Single Particle Dynamics, Lecture 7

Including Longitudinal Dynamics

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What We Learned in the Previous Lecture

In the previous lecture, we saw how to describe the transverse dynamics in a simple straight beamline, consisting of drift spaces and quadrupole magnets, using the Twiss parameters.

The Twiss parameters give the shape of an ellipse that is mapped out in phase space by plotting the transverse coordinate and momentum of a particle at a given point in a cell in a periodic beamline.

The Twiss parameters are functions of position, and vary through the periodic cell, but have the same periodicity as the beamline. It follows from Liouville's theorem that the area of the phase space ellipse is an invariant along the beamline; the area divided by 2π is called the *action* of the particle, and is a measure of the amplitude of the transverse oscillations.

The action J_x of a particle can be used as a dynamical variable: it is the conjugate momentum to a coordinate variable ϕ_x called the *angle*. The action-angle variables (ϕ_x, J_x) are related to the cartesian variables (x, p_x) by:

$$x = \sqrt{2\beta_x J_x} \cos\phi_x \tag{1}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}} (\sin\phi_x + \alpha_x \cos\phi_x) \tag{2}$$

where β_x and α_x are Twiss parameters.

The linear dynamics of a particle are particularly easy to describe in terms of action-angle variables: the action is constant, and the angle increases as:

$$\frac{d\phi_x}{ds} = \frac{1}{\beta_x} \tag{3}$$

Part II (Lectures 6 – 10): Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines
7. Including longitudinal dynamics
8. Bunches of many particles
9. Coupled optics
10. Effects of linear imperfections

What We Shall Learn in this Lecture

In the previous lecture, we did not give much attention to the longitudinal motion of a particle. This is because in a straight beamline, without RF cavities, not much happens: the particle simply "drifts", with the longitudinal coordinate increasing or decreasing uniformly, at a rate proportional to the energy deviation of the particle.

In this lecture, we shall consider what happens to the longitudinal dynamics when we include bends and RF cavities in the beamline.

We shall introduce the important concepts of *momentum compaction*, *phase slip*, and *phase stability*. Finally, we shall see that the longitudinal dynamics can be described using Twiss parameters and action-angle variables in exactly the same way as the transverse dynamics.

Recall from the previous lecture the transfer matrix for a FODO cell:

$$R = \begin{pmatrix} 1-\frac{L^2}{2f_0^2} & \frac{L}{f_0}(L+2f_0) & 0 & 0 & 0 & 0 \\ \frac{L}{4f_0^3}(L-2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f_0^2} & -\frac{L}{f_0}(L-2f_0) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f_0^3}(L+2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

If we choose the drift length L and the qudarupole focal length f_0 properly, particles oscillate in the transverse planes as they travel along the beamline. However, there are no longitudinal oscillations.

The linearised longitudinal equations of motion in a straight beamline are:

$$\frac{d\delta}{ds} = 0 \qquad \frac{dz}{ds} = \frac{\delta}{\beta_0^2 \gamma_0^2} \tag{5}$$

Let us consider first how these equations are affected if we introduce bends (dipole magnets) into the beamline...

Recall the transfer matrix for a dipole magnet:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\gamma_0^2\beta_0^2} - \frac{\omega L-\sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

where L is the length of the dipole, and $\omega = k_0$ is the dipole field B_0 normalised by the reference momentum:

$$k_0 = \frac{q}{P_0} B_0 \tag{7}$$

Note the non-zero R_{16} and R_{26} terms in this transfer matrix: these terms give the change in the horizontal coordinate and momentum with respect to changes in the energy deviation. They describe the “dispersion” introduced by the dipole. We can generalise the idea of dispersion to a *dispersion function*.

Consider a periodic beamline consisting of drifts, normal quadrupoles, and dipoles bending in the horizontal plane. In general, the transfer matrix for one periodic cell takes the form:

$$R = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & \bullet \\ \bullet & \bullet & 0 & 0 & 0 & \bullet \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

where $\bullet$ represents some non-zero value. The vertical motion is decoupled from the horizontal and the longitudinal motion, but the horizontal motion and the longitudinal motion are coupled to each other. However, the horizontal motion has no dependence on the longitudinal coordinate z: this is because, in the absence of RF cavities, the fields have no time dependence.

Since the horizontal motion is completely decoupled from the vertical motion and from the longitudinal coordinate z, the horizontal motion can be completely described in terms of a 3×3 matrix as follows:

$$\begin{pmatrix} x \\ p_x \\ \delta \end{pmatrix}_{s=s_0+C_0} = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ p_x \\ \delta \end{pmatrix}_{s=s_0} \qquad (9)$$

where C_0 is the length of a cell, measured along the reference trajectory.

Consider a particle moving through the lattice with some energy deviation δ. There exists a trajectory that the particle can follow, that has the same periodicity as the lattice. We can show the existence of this trajectory by actually calculating what it is.

Let us write the periodic trajectory $(\tilde{x}, \tilde{p}_x)$. The periodicity condition can be written:

$$\begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \delta \end{pmatrix}_{s=s_0+C_0} = \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \delta \end{pmatrix}_{s=s_0} \qquad (10)$$

We then find from equations (9) and (10) that the periodic trajectory is given by:

$$\begin{pmatrix} \tilde{x} \\ \tilde{p}_x \end{pmatrix} = \begin{pmatrix} 1 - R_{11} & -R_{12} \\ -R_{21} & 1 - R_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \delta \qquad (11)$$

The *dispersion function* η_x, (and its "conjugate", η_{p_x}) is defined as the change in the periodic trajectory with respect to the energy deviation, i.e.

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix} = \frac{\partial}{\partial \delta} \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \end{pmatrix} \tag{12}$$

From equation (11), we can calculate the dispersion in a periodic beamline from the transfer matrix R for one periodic cell:

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix} = \begin{pmatrix} 1 - R_{11} & -R_{12} \\ -R_{21} & 1 - R_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \tag{13}$$

If the matrix inversion appearing in equation (13) exists, then the dispersion function also exists.

Strictly speaking, equation (13) is only valid if there are no RF cavities in the ring, and the particle energy is constant.

