_0}\Delta\dot{\Psi}\cos\Psi_0 - \frac{dW}{dE}\frac{\Delta\dot{E}}{T_0}\\ \Delta\dot{\Psi} &= \frac{\Delta\Psi}{T_0} = \frac{2\pi q}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E}\end{aligned}\right\}$$

We can eliminate the phase differential to obtain

$$\Delta\ddot{E} + 2a_s\Delta\dot{E} + \Omega^2\Delta E = 0$$

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Longitudinal motion

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$$\Delta\ddot{E} + 2a_s\Delta\dot{E} + \Omega^2\Delta E = 0$$

Here we have defined a damping term as

$$a_s = \frac{1}{2T_0}\frac{dW}{dE}$$

and the 'frequency' of the motion as

$$\Omega = \omega_{\mathrm{rev}}\sqrt{-\frac{eU_0 q\cos\Psi_0}{2\pi\beta^2 E}\left(\alpha_c - \frac{1}{\gamma^2}\right)}$$

This differential equation is the same form as the one for a damped harmonic oscillator, and means our motion in longitudinal phase space is (at least for small values of the energy deviation) oscillatory around the nominal energy provided the damping term is not too large.

We can use our knowledge of differential equations to find the solutions to this equation…

Longitudinal motion

We adopt a trial solution for the energy deviation, where the constant ω may in general be complex

$$\Delta E(t) = \Delta E_0 \exp^{\omega t}$$

This gives the characteristic equation for ω,

$$\omega^2 + 2a_s\omega + \Omega^2 = 0$$

Which can be solved to get

$$\omega = -a_s \pm \sqrt{a_s^2 - \Omega^2}$$

If we assume light damping, then the second term is purely imaginary and we obtain an oscillatory solution for the energy deviation

$$\Delta E(t) = \Delta E_0 \exp^{-a_s t} \exp^{i\Omega t}$$

Ω Frequency

a_s Damping constant

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Above and below transition

Recalling the expression for the frequency,

$$\Omega = \omega_{\text{rev}} \sqrt{-\frac{eU_0 q \cos\Psi_0}{2\pi\beta^2 E}\left(\alpha_c - \frac{1}{\gamma^2}\right)}$$

Stable solutions require this to be a real number, and so we require that

$$\left(\alpha_c - \frac{1}{\gamma^2}\right)\cos\Psi_0 < 0$$

There are two solutions for this stable phase, namely the ones we derived previously when we talked about the transition energy. This solution below transition is

$$-\frac{\pi}{2} < \Psi_0 < \frac{\pi}{2} \quad \text{and} \quad \alpha_c < \frac{1}{\gamma^2}$$

And above transition is

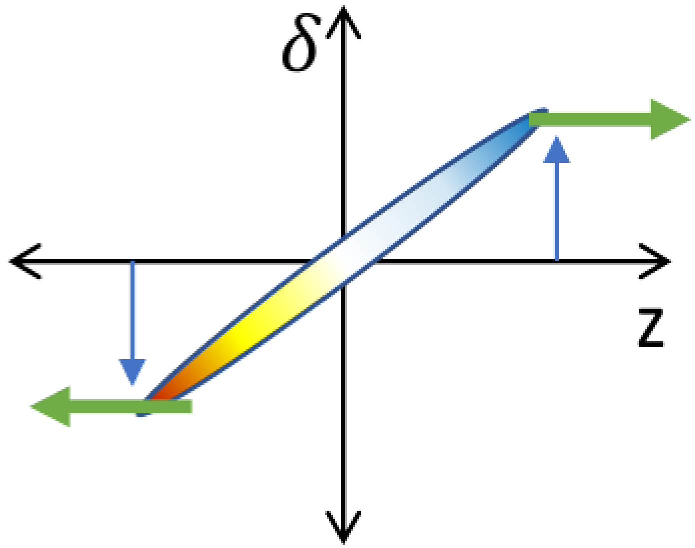
$$\frac{\pi}{2} < \Psi_0 < \frac{3\pi}{2} \quad \text{and} \quad \alpha_c > \frac{1}{\gamma^2}$$

longitudinal phase analysis - summary

1) Choose required energy gain per turn. Knowing the RF voltage we obtain the synchronous phase. There are two possible phase solutions between 0 and 2π.
2) Here we assume we want acceleration, so $U_0 > 0$
3) We now check the phase slippage factor. I.e. how does the frequency of revolution change for a small change dp/p? Positive means above transition, meaning the synchronous phase lies between π/2 and π. Negative means below transition, meaning the synchronous phase lies between 0 and π/2.
4) The curve in phase space which divides the stable and unstable regions of flow is called the **separatrix**. The stable region surrounds the stable fixed point. The separatrix can be found by an Hamiltonian analysis of the system (beyond this course).
5) The region inside the separatrix, where stable motion exists, is called the bucket. Expressions exist (beyond this course) for the width (in phase), height (in energy) and the area (maximum longitudinal emittance) of this bucket.
6) Motion is stable near the stable fixed point and unstable near the unstable fixed point. So particles cluster around the SFP and stay away from the UFP. **Hence we observe the fact that beams in a synchrotron when RF is applied are necessarily bunched**.

longitudinal phase analysis – the need for compression

Bunches of particles accelerated in this way have a spread of momenta imprinted by the change of rf phase with time.
This can lead to a spread in the bunch length.
Some (many) applications require very short bunches.



