Hamiltonian Dynamics, Lecture 9

Coupled Optics

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Review of Uncoupled Optics

In the previous lecture, we saw how the size of a bunch of many particles could be described by a *sigma matrix*, constructed from the second order moments of the phase space variables of particles in the bunch.

In the case that the sigma matrix is block diagonal, we say that the bunch was "uncoupled": there are no correlations between the variables in different degrees of freedom.

For an uncoupled bunch, we saw how to derive the bunch emittances, which were invariant quantities for a bunch undergoing linear symplectic transport. The variation in the size of the bunch along a beamline is found by combining the emittances with the Twiss parameters.

Part II (Lectures 6 – 10): Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines

7. Including longitudinal dynamics

8. Bunches of many particles

9. Coupled optics

10. Effects of linear imperfections

Many beamlines are designed to transport uncoupled bunches; but in reality, errors that introduce coupling cannot be avoided. It is therefore of great practical importance to have a means for describing the optics of coupled bunches.

There are many methods that have been derived for describing coupled optics. Sometimes, people feel quite strongly about the relative merits or failings of different methods.

In this lecture, I shall present one method that describes the coupled optics in a way that I find simple and intuitive. Essentially, we rederive the results that we have seen already in the uncoupled case for the lattice functions, phase advances, emittances etc. using a general 6×6 matrix formalism, with no assumptions about the absence or presence of coupling. We start by deriving some useful properties of symplectic matrices...

A *symplectic matrix* M is one that satisfies:

$$M^{\mathsf{T}} \cdot S \cdot M = S \tag{1}$$

where S is a block-diagonal matrix, composed of the 2×2 sub-matrices:

$$S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

If M is a symplectic matrix, then:

- The eigenvalues of M occur in reciprocal pairs, $\lambda_{\pm k}$, such that:

$$\lambda_{-k}\lambda_k = 1 \tag{3}$$

- The eigenvectors can be normalised and arranged to form a matrix E, such that:

$$E^{\mathsf{T}} \cdot S \cdot E = iS \tag{4}$$

If $\vec{e}_i$ is an eigenvector of the matrix M with corresponding eigenvalue λ_i then:

$$M \cdot \vec{e}_i = \lambda_i \vec{e}_i \tag{5}$$

It follows that:

$$\vec{e}_i^{\mathsf{T}} \cdot M^{\mathsf{T}} \cdot S \cdot M \cdot \vec{e}_j = \lambda_i \lambda_j \vec{e}_i^{\mathsf{T}} \cdot S \cdot \vec{e}_j \tag{6}$$

But if M is symplectic, then

$$M^{\mathsf{T}} \cdot S \cdot M = S \tag{7}$$

From equations (6) and (7) it follows that:

$$\vec{e}_i^{\mathsf{T}} \cdot S \cdot \vec{e}_j = \lambda_i \lambda_j \vec{e}_i^{\mathsf{T}} \cdot S \cdot \vec{e}_j \tag{8}$$

Hence, we must have either:

$$\lambda_i \lambda_j = 1 \tag{9}$$

or:

$$\vec{e}_i^{\mathsf{T}} \cdot S \cdot \vec{e}_j = 0 \tag{10}$$

From equation (9) we can sort the eigenvalues into pairs such that:

$$\lambda_{-k}\lambda_k = 1, \qquad k = \mathrm{I}, \mathrm{II}, \mathrm{III} \tag{11}$$

In general, we have from equation (10) for the eigenvectors (with appropriate normalisation):

$$\vec{e}_j^{\,\mathsf{T}} \cdot S \cdot \vec{e}_k = \begin{cases} 0 & \text{if } j \neq -k \\ \pm i & \text{if } j = -k \end{cases} \tag{12}$$

for $j, k = \mathrm{I}, \mathrm{II}, \mathrm{III}$. If we then arrange the eigenvectors into a matrix E in an appropriate order, equation (4) then follows that:

$$E^{\mathsf{T}} \cdot S \cdot E = iS \tag{13}$$

We define the matrix Q that is block diagonal, with block-diagonal components Q_2 given by:

$$Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{14}$$

Using Q, we construct the *normalising matrix*, N:

$$N = E \cdot Q \tag{15}$$

where E is the matrix of eigenvectors of R. N has the property:

$$N^{-1} \cdot R \cdot N = \tilde{R}(\mu_{\mathrm{I}}, \mu_{\mathrm{II}}, \mu_{\mathrm{III}}) \tag{16}$$

where R is the transfer matrix for one periodic cell of a beamline, and $\tilde{R}$ is a (block-diagonal) rotation matrix with rotation angles μ_k, where the eigenvalues of R are $\exp(\pm i\mu_k)$.