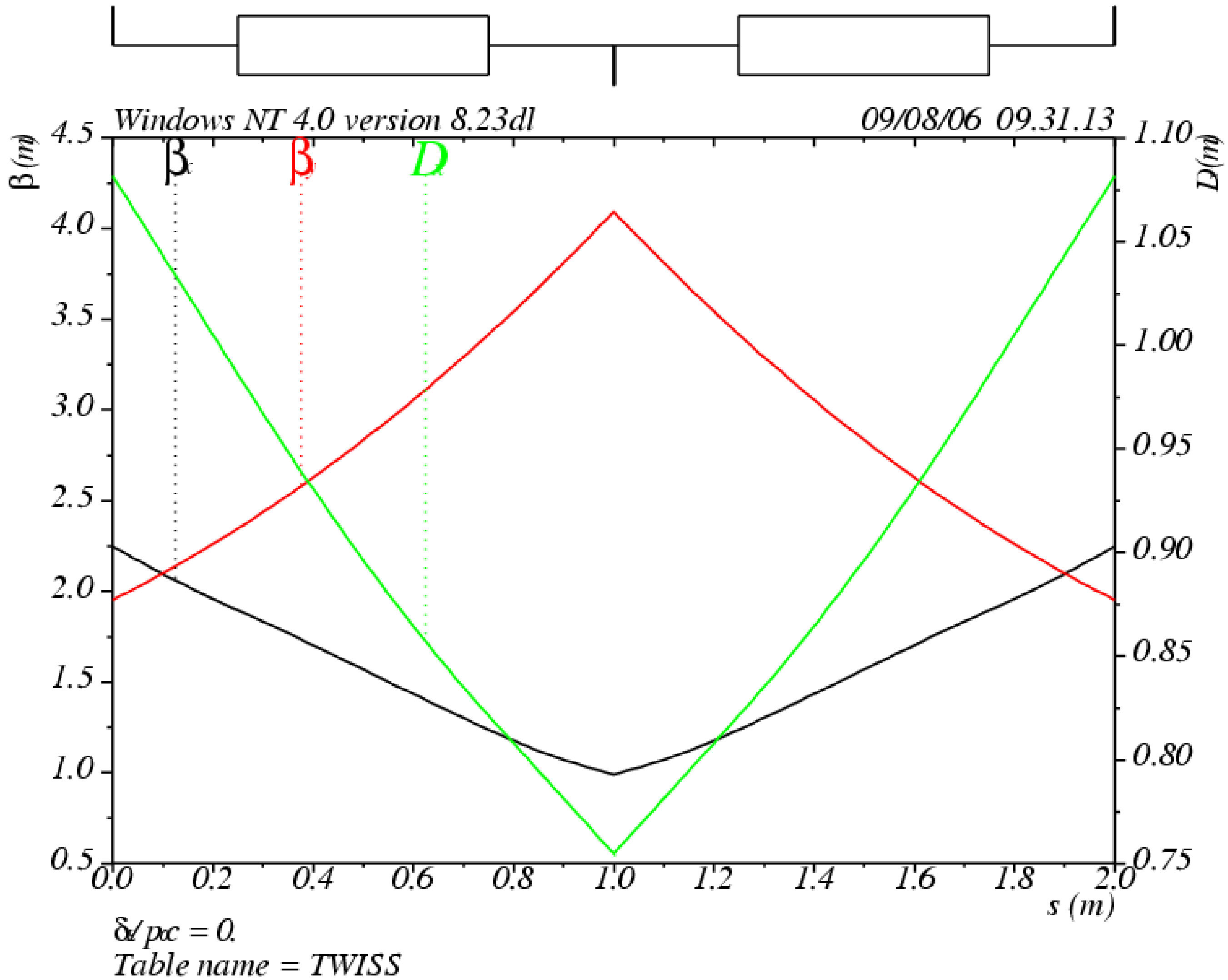
Since we defined the dispersion as the trajectory followed by a particle with some non-zero energy deviation, we can evolve the dispersion function simply using the transfer matrices. In drift spaces and normal quadrupoles, the dispersion transforms the same way as the horizontal trajectory. In bending magnets, there are additional ("zeroth-order") terms to account for the dispersive effects of dipoles.

In a beamline consisting of drift spaces, normal quadrupoles, and bending magnets in the horizontal plane, we can write:

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix}_{s_1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix}_{s_0} + \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \tag{14}$$

where R is the transfer matrix from point s_0 in the beamline to point s_1.

Dispersion Function in a FODO Lattice



Black line = β_x; red line = β_y; green line = η_x.

Now consider just the longitudinal part of the transfer matrix for a single dipole. If the initial horizontal and vertical coordinates and momenta are zero, then the linear map is:

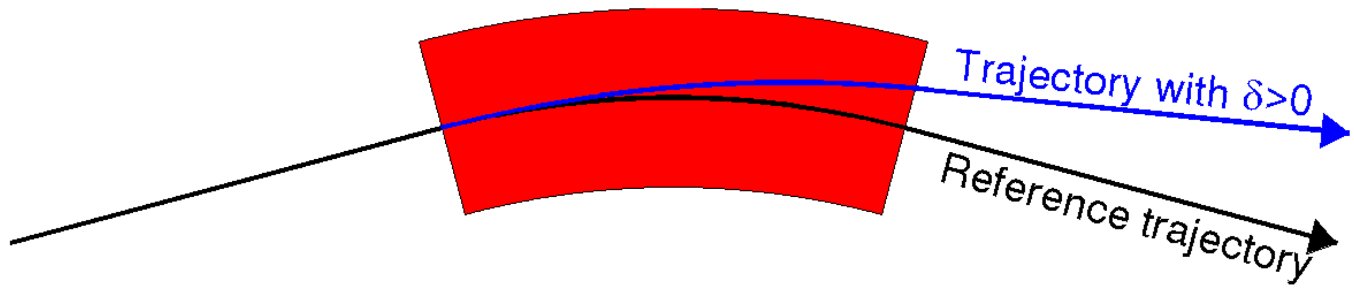
$$\Delta\delta = 0 \tag{15}$$

$$\Delta z = \left(\frac{L}{\beta_0^2 \gamma_0^2} - \frac{\omega L - \sin \omega_L}{\omega \beta_0^2} \right) \delta \tag{16}$$

Note that if

$$\frac{L}{\beta_0^2 \gamma_0^2} < \frac{\omega L - \sin \omega_L}{\omega \beta_0^2} \tag{17}$$

then a particle with energy *higher* than the reference energy *slips back* with respect to the reference particle; i.e. higher energy particles effectively travel more slowly. This is a consequence of the *dispersion*, the fact that higher energy particles take a longer path through the dipole than lower energy particles.



A particle with positive energy deviation ($\delta > 0$) follows a *longer* trajectory in the dipole than a particle on the reference trajectory. If the particles enter with the same z, and the energy is sufficiently large (both particles travel close to the speed of light), the higher energy particle falls behind the particle with the reference energy.