Therefore in some machines a “bunch compressor” is necessary.
This often takes the form of a chicane or an arc.

Figure courtesy of G. Perez Segurana

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SYNCHROTRON RADIATION AND DAMPING

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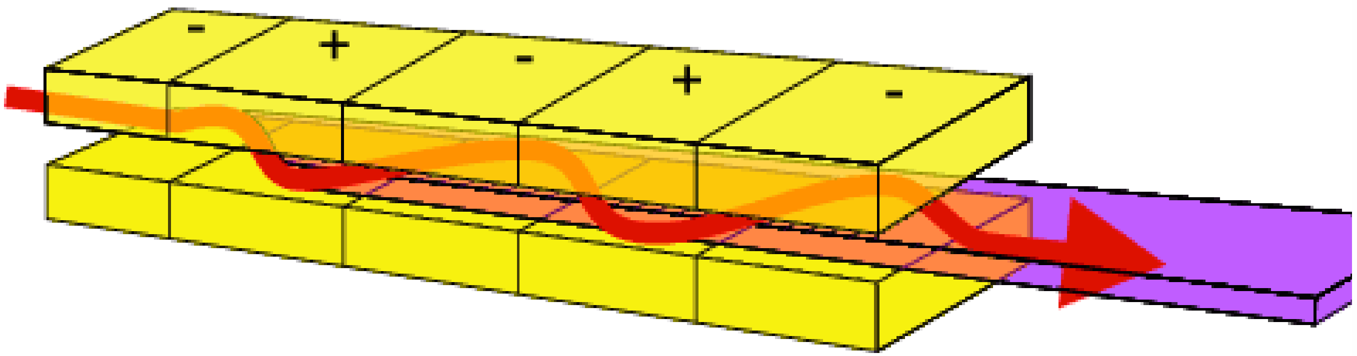
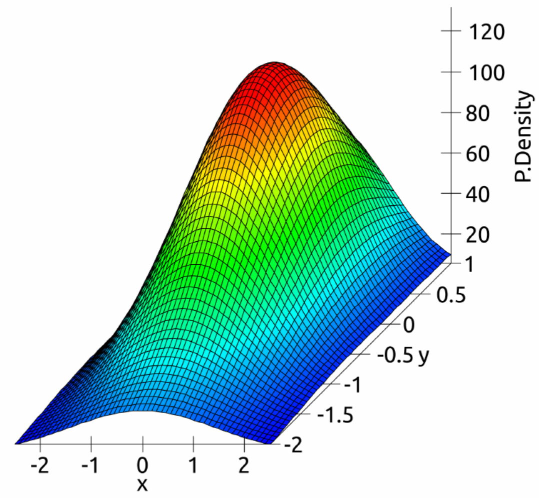
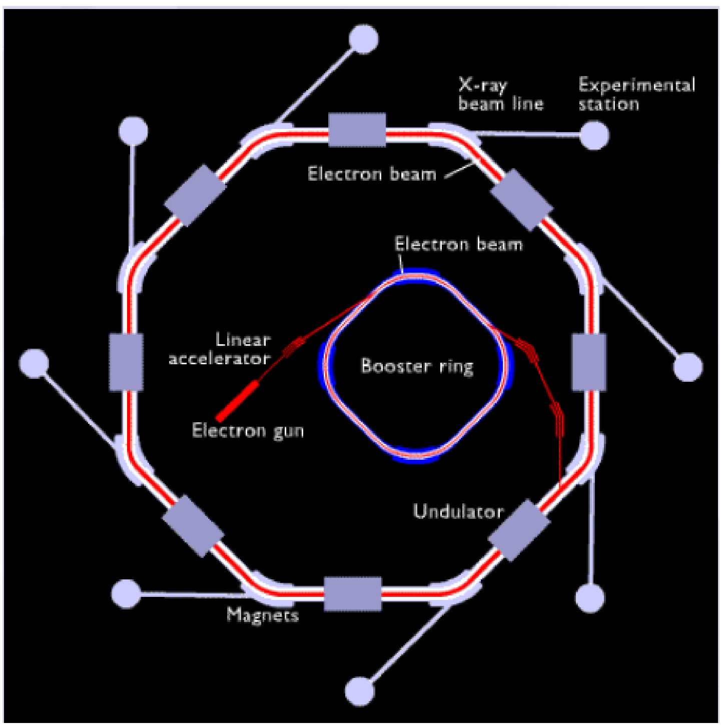
Emission of synchrotron radiation