Since E is an matrix constructed from the eigenvectors of R with appropriate ordering, it follows that E diagonalises R, with the eigenvalues forming the components on the diagonal:

$$E^{-1} \cdot R \cdot E = \begin{pmatrix} e^{-i\mu_{\mathrm{I}}} & & & & & \\ & e^{i\mu_{\mathrm{I}}} & & & & \\ & & e^{-i\mu_{\mathrm{II}}} & & & \\ & & & e^{i\mu_{\mathrm{II}}} & & \\ & & & & e^{-i\mu_{\mathrm{III}}} & \\ & & & & & e^{i\mu_{\mathrm{III}}} \end{pmatrix} \tag{17}$$

With the definition of Q given by equation (14) it follows at once that:

$$Q^{-1} \cdot E^{-1} \cdot R \cdot E \cdot Q = \tilde{R}(\mu_{\mathrm{I}}, \mu_{\mathrm{II}}, \mu_{\mathrm{III}}) \tag{18}$$

where

$$\tilde{R}(\mu_{\mathrm{I}}, \mu_{\mathrm{II}}, \mu_{\mathrm{III}}) = \begin{pmatrix} \tilde{R}_2(\mu_{\mathrm{I}}) & & \\ & \tilde{R}_2(\mu_{\mathrm{II}}) & \\ & & \tilde{R}_2(\mu_{\mathrm{III}}) \end{pmatrix} \tag{19}$$

and $\tilde{R}_2(\mu_k)$ are 2×2 rotation matrices:

$$\tilde{R}_2(\mu_k) = \begin{pmatrix} \cos \mu_k & \sin \mu_k \\ -\sin \mu_k & \cos \mu_k \end{pmatrix} \tag{20}$$

Since $N = E \cdot Q$ (15), we have finally (16):

$$N^{-1} \cdot R \cdot N = \tilde{R}(\mu_{\mathrm{I}}, \mu_{\mathrm{II}}, \mu_{\mathrm{III}}) \tag{21}$$

The normalizing matrix is symplectic, i.e.:

$$N^{\mathsf{T}} \cdot S \cdot N = S \tag{22}$$

The proof of equation (22) is straightforward. Since $N = E \cdot Q$, we can write:

$$N^{\mathsf{T}} \cdot S \cdot N = Q^{\mathsf{T}} \cdot E^{\mathsf{T}} \cdot S \cdot E \cdot Q \tag{23}$$

Then from equation (4):

$$E^{\mathsf{T}} \cdot S \cdot E = iS \tag{24}$$

we have:

$$N^{\mathsf{T}} \cdot S \cdot N = iQ^{\mathsf{T}} \cdot S \cdot Q \tag{25}$$

Equation (22) then follows directly from the definition (14) of Q. Note that since N is symplectic, its inverse N^{-1} is also symplectic.

We use the normalising matrix N to define a vector $\vec{J}$ of new dynamical variables:

$$\vec{J} = N^{-1} \cdot \vec{x} \tag{26}$$

where

$$\vec{J} = \begin{pmatrix} X_{\mathrm{I}} \\ P_{\mathrm{I}} \\ X_{\mathrm{II}} \\ P_{\mathrm{II}} \\ X_{\mathrm{III}} \\ P_{\mathrm{III}} \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \tag{27}$$

Since the variables (x, p_x) etc. form canonically conjugate pairs, and the new variables (X_k, P_k) are derived from them by a symplectic transformation (N^{-1}), it follows that the new variables (X_k, P_k) are also canonically conjugate (see Goldstein, section 9-3).

Using the mixed-variable generating function:

$$F_1(X_k, \phi_k) = -\frac{1}{2} \sum_{k=\mathrm{I,II,III}} X_k^2 \tan \phi_k \tag{28}$$

and the usual relations:

$$X_k = \frac{\partial F_1}{\partial P_k}, \qquad J_k = -\frac{\partial F_1}{\partial \phi_k} \tag{29}$$

we derive new canonical (action-angle) variables (ϕ_k, J_k):

$$J_k = \frac{1}{2}(X_k^2 + P_k^2) \qquad \tan \phi_k = -\frac{P_k}{X_k} \tag{30}$$

in terms of which:

$$\vec{J} = \begin{pmatrix} \sqrt{2J_{\mathrm{I}}} \cos \phi_{\mathrm{I}} \\ -\sqrt{2J_{\mathrm{I}}} \sin \phi_{\mathrm{I}} \\ \sqrt{2J_{\mathrm{II}}} \cos \phi_{\mathrm{II}} \\ -\sqrt{2J_{\mathrm{II}}} \sin \phi_{\mathrm{II}} \\ \sqrt{2J_{\mathrm{III}}} \cos \phi_{\mathrm{III}} \\ -\sqrt{2J_{\mathrm{III}}} \sin \phi_{\mathrm{III}} \end{pmatrix} \tag{31}$$