Let us extend the effects of dispersion on the path length to a complete storage ring. We don't need to know the details of the design of the storage ring; we shall carry out a general analysis. We assume that there are no RF cavities in the ring, so the energy deviation δ is constant.

Let C be the path length for one complete turn for a particle that has zero horizontal or vertical action. In other words, the particle performs no betatron oscillations, and after one complete turn returns to its starting conditions in transverse phase space. In general, because of the presence of dipoles, the path length C will be a function of the energy deviation δ. We write this as a series:

$$C = C_0 \left(1 + \alpha_p \delta + \frac{1}{2} \alpha_p^{(2)} \delta^2 + \cdots \right) \tag{18}$$

The path length in terms of the energy deviation is written as a series (18):

$$C = C_0 \left(1 + \alpha_p \delta + \frac{1}{2} \alpha_p^{(2)} \delta^2 + \cdots \right) \tag{19}$$

C_0 is the path length of a particle with zero energy deviation: this is sometimes referred to as the "circumference" of the storage ring. The coefficient α_p is called the *momentum compaction factor*. From equation (19) we can write:

$$\alpha_p = \left(\frac{1}{C} \frac{dC}{d\delta} \right)_{\delta=0} \tag{20}$$

Notice that there are "higher-order momentum compaction factors", $\alpha_p^{(2)}$ etc. Since this is a course on *linear* dynamics, we shall consider only the first-order momentum compaction factor α_p.

The momentum compaction is a property of the *lattice*. Given a design for a beamline involving drifts, dipoles and quadrupoles, one can calculate the momentum compaction. Let us find a general expression for it.

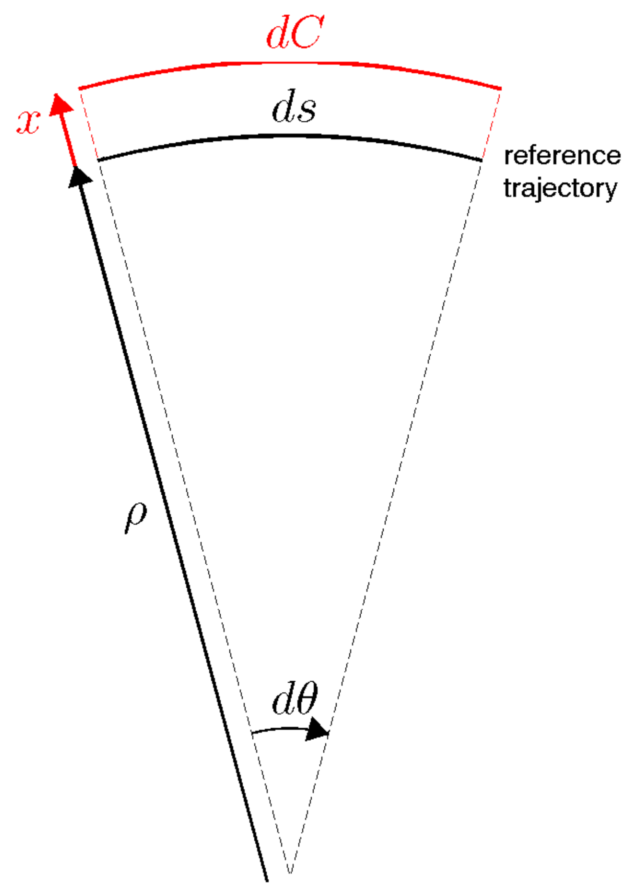
Consider a curved reference trajectory, with radius of curvature ρ, and consider a particle moving with displacement x in the plane of the reference trajectory. The infinitesimal path length dC of the particle for a path length $ds = \rho\, d\theta$ along the reference trajectory is:

$$dC = (\rho + x)d\theta = (\rho + x)\frac{ds}{\rho} \tag{21}$$

The total path length C is obtained by integrating over all infinitesimal path lengths:

$$C = \int_0^{C_0} ds + \int_0^{C_0} \frac{x}{\rho} ds = C_0 + \int_0^{C_0} \frac{x}{\rho} ds \tag{22}$$

Calculating the Momentum Compaction in a Lattice



$$dC = (\rho + x)d\theta = (\rho + x)\frac{ds}{\rho}$$

Now, consider the case that the offset x arises purely from *dispersion*, i.e. the transverse action is zero and x is a function only of the energy deviation δ. In this case, we can write:

$$x = \eta_x \delta + \frac{1}{2}\eta_x^{(2)}\delta^2 + \cdots \tag{23}$$

where η_x is the dispersion function that we saw how to calculate earlier from the transfer matrix for a single cell. (Note that there are also "higher-order dispersion functions", $\eta_x^{(2)}$ etc. Since this is a course in linear dynamics, we shall consider only the linear dispersion, η_x.)

From equations (22) and (23) it follows that:

$$\alpha_p = \left(\frac{1}{C}\frac{dC}{d\delta}\right)_{\delta=0} = \frac{1}{C_0}\int_0^{C_0} \frac{\eta_x}{\rho} ds \tag{24}$$

Equation (25) for the momentum compaction is:

$$\alpha_p = \left(\frac{1}{C} \frac{dC}{d\delta} \right)_{\delta=0} = \frac{1}{C_0} \int_0^{C_0} \frac{\eta_x}{\rho} ds \tag{25}$$

In other words, the proportional change in path length (over one complete turn through the storage ring) with respect to energy deviation is equal to the integral around the storage ring of the dispersion divided by the local radius of curvature of the reference trajectory, divided by the nominal path length.

Note that straight sections, where the reference trajectory is not curved, effectively have a radius of curvature that is infinite, $\rho \to \infty$. Thus, it is only regions where the reference trajectory is curved (generally, dipoles) that contribute to the momentum compaction.

We have seen that the momentum compaction factor α_p describes the proportional change in path length of a particle moving round a storage ring with respect to energy deviation, and the α_p is a function only of the lattice design. The *revolution period* T depends on the path length and the speed of the particle, so involves the reference energy as well as the lattice design.

To describe the proportional change in revolution period, we introduce a quantity called the *phase slip factor* η_p, analagous to the momentum compaction factor:

$$T = T_0 \left(1 + \eta_p \delta + \frac{1}{2} \eta_p^{(2)} \delta^2 + \cdots \right) \tag{26}$$

The phase slip factor η_p should not be confused with the dispersion function η_x: they are different quantities.