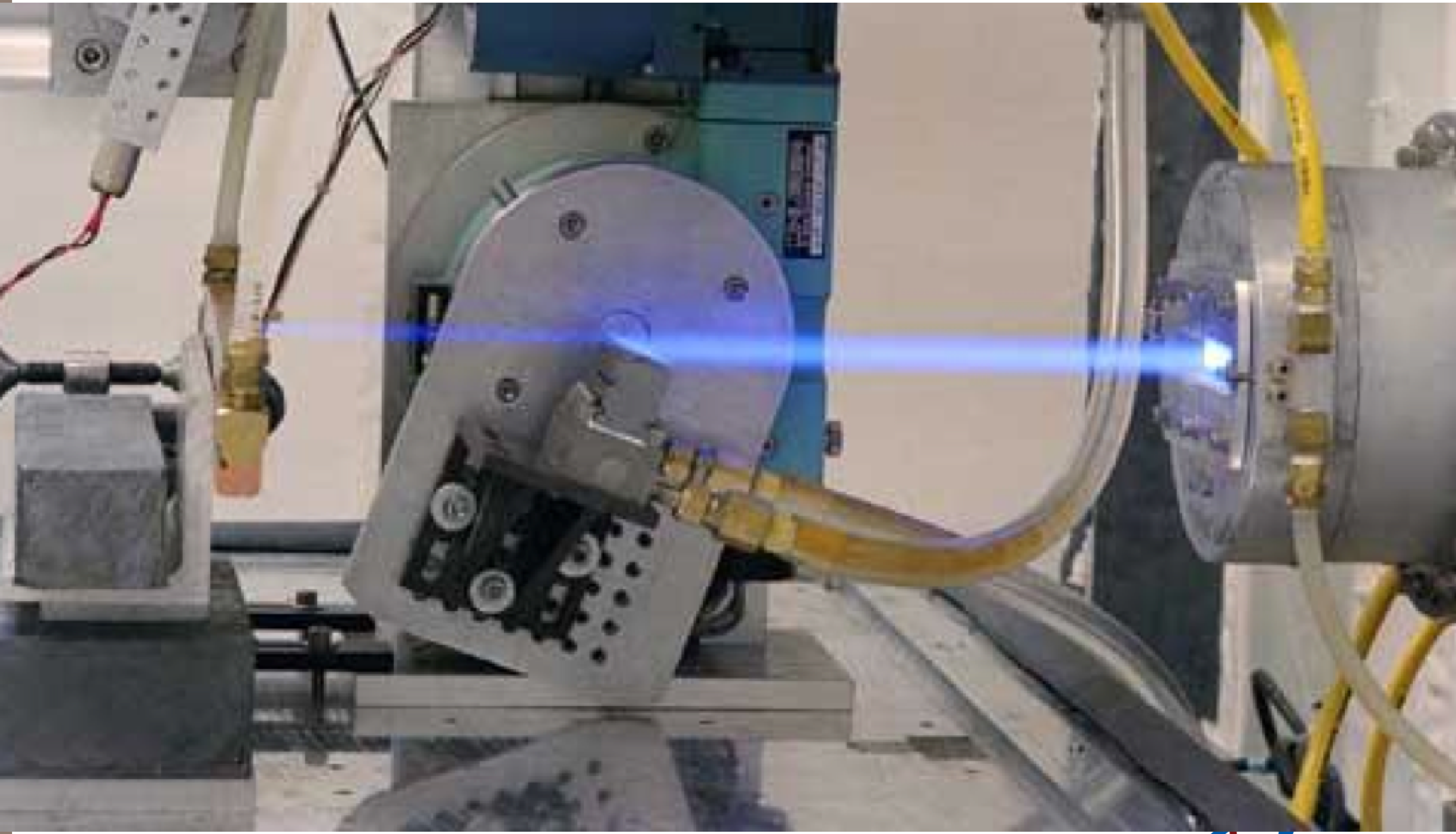
undulator



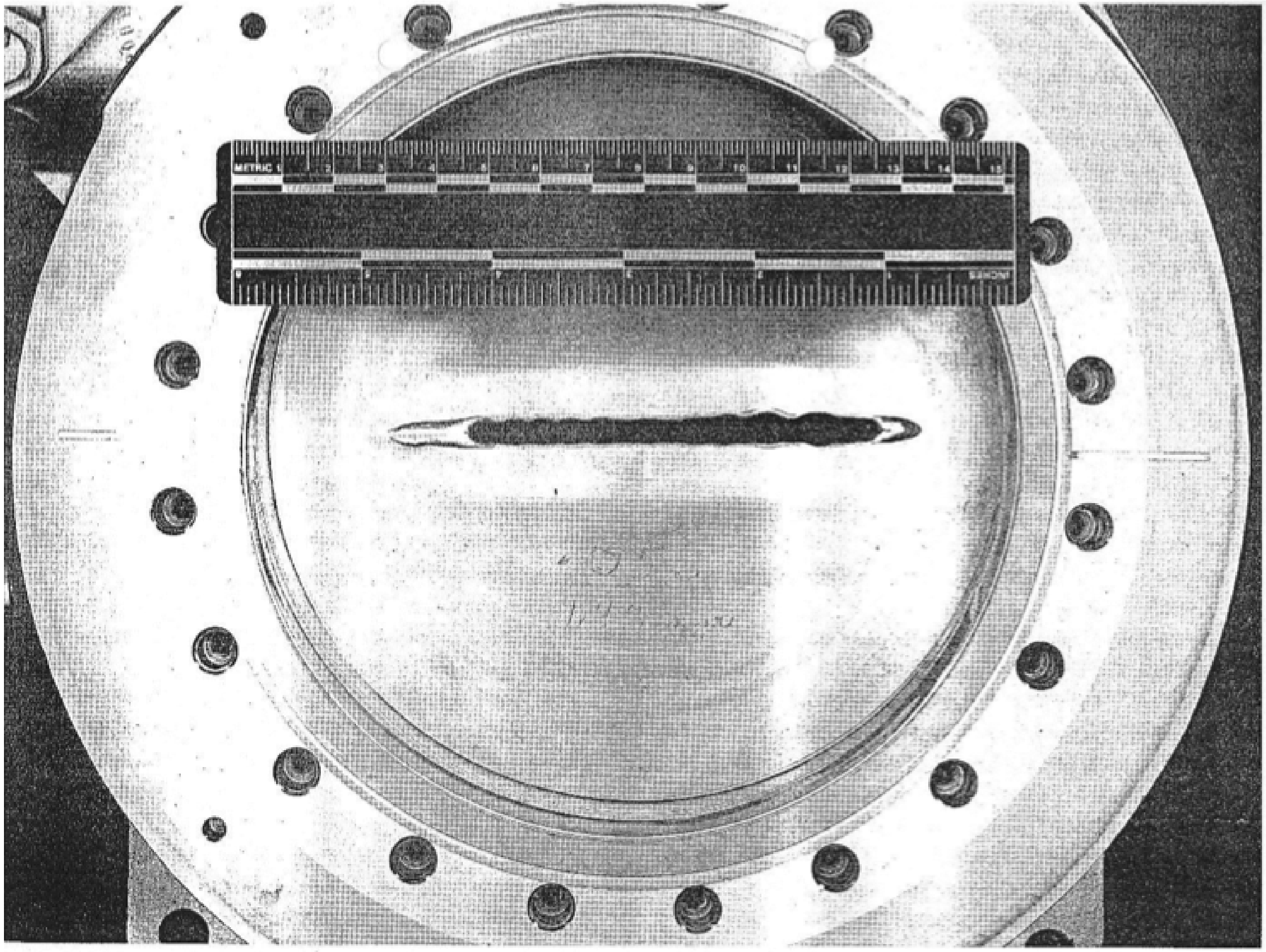


All accelerating charges radiate electromagnetic radiation.

It can be intense…



And can interact with material…



ig. 12. Damaged X-ray ring front end gate valve. The power incident on the valve was approximately 1 kW for a duration estimated to 2–10 min and drilled a hole through the valve plate.

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Retarded potentials

The radiation field is given in terms of the potentials by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

$$\vec{B} = \nabla \times \vec{A}$$

Where the potentials are the retarded potentials (see EM course).