Under the transfer matrix R for a single periodic cell of the beamline, the action-angle variables transform as:

$$J_k \to J_k, \qquad \phi_k \to \phi_k + \mu_k \tag{32}$$

The transformations (32) may be shown as follows. Consider a transformation $R = R(s_0, s_1)$ across one complete periodic cell from point s_0 along the reference trajectory to point s_1:

$$\vec{x}(s_1) = R \cdot \vec{x}(s_0) \tag{33}$$

Then:

$$\begin{aligned} \vec{J}(s_1) &= N^{-1} \cdot \vec{x}(s_1) && (34)\\ &= N^{-1} \cdot R \cdot \vec{x}(s_0) && (35)\\ &= N^{-1} \cdot R \cdot N \cdot N^{-1}\vec{x}(s_0) && (36)\\ &= N^{-1} \cdot R \cdot N \cdot \vec{J}(s_0) && (37) \end{aligned}$$

Using equation (16) in (37), we have:

$$\vec{J}(s_1) = \tilde{R}(\mu_{\rm I}, \mu_{\rm II}, \mu_{\rm III}) \cdot \vec{J}(s_0) \tag{38}$$

Since $\tilde{R}(\mu_{\rm I}, \mu_{\rm II}, \mu_{\rm III})$ is a rotation matrix (19), we find that:

$$\tilde{R}(\mu_{\rm I}, \mu_{\rm II}, \mu_{\rm III}) \cdot \begin{pmatrix} \sqrt{2J_{\rm I}}\cos\phi_{\rm I} \\ -\sqrt{2J_{\rm I}}\sin\phi_{\rm I} \\ \sqrt{2J_{\rm II}}\cos\phi_{\rm II} \\ -\sqrt{2J_{\rm II}}\sin\phi_{\rm II} \\ \sqrt{2J_{\rm III}}\cos\phi_{\rm III} \\ -\sqrt{2J_{\rm III}}\sin\phi_{\rm III} \end{pmatrix} = \begin{pmatrix} \sqrt{2J_{\rm I}}\cos(\phi_{\rm I}+\mu_{\rm I}) \\ -\sqrt{2J_{\rm I}}\sin(\phi_{\rm I}+\mu_{\rm I}) \\ \sqrt{2J_{\rm II}}\cos(\phi_{\rm II}+\mu_{\rm II}) \\ -\sqrt{2J_{\rm II}}\sin(\phi_{\rm II}+\mu_{\rm II}) \\ \sqrt{2J_{\rm III}}\cos(\phi_{\rm III}+\mu_{\rm III}) \\ -\sqrt{2J_{\rm III}}\sin(\phi_{\rm III}+\mu_{\rm III}) \end{pmatrix} \tag{39}$$

The transformations (32) follow. The actions J_k are conserved, and we identify the phase angles μ_k derived from the eigenvalues of the transfer matrix R for one periodic cell of the beamline with the phase advance across one periodic cell.

The *tune* of a storage ring is defined as the total phase advance for one turn round the ring, divided by 2π. Note that there are three tunes, corresponding to the three degrees of freedom: horizontal, vertical and longitudinal.

Consider a symplectic transfer matrix R that represents the map for one complete turn round a storage ring. The *fractional parts* of the tunes can be written:

$$\nu_k = \frac{\mu_k}{2\pi} = \frac{\arg \lambda_k}{2\pi} \tag{40}$$

where $\lambda_{\pm k}$ are the eigenvalues of R.