Let us find an expression for the phase slip factor for a given lattice and reference energy. From equation (26) we can write:

$$\eta_p = \left(\frac{1}{T} \frac{dT}{d\delta} \right)_{\delta=0} \tag{27}$$

Now we write:

$$T = \frac{C}{\beta c} \tag{28}$$

where βc is the speed of the particle (c is the speed of light). From equation (28) it follows that:

$$\frac{1}{T} \frac{dT}{d\delta} = \frac{1}{C} \frac{dC}{d\delta} - \frac{1}{\beta} \frac{d\beta}{d\delta} \tag{29}$$

Using the definition of the energy deviation:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{30}$$

we find:

$$\beta\gamma = \left(\frac{1}{\beta_0} + \delta \right) \beta_0 \gamma_0 \tag{31}$$

from which we find, after some algebra:

$$\frac{d\beta}{d\delta} = \frac{\beta_0 \gamma_0}{\gamma^3} \tag{32}$$

Using (32), we can write from (29):

$$\left(\frac{1}{T} \frac{dT}{d\delta} \right)_{\delta=0} = \left(\frac{1}{C} \frac{dC}{d\delta} \right)_{\delta=0} - \left(\frac{\beta_0 \gamma_0}{\beta \gamma^3} \right)_{\delta=0} \tag{33}$$

Finally, using (20) and (27), we have:

$$\eta_p = \alpha_p - \frac{1}{\gamma_0^2} \tag{34}$$

The phase slip factor is given by (34):

$$\eta_p = \alpha_p - \frac{1}{\gamma_0^2} \tag{35}$$

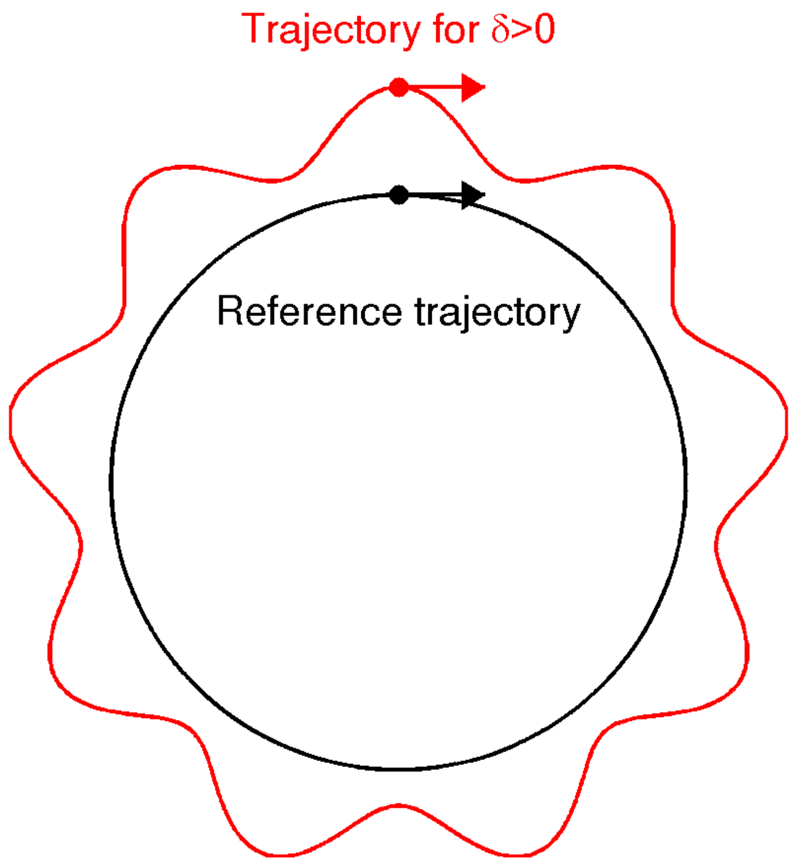
We can identify three distinct regimes:

$\eta_p < 0$	$\alpha_p < 0$ or $\gamma_0^2 < \frac{1}{\alpha_p}$	**Below Transition** revolution frequency *increases* with increasing energy
$\eta_p = 0$	$\gamma_0^2 = \frac{1}{\alpha_p}$	**At Transition** revolution frequency *independent* of energy
$\eta_p > 0$	$\gamma_0^2 > \frac{1}{\alpha_p}$	**Above Transition** revolution frequency *decreases* with increasing energy

Consider a lattice with positive momentum compaction, i.e. $\alpha_p > 0$. In such a lattice, increasing the energy of a particle leads to an increase in path length. However, if the beam is at low energy, such that the particles have speeds significantly less than the speed of light, then increasing the energy of a particle leads to an increase in its speed, which can more than compensate the increase in path length. The result is that the particle makes a larger number of revolutions per unit time.

For ultrarelativistic particles, however, an increase in energy leads to a negligible increase in speed, since the particles are already travelling very close to the speed of light. In this case, the increase in path length dominates over the increase in speed, and the particle makes a smaller number of revolutions per unit time.

At some energy in between these two regimes, the revolution frequency is independent of energy. This energy is known as the *transition energy*.



If $\alpha_p > 0$, increasing the energy of a particle increases the path length (red, "dispersive" trajectory). If the particles are already travelling at speeds close to the speed of light, increasing the energy means that a particle takes a *longer* time to go round the ring.

Most modern electron storage rings operate with ultrarelativistic particles (typically, $E > 500\,\mathrm{MeV}$, or $\gamma_0 > 1000$), and positive momentum compaction factor ($\alpha_p \sim 10^{-3}$ is typical). Such rings are always above transition.

There has been some interest recently in operating electron storage rings with very small, or even negative, momentum compaction factor. Such operating modes, in which the beam is below transition, are generally experimental, and of interest for producing very short bunches, or counteracting certain beam instabilities (collective effects).

Positron rings may inject beam at low energy, where the particle speeds are significantly less than the speed of light. Such a ring could be below transition. If the beam energy is increased, the operational energy can actually cross transition.

The transfer matrix for a "thin" ($L \to 0$) RF cavity can be written:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{f_{\parallel}} & 1 \end{pmatrix} \tag{36}$$

where the "longitudinal focusing strength" is given by:

$$\frac{1}{f_{\parallel}} = \frac{q\hat{V}}{P_0 c} k \cos\phi_0 \tag{37}$$

$\hat{V}$ is the voltage, $k = 2\pi f_{RF}/c$ where f_{RF} is the RF frequency, and ϕ_0 is the RF phase.