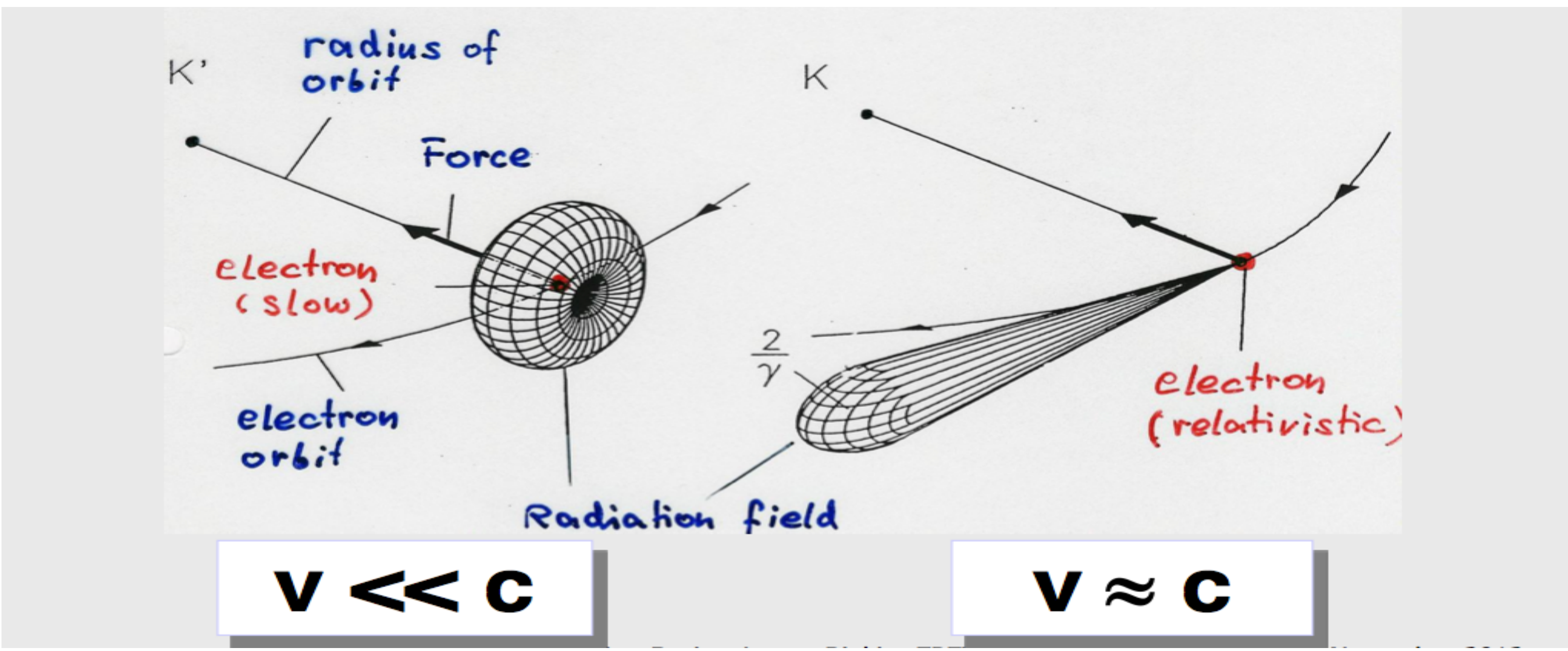
The calculations of E and B lead to a doughnut-shaped pattern of radiation power in the rest frame of the particle.

This means the pattern is also doughnut-shaped in the non-relativistic case in the lab, and becomes two back-to-back cones in the lab after the appropriate Lorentz transformation in the relativistic case.

Emission of synchrotron radiation from a dipole bend

The radiation is emitted into a cone with an opening angle of $1/\gamma$, where γ is the relativistic Lorentz factor.

(We shall assume the radiation is emitted in the direction of motion i.e. *v* is close to *c*)



The spectrum of bending radiation

A fixed observer sees a very short EM pulse of photons as the circulating electron bunch passes them turn after turn.

This gives a broad photon frequency spectrum characterised by the "critical energy", which divides the spectrum into two regions of equal power.

We won't be deriving the expression here, but for reference, the number of photons emitted as a function of frequency / energy is given by

$$N \propto \left(\frac{\omega}{\omega_c}\right) \int_{\omega/\omega_c}^{\infty} K_{5/3}(y)\,dy$$

where K is a modified Bessel function and the critical frequency is given by

$$\omega_c = \frac{E_C}{\hbar} = \frac{3c\gamma^3}{2\rho}$$

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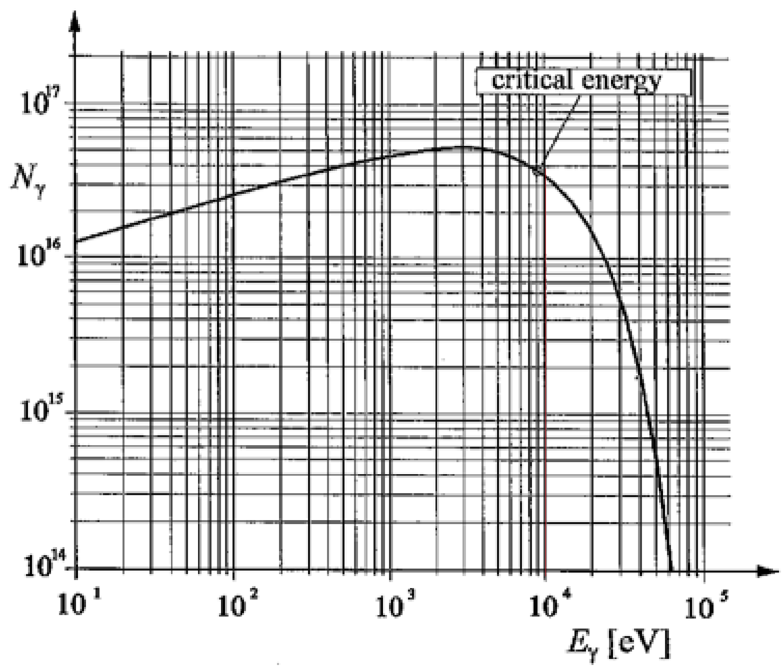