Consider the transfer matrix R for one periodic cell of a beamline. The transfer matrix is symplectic, so the eigenvalues occur in reciprocal pairs. Now suppose that a phase space vector $\vec{x}$ can be decomposed in terms of the eigenvectors of R:

$$\vec{x}_0 = \sum_{k=\mathrm{I,II,III}} c_{-k}\vec{e}_{-k} + c_k\vec{e}_k \tag{41}$$

Then after transport through n periodic cells, the phase space vector becomes:

$$\vec{x}(n) = R^n\vec{x}_0 = \sum_{k=\mathrm{I,II,III}} c_{-k}\lambda_{-k}^n\vec{e}_{-k} + c_k\lambda_k^n\vec{e}_k \tag{42}$$

If all the eigenvalue pairs $(\lambda_{-k}, \lambda_k)$ lie on the unit circle (i.e. form complex conjugate pairs) then the *amplitude* of the coefficients of the various eigenvectors in the decomposition of $\vec{x}$ remain the same size, and the motion of a particle starting from $\vec{x}_0$ remains bounded.

However, if one pair of eigenvalues $(\lambda_{-k}, \lambda_k)$ are real, then unless they are both exactly equal to ± 1, one of the eigenvalues must be greater than one. In this case, we see from (42) that the coefficient of at least one of the eigenvectors in the decomposition of the phase space vector increases without bound: the motion of the particle is unstable.

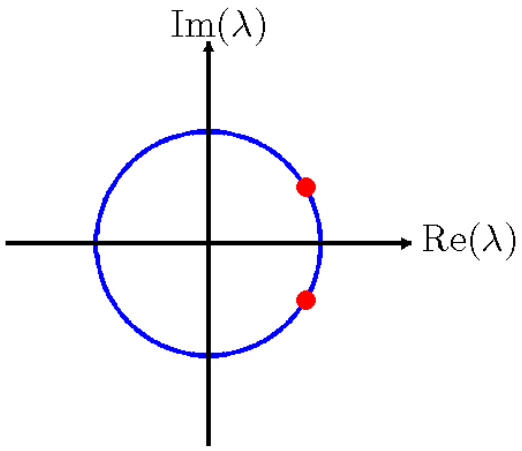
In the case of a storage ring, we see from equation (40) that real eigenvalues correspond to values for the tunes:

$$\nu_k = 0, \qquad \text{or} \qquad \nu_k = 0.5 \tag{43}$$

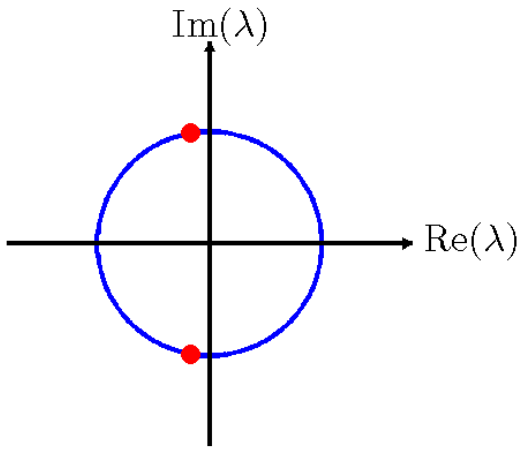
If the fractional part of the tune is 0 or 0.5, then the linear dynamics will not be stable. In accelerator jargon, we would say the lattice is "on the integer resonance", or "on the half-integer resonance".

Stable and Unstable Linear Dynamics

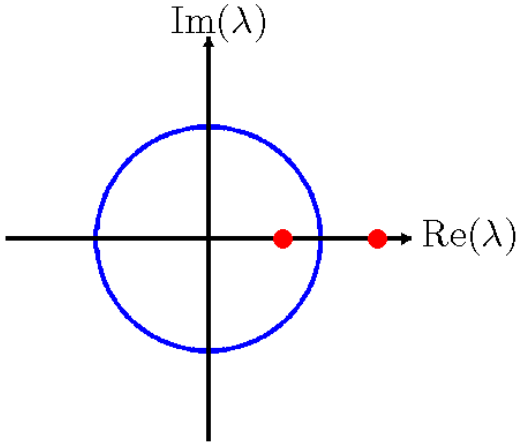
Stable



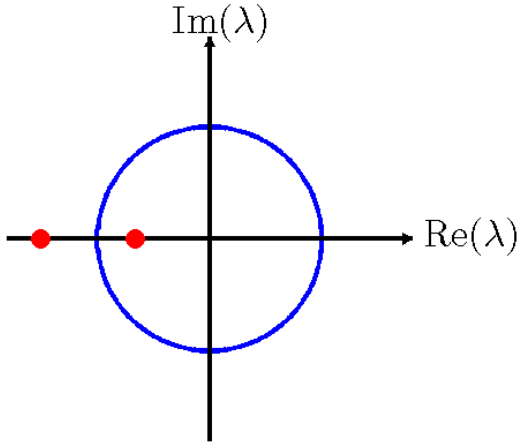
Stable



Integer Resonance: Unstable



Half-Integer Resonance: Unstable



If the angle variables are uncorrelated so that:

$$\langle \sin\phi_j \sin\phi_k \rangle = \langle \cos\phi_j \cos\phi_k \rangle = \frac{1}{2}\delta_{jk} \tag{44}$$