In Lecture 4, we saw that there were zeroth-order terms in the map for an RF cavity:

$$m_z = \frac{2}{\pi} L \sin^2\left(\frac{\psi_\parallel}{2}\right) \tan\phi_0 \tag{38}$$

$$m_\delta = \frac{q\hat{V}}{P_0 c} \frac{\sin\psi_\parallel}{\psi_\parallel} \sin\phi_0 \tag{39}$$

In the limit of a "thin" cavity ($L \to 0$), we have:

$$m_z \mapsto 0 \tag{40}$$

$$m_\delta \mapsto \frac{q\hat{V}}{P_0 c} \sin\phi_0 \tag{41}$$

For a thin cavity, the zeroth-order term for δ leads to a non-zero change in δ if $\sin\phi_0 \neq 0$, even when the initial values of z and δ are zero.

Relativistic particles in a storage ring lose energy by the emission of synchrotron radiation in the bending magnets. This energy must be replaced by the RF cavities.

In the simplest model of synchrotron radiation, the effect of the radiation on a particle is the loss of some energy U_0 over one turn. The resulting change in the energy deviation is:

$$\Delta\delta = -\frac{U_0}{P_0 c} \tag{42}$$

Therefore, if the particle crosses the RF cavity at a phase ϕ_0 such that:

$$\frac{q\hat{V}}{P_0 c}\sin\phi_0 = \frac{U_0}{P_0 c} \tag{43}$$

then the overall change in δ over one turn is zero.

The RF phase at which particles gain exactly the right amount of energy to replace the energy lost through synchrotron radiation is called the *synchronous phase*, ϕ_s. In proton storage rings, where synchrotron radiation losses are usually negligible, the synchronous phase is close to zero (or π). But in high-energy electron storage rings, the particles lose significant amounts of energy through synchrotron radiation, and the synchronous phase is therefore significantly different from zero.

Synchrotron radiation also provides a mechanism that "damps" particle oscillations so that they generally cross the RF cavities at a phase close to the synchronous phase. For the rest of this lecture, we shall always assume that the RF phase ϕ_0 is equal to the synchronous phase ϕ_s.

Let's assume that we have designed a storage ring lattice using drifts, normal quadrupoles and bending magnets in the horizontal plane. We can design the lattice so that at some point the dispersion is zero. Matching from some arbitrary dispersion (η_x, η_{p_x}) to zero dispersion ($\eta_x = 0$, $\eta_{p_x} = 0$) may be achieved using a dipole with appropriate parameters (length and bending angle) calculated from equation (14):

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix}_{s_1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix}_{s_0} + \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \qquad (44)$$

At the location with zero dispersion, there is no coupling, either between the transverse planes, or between the transverse and longitudinal planes. For our purposes, this is a good location for an RF cavity, since it simplifies the analysis.

The longitudinal part of the transfer matrix for a single turn through the storage ring, starting at the centre of the RF cavity, may be written:

$$R_{\|} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_{\|}} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\eta_p C_0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_{\|}} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{\eta_p C_0}{2f_{\|}} & -\eta_p C_0 \\ -\frac{\eta_p C_0}{4f_{\|}^2} - \frac{1}{f_{\|}} & 1 + \frac{\eta_p C_0}{2f_{\|}} \end{pmatrix} \qquad (45)$$

where η_p is the phase slip factor, and C_0 is the path length along the reference trajectory.

Note that we can describe the longitudinal single-turn matrix $R_{\|}$ (45) using Twiss parameters and a phase advance in the same way that we did for the transverse parts of the transfer matrix. In practice, however, the longitudinal focusing is usually very weak, so that:

$$\left| \frac{\eta_p C_0}{f_{\|}} \right| \ll 1 \tag{46}$$

In that case, the variation of the longitudinal Twiss parameters around the ring is very small, and it is not usual to plot them or even calculate them. However, for some specialised rings using a very high RF voltage, or with very large phase slip, the variation in Twiss parameters around the ring may be substantial.

The longitudinal part of the transfer matrix for one turn, starting at the centre of the RF cavity is (45):

$$R_{\parallel} = \begin{pmatrix} 1 + \frac{\eta_p C_0}{2 f_{\parallel}} & -\eta_p C_0 \\ -\frac{\eta_p C_0}{4 f_{\parallel}^2} - \frac{1}{f_{\parallel}} & 1 + \frac{\eta_p C_0}{2 f_{\parallel}} \end{pmatrix} \tag{47}$$

This can be written in the usual "Twiss" form:

$$R_{\parallel} = \begin{pmatrix} \cos\mu_z + \alpha_z \sin\mu_z & \beta_z \sin\mu_z \\ -\gamma_z \sin\mu_z & \cos\mu_z - \alpha_z \sin\mu_z \end{pmatrix} \tag{48}$$

From (47) and (48), assuming that the longitudinal phase advance μ_z is small, we can write:

$$\mu_z \approx \sqrt{-\frac{\eta_p C_0}{f_{\parallel}}} = \sqrt{-2\pi \frac{q\hat{V}}{P_0 c} \beta_0 h \eta_p \cos\phi_s} \tag{49}$$

where we have defined the *harmonic number*, h:

$$h = \frac{C_0}{\beta_0 \lambda_{RF}} \tag{50}$$

The *synchrotron tune*, ν_s is defined as the number of complete periods of longitudinal (synchrotron) motion per turn, i.e.:

$$\nu_s = \frac{\mu_z}{2\pi} \approx \sqrt{-\frac{1}{2\pi}\frac{q\hat{V}}{P_0 c}\beta_0 h \eta_p \cos\phi_s} \tag{51}$$