The plot shows the spectrum for an electron beam of energy 5 GeV passing through a dipole of bend radius 12.2 m but the shape is universal.

$$\omega_c = \frac{E_C}{\hbar} = \frac{3c\gamma^3}{2\rho}$$

Larmor's formula

To calculate the SR power in colliders, we'll jump from the retarded potentials to the relativistic version of Larmor's formula, giving the power emitted by a charged particle

$$P_s = \frac{e^2 c}{6\pi\epsilon_0} \frac{1}{(m_0 c^2)^2} \left[\left(\frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{d\tau} \right)^2 \right]$$

$$\vec{p} = \gamma m_0 \vec{v}$$

This tells us that power is emitted as SR if the momentum changes or the total energy changes.

We'll deal with two limiting cases:

Linear acceleration $\frac{d\vec{p}}{d\tau} \parallel \vec{p}$

Circular acceleration $\frac{d\vec{p}}{d\tau} \perp \vec{p}$

The linear case

For the case in linear acceleration (linacs), the spatial energy gradient is often known, and a short calculation gives a formula for the power radiated from linearly-accelerating particles:

$$P_s = \frac{e^2 c}{6\pi\epsilon_0} \frac{1}{(m_0 c^2)^2} \left(\frac{dE}{dx} \right)^2$$

Even a high field gradient such as dE/dx=35 MeV / m gives P_s around 10^{-16} W (for electrons).

For ultra-relativistic particles we can generally neglect this effect.

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The circular case

Assume the motion is circular and the total energy stays the same.
Larmor's formula becomes

$$P_s = \frac{e^2 c}{6\pi\epsilon_0}\frac{1}{(m_0c^2)^2}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 \cancel{- \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2}\right] \qquad P_s = \frac{e^2 c\gamma^2}{6\pi\epsilon_0}\frac{1}{(m_0c^2)^2}\left(\frac{dp}{dt}\right)^2$$

This can be written in terms of the energy of the particle E (for a relativistic particle) and the bending radius of the motion R

$$P_s = \frac{e^2 c}{6\pi\epsilon_0}\frac{1}{(m_0c^2)^4}\frac{E^4}{R^2}$$

This formula was found by Lienard. Note it depends on the fourth power of the particle energy and the second power of the radius.
Comparing electrons to protons

$$\frac{P_{s,e}}{P_{s,p}} = \left(\frac{m_p c^2}{m_e c^2}\right)^4 = 1.13\times 10^{13}$$

i.e. SR much more important in electron machines

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This formula was found by Lienard. Note it depends on the fourth power of the particle energy and the second power of the radius.
Comparing electrons to protons

$$\frac{P_{s,e}}{P_{s,p}} = \left(\frac{m_pc^2}{m_ec^2}\right)^4 = 1.13\times10^{13}$$

i.e. SR much more important in electron machines

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The circular case

We'd like to know the energy loss per turn i.e. for one complete revolution. We assume constant R (i.e. a circular machine) and P_s

$$U_0 = \oint P_s dt = P_s \frac{2\pi R}{c}$$

We obtain a useful expression for the energy loss per turn of a circular machine

$$U_0 = \frac{e^2}{3\epsilon_0 (m_0 c^2)^4} \frac{E^4}{R} = \frac{4\pi m_0 c^2 \gamma^4}{3R}$$

Choosing values for an electron (or positron) we arrive at

$$U_0 = 0.0855[\mathrm{MeV}] \frac{(E[\mathrm{GeV}])^4}{R[\mathrm{m}]}$$

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Orbit period.