(where δ_{jk} is the Kronecker delta symbol) and:

$$\langle \sin\phi_j \cos\phi_k \rangle = 0 \tag{45}$$

then the sigma matrix can be written:

$$\Sigma = \sum_{k=\mathrm{I,II,III}} B^k \epsilon_k \tag{46}$$

where the matrices B^k are defined:

$$B^k = N \cdot T^k \cdot N^{\mathsf{T}} \tag{47}$$

the *emittances* ϵ_k are given by:

$$\epsilon_k = \langle J_k \rangle \tag{48}$$

and the distribution is matched in the sense that the sigma matrix is invariant under transport through one periodic cell:

$$R \cdot \Sigma \cdot R^{\mathsf{T}} = \Sigma \tag{49}$$

The sigma matrix may be written as the average over all particles in the bunch of the outer product of the phase space variables:

$$\Sigma = \langle \vec{x} \cdot \vec{x}^{\mathsf{T}} \rangle \tag{50}$$

Using the definition of the vector $\vec{J}$ (26), we can write:

$$\Sigma = N \cdot \langle \vec{J} \cdot \vec{J}^{\mathsf{T}} \rangle \cdot N^{\mathsf{T}} \tag{51}$$

Let us assume that the angle variables of particles in the bunch are uncorrelated, in the sense that they satisfy (44):

$$\langle \sin\phi_j \sin\phi_k \rangle = \langle \cos\phi_j \cos\phi_k \rangle = \frac{1}{2}\delta_{jk} \tag{52}$$

and (45):

$$\langle \sin\phi_j \cos\phi_k \rangle = 0 \tag{53}$$

Then, using the expression for $\vec{J}$ (31), we find that:

$$\Sigma = \sum_{k=\mathrm{I,II,III}} N \cdot T^k \cdot N^{\mathsf{T}} \epsilon_k \tag{54}$$

where

$$T^{\mathrm{I}} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \quad T^{\mathrm{II}} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \quad T^{\mathrm{III}} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \tag{55}$$

and

$$\epsilon_k = \langle J_k \rangle \tag{56}$$

If we define:

$$B^k = N \cdot T^k \cdot N^{\mathsf{T}} \tag{57}$$

then equation (54) may be written:

$$\Sigma = \sum_{k=\mathrm{I,II,III}} B^k \epsilon_k \tag{58}$$

The components B^k_{ij} of the matrices B^k are the generalisations of the Twiss parameters to coupled optics; we refer to them as the *coupled lattice functions*, or the *generalised Twiss parameters*.

If the phase angles of all particles in the bunch are uncorrelated (44), (45), then equation (49):

$$R \cdot \Sigma \cdot R^{\mathsf{T}} = \Sigma \tag{59}$$

tells us that the bunch distribution is matched to the transfer matrix for one periodic cell. This may be shown as follows.

First, using equation (54) we write:

$$R \cdot \Sigma \cdot R^{\mathsf{T}} = \sum_{k=\mathrm{I,II,III}} R \cdot N \cdot T^k \cdot N^{\mathsf{T}} \cdot R^{\mathsf{T}} \epsilon_k \tag{60}$$

But from equation (16) we can write:

$$R \cdot N = N \cdot \tilde{R} \tag{61}$$

Hence:

$$R \cdot \Sigma \cdot R^{\mathsf{T}} = \sum_{k=\mathrm{I,II,III}} N \cdot \tilde{R} \cdot T^k \cdot \tilde{R}^{\mathsf{T}} \cdot N^{\mathsf{T}} \epsilon_k \tag{62}$$

But since $\tilde{R}$ is just a rotation matrix:

$$\tilde{R}^{\mathsf{T}} = \tilde{R}^{-1} \tag{63}$$

and:

$$\tilde{R} \cdot T^k \cdot \tilde{R}^{\mathsf{T}} = T^k \tag{64}$$

Hence:

$$\sum_{k=\mathrm{I,II,III}} R \cdot N \cdot T^k \cdot N^{\mathsf{T}} \cdot R^{\mathsf{T}} \epsilon_k = \sum_{k=\mathrm{I,II,III}} N \cdot T^k \cdot N^{\mathsf{T}} \epsilon_k = \Sigma \tag{65}$$

So finally, from equation (60), equation (49) follows:

$$R \cdot \Sigma \cdot R^{\mathsf{T}} = \Sigma \tag{66}$$

The eigenvalues of $\Sigma \cdot S$ are $\pm i\epsilon_k$ where the emittances ϵ_k are given by equation (48).

