Cockcroft Lectures: Intermediate Beam Dynamics for Particle Accelerators

Problem Set 9

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1. In one degree of freedom, the beam distribution matrix (the Σ matrix) is a 2×2 symmetric matrix, which is parameterised in terms of the Twiss parameters α_x, β_x, γ_x, and the emittance ϵ_x:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \tag{1}$$

   The matrix Σ has three independent components. Using the parameterisation in terms of the Twiss parameters and the emittance, there are apparently $3 + 1 = 4$ variables. However, the condition:

$$\beta_x \gamma_x - \alpha_x^2 = 1 \tag{2}$$

   imposes one constraint, allowing us to eliminate one of the Twiss parameters. The number of independent variables in the Twiss/emittance parameterisation is therefore equal to the number of independent components in the beam distribution matrix, Σ.

   (a) In three degrees of freedom, how many independent components are there of the beam distribution matrix, Σ?

   (b) In three degrees of freedom, the "generalised Twiss parameters" are the components of the 6×6 symmetric matrices B^k. How many generalised Twiss parameters are there in three degrees of freedom? How many emittances are there?

   (c) How many relationships must there be between the generalised Twiss parameters in three degrees of freedom (analogous to the relationship $\beta_x \gamma_x - \alpha_x^2 = 1$ in one degree of freedom)?

2. Assume that the horizontal coordinate of a particle in a bunch is given by:

$$x = \sqrt{2\beta_x J_x} \cos \phi_x + \eta_x \delta \tag{3}$$

where the angle variables ϕ_x are uncorrelated with any other variables, and η_x is the dispersion. Assume also that the action variables J_x are uncorrelated with the energy deviations δ.

(a) Using equation (3), find a relationship between the moments $\langle x\delta \rangle$ and $\langle \delta^2 \rangle$.

(b) Write the moments $\langle x\delta \rangle$ and $\langle \delta^2 \rangle$ in terms of generalised Twiss parameters β_{ij}^k, and the emittances ϵ_k.

(c) Write down an expression for the dispersion η_x in terms of generalised Twiss parameters.