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The circular case

We define the useful constant

$$c_\gamma = \frac{4\pi}{3}\frac{r_0}{(m_0c^2)^3} = \frac{1}{3}\frac{e^2}{\epsilon_0(m_0c^2)^4}$$

r_0=classical electron radius

giving the compact formula for the energy radiated per revolution

$$U_0 = c_\gamma \frac{E^4}{R}$$

Rewriting the radius *R* using the beam rigidity

$$\frac{1}{R} = \frac{eB}{p} \qquad \frac{1}{R} = \frac{ecB}{\beta E}$$

we end up with our final expressions for the energy lost per turn,

$$U_0 = c_\gamma \frac{ec}{\beta}E^3B$$

and the power radiated:

$$P_s = c_\gamma \frac{ce^2c^2}{2\pi\beta^2}E^2B^2$$

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$$\frac{U_0 c}{2\pi R}$$

and the power radiated:

$$P_s = \boxed{c_\gamma \frac{ce^2c^2}{2\pi\beta^2}E^2B^2}$$

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Energy losses

Table 2.1 A few important circular electron accelerators. L is the circumference of the machine, E the maximum beam energy, R the bending radius, B the field in the bending magnets and ΔE the energy loss per revolution.

accelerator	L [m]	E [GeV]	R [m]	B [T]	ΔE [keV]
BESSY I (Berlin)	62.4	0.80	1.78	1.50	20.3
DELTA (Dortmund)	115	1.50	3.34	1.50	134.1
DORIS II (Hamburg)	288	5.00	12.2	1.37	4.53×10^3
ESRF (Grenoble)	844	6.00	23.4	0.855	4.90×10^3
PETRA (Hamburg)	2304	23.50	195	0.40	1.38×10^5
LEP (Geneva)	27×10^3	70.00	3000	0.078	7.08×10^5

Wille

At around 100 GeV the SR losses get so large that it becomes too expensive to replace them with the RF systems (e.g. LEP).

We need to use either heavier particles (protons, muons, …) or a larger bending radius.

Energy losses – e.g. the proposed FCC-ee at CERN

	Z	**WW**	**ZH**	**$t\bar{t}$**	
Circumference (km)			97.756		
Bending radius (km)			10.760		
Free length to IP l^* (m)			2.2		
Solenoid field at IP (T)			2.0		
Full crossing angle at IP θ (mrad)			30		
SR power/beam (MW)			50		
Beam energy (GeV)	45.6	80	120	175	182.5
Beam current (mA)	1390	147	29	6.4	5.4

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Away from the design orbit

$$P_s = c_\gamma \frac{ce^2c^2}{2\pi\beta^2}E^2B^2$$

Note that we've implicitly assumed that we're considering a particle on a circular reference orbit moving in an isomagnetic ring.

What if a particle is not on the reference orbit?

If the particle is not on the reference orbit but does have the reference energy, E, then the effect of the betratron oscillations on the synchrotron radiation emission will average out, **as long as** the fields vary linearly with displacement from the design orbit.

What if a particle does not have the reference energy / momentum?

That will affect the amount of synchrotron radiation emitted, and that's what we'll consider next.

Effect of radiation on beam

So far all our analysis of beams has assumed motion in dipoles, quadrupoles and sextupoles without any effects of radiation.

We now know a charged particle moving in an electromagnetic field (e.g. a dipole magnet or an rf cavity), will radiate.

What does this mean for the particle motion?

The radiation can produce damping effects in all planes, and lead to concepts such as equilibrium emittance, growing energy spread and beam polarization.

Radiation damping for longitudinal motion (rough calculation)

Synchrotron radiation gives a natural mechanism to damp the synchrotron oscillations of a particle:

- Let E_s be the synchronous particle energy.
- Imagine some particle has more energy by ΔE $\boxed{\Delta E \ll E_s}$
- Then this particle will radiate more, moving its energy close to E_s.

The radiation power is

$$P_\gamma(\Delta E) = c_\gamma \frac{ce^2c^2}{2\pi\beta^2}(E_s + \Delta E)^2 B^2$$

Hence we can show that for a period of orbit T_0

$$\begin{aligned}\text{change of } \Delta E \text{ per turn} &= -[P_\gamma(\Delta E) - P_\gamma(0)]T_0 \\ &\approx -2\frac{P_\gamma(0)\Delta E}{E_s}T_0 = -2\frac{U_0}{E_s}\Delta E\end{aligned}$$

Showing that the rate of energy loss varies linearly with ΔE and hence exponential damping will occur.

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Transverse damping

So the synchrotron radiation provides a natural damping mechanism to the energy of a particle scaling as U_0/E_s.

There is also a natural damping in the transverse plane too. The radiation takes energy from all three spatial directions, so p_x and p_y reduce. Momentum is regained longitudinally in the RF cavity.

This means the ratios p_x/p_z and p_y/p_z drop, and so the slopes *x' and y'* reduce.

Hence radiation damps the beam motion in all three planes.

The full calculation of how fast the emittance drops is too long for this course, but let's look at the general procedure and some of the results.

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Result of a more detailed calculation vertically

To get the correct answer for how the emittance reduces with time due to damping, we need to be a little more careful in our calculation. Let's look at the answer for all three planes separately.