The eigenvectors of $\Sigma \cdot S$ are contained in the same matrix E that contains the eigenvectors of the transfer matrix R.

These statements may be proved as follows. First we use equation (46) to write:

$$\Sigma \cdot S = \sum_{k=\mathrm{I,II,III}} B^k \cdot S\epsilon_k \tag{67}$$

Now, using the definition (47) for the matrices B^k we write:

$$B^k \cdot S = N \cdot T^k \cdot N^{\mathsf{T}} \cdot S \tag{68}$$

But since N is symplectic (22), it follows that:

$$B^k \cdot S = N \cdot T^k \cdot S \cdot N^{-1} \tag{69}$$

Now, from the definition (15) of the normalising matrix N:

$$N = E \cdot Q \tag{70}$$

it follows that:

$$B^k \cdot S = E \cdot Q \cdot T^k \cdot S \cdot Q^{-1} \cdot E^{-1} \tag{71}$$

or:

$$E^{-1} \cdot B^k \cdot S \cdot E = Q \cdot T^k \cdot S \cdot Q^{-1} \tag{72}$$

The matrices on the right hand side of equation (72) are all constant. Performing the matrix multiplications, we find:

$$E^{-1} \cdot B^k \cdot S \cdot E = -i\bar{T}^k \tag{73}$$

where

$$\bar{T}^{\mathrm{I}} = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \quad \bar{T}^{\mathrm{II}} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \quad \bar{T}^{\mathrm{III}} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix} \tag{74}$$

Finally, from equation (67) and equation (73) we can write:

$$E^{-1} \cdot \Sigma \cdot S \cdot E = -i \begin{pmatrix} \epsilon_{\mathrm{I}} & & & & & \\ & -\epsilon_{\mathrm{I}} & & & & \\ & & \epsilon_{\mathrm{II}} & & & \\ & & & -\epsilon_{\mathrm{II}} & & \\ & & & & \epsilon_{\mathrm{III}} & \\ & & & & & -\epsilon_{\mathrm{III}} \end{pmatrix} \tag{75}$$

Since E diagonalises $\Sigma \cdot S$, it must be constructed from the eigenvectors of $\Sigma \cdot S$. The components on the diagonal of the diagonalised matrix are the eigenvalues of $\Sigma \cdot S$, and we see that these are $\pm i\epsilon_k$, where ϵ_k are the emittances.

Consider the transformation of the sigma matrix under the action of a transfer matrix $R = R(s_0, s_1)$, corresponding to the transport of a bunch from a point s_0 along the reference trajectory to a point s_1. We do not now require that the transport be through a complete periodic cell. Since the coordinates of each particle in the bunch transform:

$$\vec{x}(s_1) = R \cdot \vec{x}(s_0) \tag{76}$$

It follows from (50) that the sigma matrix transforms as:

$$\Sigma(s_1) = R \cdot \Sigma(s_0) \cdot R^{\mathsf{T}} \tag{77}$$

Hence we have:

$$\begin{aligned} \Sigma(s_1) \cdot S &= R \cdot \Sigma(s_0) \cdot R^{\mathsf{T}} \cdot S & (78) \\ &= R \cdot \Sigma(s_0) \cdot S \cdot R^{-1} & (79) \end{aligned}$$

where the last step follows from the fact that the transfer matrix R is symplectic.

We have shown that:

$$\Sigma(s_1) \cdot S = R \cdot \Sigma(s_0) \cdot S \cdot R^{-1} \tag{80}$$

But for *any* matrices U and V, the eigenvalues of V are the same as the eigenvalues of $U \cdot V \cdot U^{-1}$. Hence the eigenvalues of $\Sigma(s_1) \cdot S$ are equal to the eigenvalues of $\Sigma(s_0) \cdot S$. But these eigenvalues are just $\pm i\epsilon_k$, where ϵ_k are the bunch emittances. Hence, the bunch emittances are conserved under any linear symplectic transformation: this is just a consequence of Liouville's theorem.