Note that for stable synchrotron oscillations, the right-hand side must be a real number, i.e.:

$$\frac{q\hat{V}}{P_0 c} h \eta_p \cos\phi_s < 0 \tag{52}$$

Assuming a positively charged beam, with positive RF voltage and reference momentum, the condition for stable synchrotron oscillations (52) reduces to:

$$\eta_p \cos\phi_s < 0 \tag{53}$$

If the storage ring is operating above the transition energy, then $\eta_p > 0$ and the longitudinal stability condition implies:

$$\frac{\pi}{2} < \phi_s < \frac{3\pi}{2} \tag{54}$$

In equation (50) we introduced the harmonic number:

$$h = \frac{C_0}{\beta_0 \lambda_{RF}} \tag{55}$$

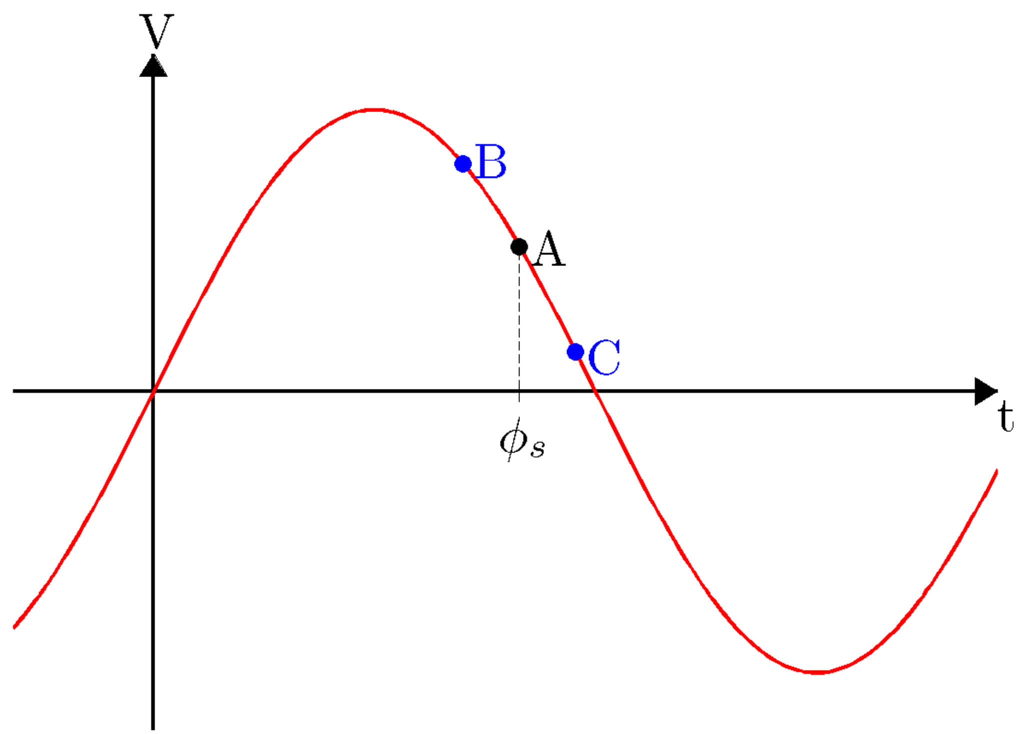
Note that this can be written:

$$h = \frac{C_0}{\beta_0 c} f_{RF} = \frac{T_0}{T_{RF}} \tag{56}$$

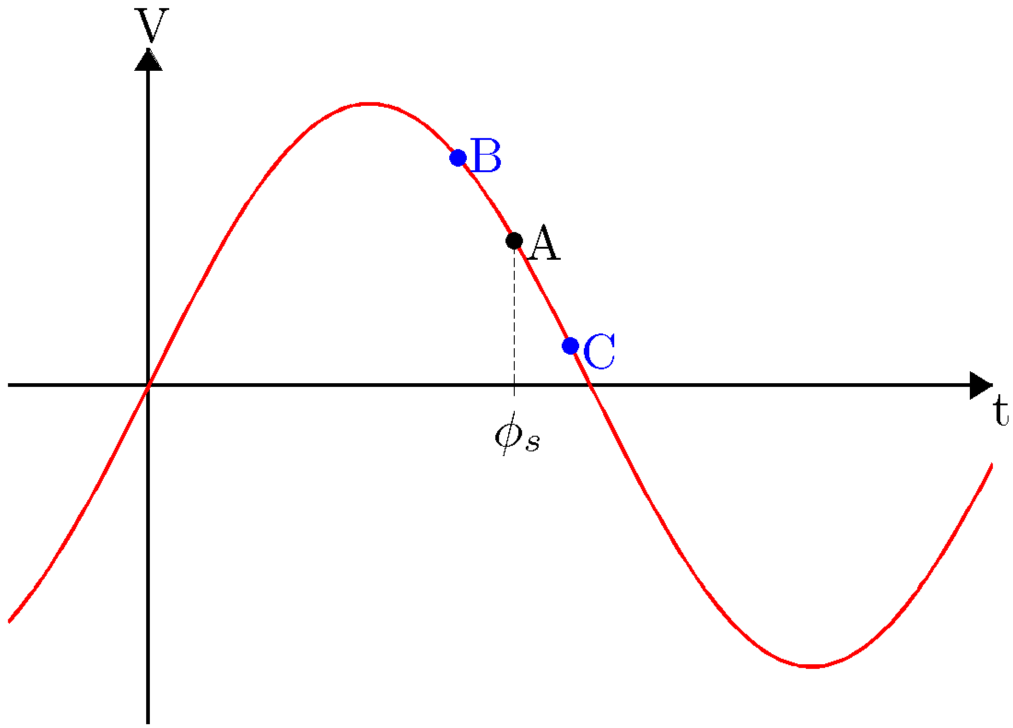
In other words, the harmonic number is the ratio of T_0 (the revolution period for the reference particle) and T_{RF} (the RF period). In a *synchrotron*, the particle motion and the RF oscillations are synchronised, in the sense that particles arrive at a fixed phase of the RF oscillations after each successive turn around the storage ring. The "synchrotron condition" can be expressed as:

