Cockcroft Lectures: Intermediate Beam Dynamics for Particle Accelerators

Problem Set 8

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1. Write down expressions for the phase-space coordinates (x, p_x) of a single particle in terms of the action-angle variables J_x and ϕ_x for that particle.

2. Consider a bunch of a large number of particles in which the angle variables for the different particles have a uniform random distribution between 0 and 2π. Show that:

$$\langle x^2 \rangle = \beta_x \epsilon_x \tag{1}$$

$$\langle x p_x \rangle = -\alpha_x \epsilon_x \tag{2}$$

$$\langle p_x^2 \rangle = \gamma_x \epsilon_x \tag{3}$$

   where the brackets $\langle \cdot \rangle$ indicate an average over all particles in the bunch, and the Twiss parameters α_x, β_x, γ_x relate the action variable to the variables x and p_x in the usual way. How is the bunch emittance ϵ_x related to the action variables of the particles in the bunch?

3. Using the expressions given in Problem 2 and the relation between the Twiss parameters:

$$\beta_x \gamma_x - \alpha_x^2 = 1 \tag{4}$$

   derive an expression for the emittance in terms of the second-order moments of the bunch distribution.

4. Using your result from Problem 3, show that the eigenvalues $\lambda_\pm$ of the matrix $\Sigma \cdot S$ where:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5}$$

   are $\lambda_\pm = \pm i \epsilon_x$.

5. Consider a 2×2 transfer matrix R given by:

$$R = I \, \cos \mu_x + S \cdot A_x \, \sin \mu_x \tag{6}$$

where I is the 2×2 identity matrix, μ_x is a real number, and the matrix A_x is constructed from the Twiss parameters:

$$A_x = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix} \tag{7}$$

Show that, if the second-order moments of the beam distribution satisfy equations (1), (2) and (3), then the beam distribution is "matched" to the transfer matrix, i.e. the beam distribution matrix is invariant under the transformation:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to R \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{8}$$