Vertically, it turns out the rough estimate we derived for longitudinal damping is quite accurate, and the vertical emittance damps according to

$$\epsilon_y(t) = \epsilon_y(0)\exp\left(-2\frac{t}{\tau_y}\right)$$

With the damping time given by

$$\tau_y = 2\frac{E_s}{U_0}T_0$$

This is valid for v≈c, such that E=pc.

Result of a more detailed calculation longitudinally

We have to do a bit more work to get the correct effects for longitudinal damping but the results are given here.

We introduce the longitudinal emittance:

$$\epsilon_z^2 = < z^2 >< \delta^2 > - < z\delta >^2$$

Then

$$\epsilon_z(t) = \epsilon_z(0) \exp\left(-2\frac{t}{\tau_z}\right)$$

where we define the longitudinal damping time

$$\tau_z = \frac{2}{j_z}\frac{E_s}{U_0}T_0$$

and introduce j_z which is an example of a 'damping partition number'.

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Damping partition numbers and radiation integrals

The full analysis to calculate j_z starts by considering the radiated energy per turn

$$U_{rad} = \oint P_\gamma dt = \oint P_\gamma \frac{dt}{ds} ds = \frac{1}{c} \oint P_\gamma \left(1 + \frac{x}{\rho}\right) ds = \frac{1}{c} \oint P_\gamma \left(1 + \frac{D}{\rho}\frac{\Delta E}{E_0}\right) ds$$

Recall this?

Longitudinal motion

We can eliminate the phase differential to obtain

$$\Delta \ddot{E} + 2a_s \Delta \dot{E} + \Omega^2 \Delta E = 0$$

Here we have defined a damping term as

$$a_s = \frac{1}{2T_0}\frac{dW}{dE}$$

$$\boxed{\Delta E(t) = \Delta E_0 \exp^{-a_s t} \exp^{i\Omega t}}$$

We've identified our energy loss W with radiation emission U_{rad}. So to find the damping time we need to calculate

$$\frac{\mathrm{d}U_{\mathrm{rad}}}{\mathrm{d}E}$$

Damping partition numbers and radiation integrals

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Now we need to calculate

$$\frac{\mathrm{d}U_{\mathrm{rad}}}{\mathrm{d}E}$$

We won't carry out all of the steps here, but using $P_\gamma \sim E^2 B^2$

$$x = D\frac{\Delta E}{E_0}$$

we can calculate

$$\frac{dB}{dE}$$

$$\frac{\mathrm{d}P_\gamma}{\mathrm{d}E} = \frac{2P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{D}{E_0}\frac{\mathrm{d}B}{\mathrm{d}x}$$

evaluated at $E=E_0$

which helps us see where some of the terms come from in the answer (on the next slide).

Damping partition numbers and radiation integrals

For the longitudinal case the full analysis finds

$$\tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

$$\frac{\mathrm{d}P_\gamma}{\mathrm{d}E} = \frac{2P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{D}{E_0}\frac{\mathrm{d}B}{\mathrm{d}x}$$

$$j_z = 2 + \frac{I_4}{I_2}$$

The discussion of damping partition numbers is beyond this course, but there is one for each plane. It is a function of the lattice (not the beam).

This result is phrased in terms of “radiation integrals”, where we need numbers 2 and 4

$$I_2 = \oint \frac{\mathrm{d}s}{\rho^2} \qquad I_4 = \oint \frac{D_x}{\rho}\left(\frac{1}{\rho^2} + 2k\right)\mathrm{d}s$$

Each is an integral over the lattice.

Damping partition numbers and radiation integrals

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K_x

Each is an integral over the lattice.

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Radiation damping for transverse

The horizontal emittance is a very involved calculation due to dispersion, coupling and other factors.
The complete calculation shows the horizontal emittance also decays exponentially,

$$\frac{d\epsilon_x}{dt} = -\frac{2}{\tau_x}\epsilon_x$$

Where we define the damping constant,

$$\tau_x = \frac{2}{j_x}\frac{E_s}{U_0}T_0$$

With the damping partition number

$$j_x = 1 - \frac{I_4}{I_2}$$

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Quantum excitation

So, we have damping times in all three planes of order E_S/U_0.

According to this picture, a beam of particles in a storage ring would damp down to zero size in a few thousand turns. Which cannot be true!

Lots of things happen to stop this (e,g intra-beam scattering, beam-beam effects) but often the first physics to limit the shrinking beam is the synchrotron radiation itself.

The discrete quantum nature of photon emission acts as a noise source on the beam through little kicks causing the energy oscillation to grow, providing a growth on the emittance.

The result is that some kind of equlibrium, where beam sizes (length, height, energy spread, etc) all tend to some non-zero values.

See later courses for more details!