$$h = \text{integer} \tag{57}$$

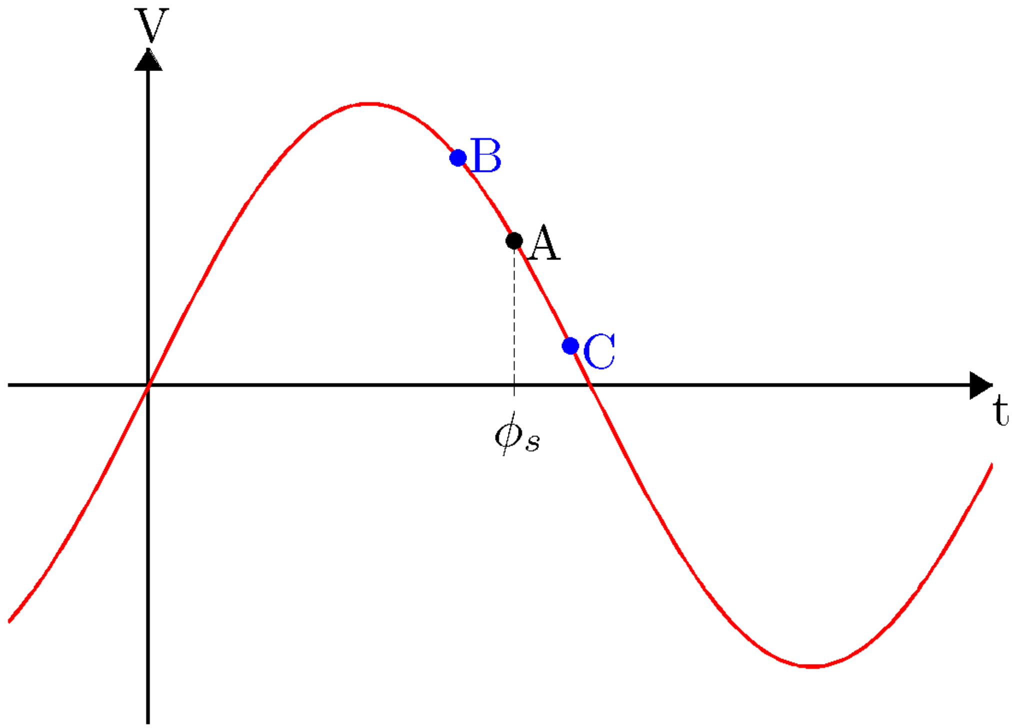
In a synchrotron, there are two ways to change the energy of the beam: one is to change the strengths of all the bending magnets; the other is to change the RF frequency.



Particle A arrives at the RF cavity at the "correct" time to restore the energy lost through synchrotron radiation. The overall change in energy of particle A over one turn of the ring is zero.



Particle B arrives at the RF cavity "early", and receives extra energy in addition to that lost through synchrotron radiation. If the ring is above transition, the extra energy will tend to "slow down" particle B. The effect is analogous to a restoring force, acting in the direction of the synchronous phase ϕ_s.



Particle C arrives at the RF cavity "late", and receives too little energy to make up that lost through synchrotron radiation. If the ring is above transition, the extra energy will tend to "speed up" particle C. The effect is again analogous to a restoring force, acting in the direction of the synchronous phase ϕ_s.

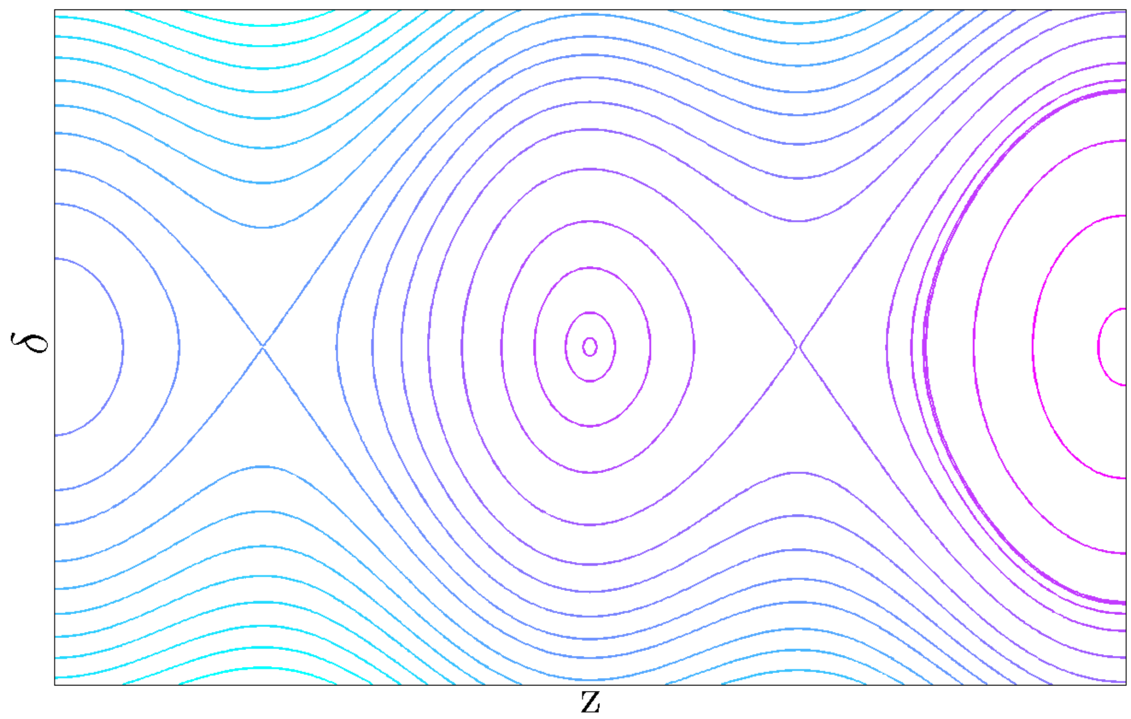
For small synchrotron oscillations, the slope of the RF voltage is approximately linear. But for large oscillations, the sinusoidal shape of the RF voltage as a function of time cannot be neglected. To investigate some of the effects of this RF "curvature", let us assume that the synchrotron oscillations are slow compared to the revolution period, i.e. the synchrotron tune $\nu_s \ll 1$. The longitudinal equations of motion can then be approximated:

$$\frac{dz}{ds} \approx -\eta_p \delta \qquad \frac{d\delta}{ds} \approx \frac{1}{C_0} \frac{q\hat{V}}{P_0 c} [\sin(\phi_s - kz) - \sin\phi_s] \tag{58}$$