Since the bunch emittances are conserved, and the sigma matrix is related to the coupled lattice functions through (46):

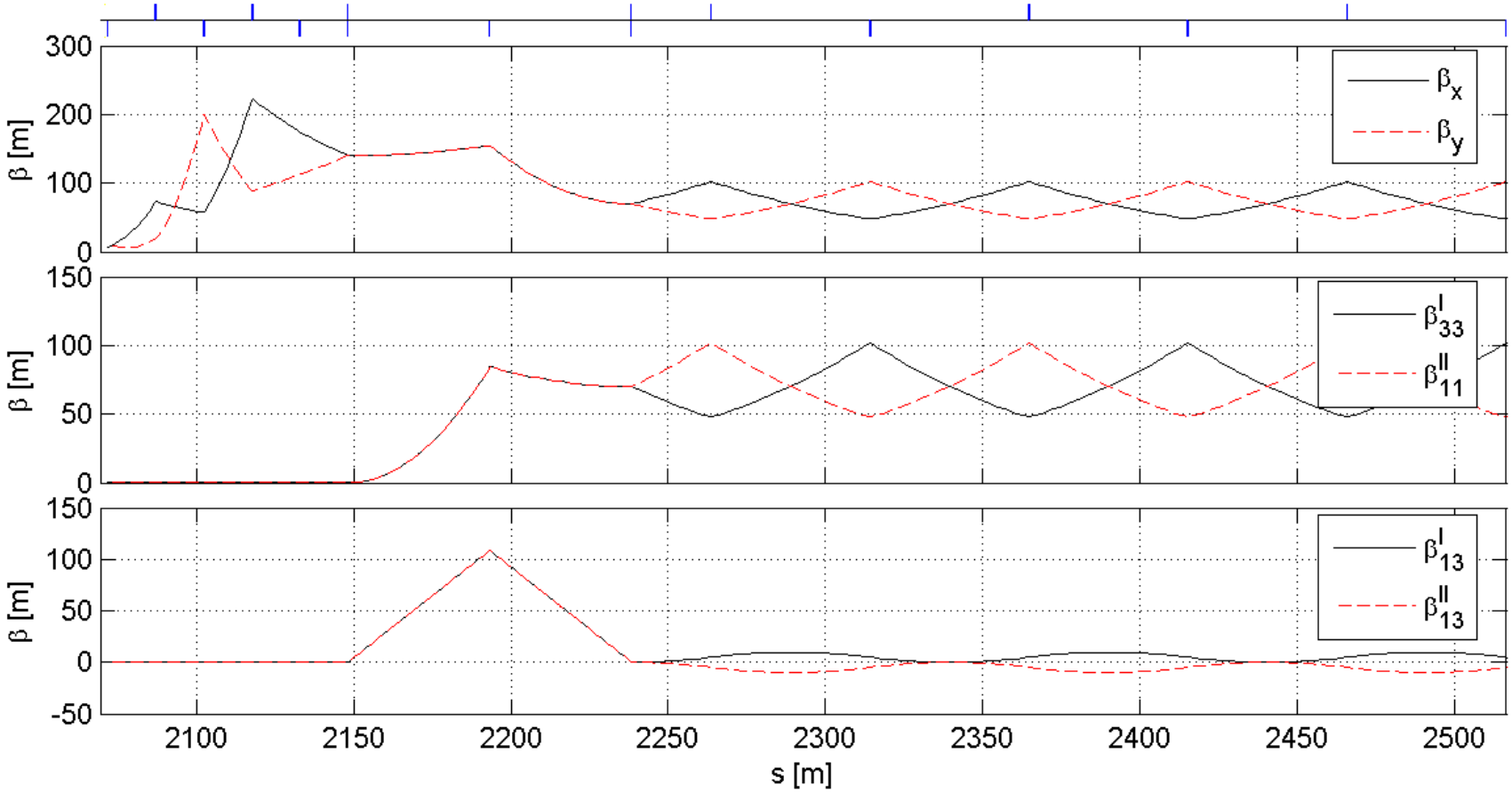
$$\Sigma = \sum_{k=\mathrm{I,II,III}} B^k \epsilon_k \tag{81}$$

the coupled lattice functions must transform in *the same way* as the sigma matrix, i.e.

$$B^k(s_1) = R \cdot B^k(s_0) \cdot R^{\mathsf{T}} \tag{82}$$

Here, R is *any* symplectic transfer matrix: there is no requirement that the beamline be periodic. Hence, as in the uncoupled case, we can determine the initial lattice functions from the bunch distribution at the start of the beamline, and then propagate the lattice functions along the beamline using the appropriate transfer matrices.