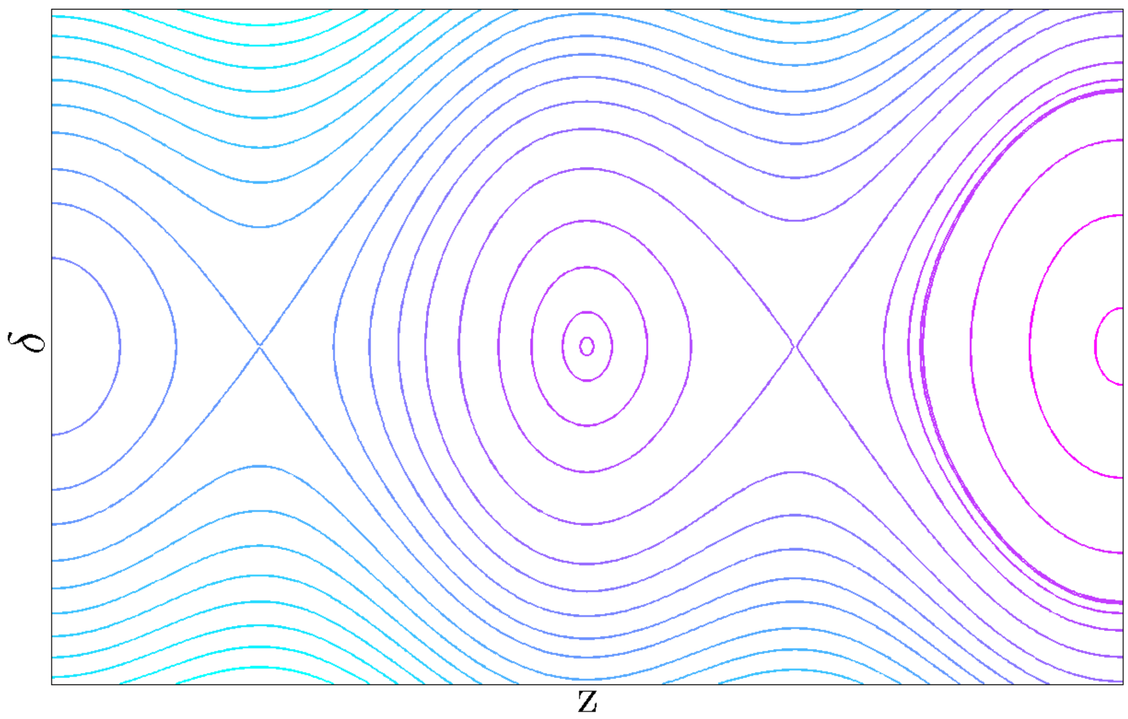
Equations (58) can be derived from the Hamiltonian:

$$H = -\frac{\eta_p}{2}\delta^2 - \frac{1}{kC_0}\frac{q\hat{V}}{P_0 c}[\cos(\phi_s - kz) - kz\sin\phi_s] \tag{59}$$

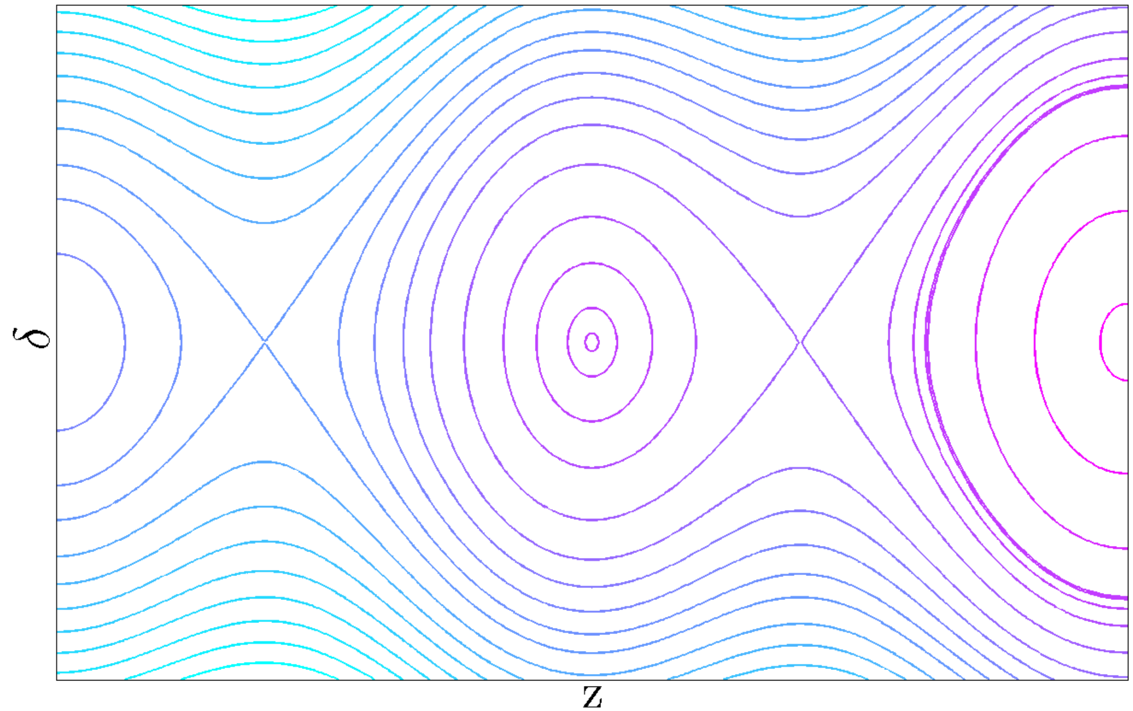
Since the evolution of the dynamical variables in phase space follow a contour of fixed value for H, we can draw a "phase space portrait"...



Particles follow contours in phase space of fixed value for the Hamiltonian H. Note that there are bounded regions where the contours form closed loops: particles in these regions perform stable oscillations. The line forming the boundary of each stable region is called a *separatrix*.



A region within a separatrix, where the particles perform stable oscillations, is known as an *RF bucket*.



There is a maximum value for the energy deviation δ for any particle within an RF bucket: this value is known as the *RF bucket height*, δ_{max}. It can be shown that:

$$\delta_{\text{max}} = \frac{2\nu_s}{\beta_0 h \eta_p}\sqrt{1 + \left(\phi_s - \frac{\pi}{2}\right)\tan\phi_s} \qquad (60)$$

The *dispersion* describes the change in trajectory of a particle with energy. Dipoles introduce dispersion into a beamline. The combination of dispersion with a curved reference trajectory leads to a change in path length with energy: this is quantified by the *momentum compaction factor*. The *phase slip factor* quantifies the change in revolution period with energy in a storage ring.

Because of relativistic limits on velocity, increasing the energy of a particle can increase the path length without a significant increase in speed; the result can be an increase in revolution period. The energy at which the change in path length exactly balances the increase in speed is called the *transition* energy: at this energy, the phase slip factor is zero.

Summary II

In a high-energy electron storage ring, RF cavities are needed to replace the energy lost by synchrotron radiation. Particles must cross the RF cavities at the correct RF phase (the *synchronous phase*) to restore the exact amount of energy lost through synchrotron raditaion. The gradient of RF voltage with time has the effect of a "longitudinal focusing" force, and results in synchrotron oscillations.

Phase stability results in particles making synchrotron oscillations around the synchronous phase. The number of synchrotron oscillations performed in each turn around the ring is the *synchrotron tune*.

The "curvature" of the RF voltage resulting from the sinusoidal waveform leads to a distortion of longitudinal phase space, and a limit on the maximum energy deviation, beyond which synchrotron oscillations are no longer stable.