Example: Lattice Functions with a "Coupling Bump"



Consider a bunch of particles in a storage ring. If we know the transfer matrix R for one full turn of the ring starting at some point, then we can calculate (from the eigenvectors of R) the coupled lattice functions B^k at that point. If we also know the bunch emittances, then we can calculate the sigma matrix for the matched distribution at that point. In particular, the square of the horizontal beam size is given by:

$$\langle x^2 \rangle = \beta_{11}^{\mathrm{I}} \epsilon_{\mathrm{I}} + \beta_{11}^{\mathrm{II}} \epsilon_{\mathrm{II}} + \beta_{11}^{\mathrm{III}} \epsilon_{\mathrm{III}} \tag{83}$$

This equation is exact. However, you often see an alternative expression used that requires some assumptions...

If we assume that there is no coupling between the horizontal and vertical planes, then $\beta^{\mathrm{II}}_{11} = 0$. Also, if the coupling between the horizontal and longitudinal planes is weak, then we can make the approximation:

$$\beta^{\mathrm{I}}_{11} \approx \beta_x \tag{84}$$

where β_x is the horizontal beta function (Twiss parameter) describing the horizontal motion in the absence of any coupling. Furthermore, if the longitudinal motion is slow, so that the energy deviation δ of each particle can be treated as constant, then the horizontal position of a particle at the chosen point in the storage ring is:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x + \eta_x\delta \tag{85}$$

where η_x is the dispersion.

Finally, we assume that the angle variables ϕ_x of all particles in the beam are uncorrelated, and that the energy deviation δ is uncorrelated with the horizontal variables J_x or ϕ_x. If all the above assumptions are valid, then the square of the horizontal beam size may be written:

$$\sigma_x^2 = \beta_x \epsilon_x + \eta_x^2 \sigma_\delta^2 \tag{86}$$

where

$$\sigma_x^2 = \langle x^2 \rangle \tag{87}$$
$$\sigma_\delta^2 = \langle \delta^2 \rangle \tag{88}$$

In many practical cases, the above assumptions are satisfied well enough to allow us to calculate the beam size in this way. However, accelerator physicists are becoming increasingly ambitious in the systems they design and build, and are constantly pushing into new regimes. Caution is advised!

Given the transfer matrix R for one periodic cell of a beamline, and the sigma matrix Σ for the matched distribution:

- The eigenvalues of R are $\exp(\pm i\mu_k)$, where μ_k are the phase advances across the cell in each degree of freedom.

- The eigenvectors of R are the same as the eigenvectors of $\Sigma \cdot S$.

- The eigenvectors may be used to construct coupled lattice functions B^k for coupled optics, which relate the sigma matrix directly to the bunch emittances.

- The eigenvalues of $\Sigma \cdot S$ are $\pm i\epsilon_k$ where ϵ_k are the beam emittances.

In terms of the coupled lattice functions and the bunch emittances, the sigma matrix is given by (46):

$$\Sigma = \sum_{k=\mathrm{I,II,III}} B^k \epsilon_k \tag{89}$$

The coupled lattice functions B^k relate the components of the sigma matrix to the bunch emittances; they are given by (47):

$$B^k = N \cdot T^k \cdot N^{\mathsf{T}} \tag{90}$$

where the normalising matrix, N is given by (15):

$$N = E \cdot Q \tag{91}$$

where E are the eigenvectors of R (or $\Sigma \cdot S$), normalised and ordered so that E satisfies equation (4):

$$E^{\mathsf{T}} \cdot S \cdot E = iS \tag{92}$$

The constant matrices Q and T^k are given by equations (14) and (55) respectively.

The bunch emittances are conserved as the bunch is transported along a beamline, as long as the transport is linear and symplectic. The lattice functions transform as:

$$B^k(s_1) = R \cdot B^k(s_0) \cdot R^{\mathsf{T}} \tag{93}$$

where $R = R(s_0, s_1)$ is the transfer matrix from point s_0 to the point s_1 along the (not necessarily periodic) beamline.

Finally, be warned that, as I said at the start, there are many different ways of describing coupled optics. Unfortunately, no single approach has yet become standard, so many different definitions of the coupled lattice functions are in use. When working on a project involving coupled lattice functions, always make sure you understand which definitions are in use.