Single Particle Dynamics, Lecture 8

Bunches of Many Particles

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We have seen how the motion of a particle through a periodic beamline can be described in terms of the Twiss parameters, α_x, β_x, γ_x. These parameters are defined for an uncoupled beamline by the 2×2 transfer matrix R_2 for *one periodic cell* of the beamline:

$$R_2 = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \tag{1}$$

where

$$\beta_x \gamma_x - \alpha_x^2 = 1 \tag{2}$$

Using the Twiss parameters, the horizontal coordinate and normalized momentum can be written in terms of *action-angle variables* (J_x, ϕ_x):

$$\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2J_x}\cos\phi_x \\ -\sqrt{2J_x}\sin\phi_x \end{pmatrix} \tag{3}$$

Using action-angle variables, tracking a particle through a lattice is very easy: the action J_x is constant, and the angle ϕ_x increases according to:

$$\frac{d\phi_x}{ds} = \frac{1}{\beta_x} \tag{4}$$

where s is the path length along the reference trajectory. We have assumed that the beamline is *uncoupled*, that is the transfer matrix between any two points contains non-zero components only in the 2×2 block diagonals. If this is true, then the optics description we have developed can be applied to any of the degrees of freedom: horizontal, vertical or longitudinal.

Of course, we have to know the value of β_x at every point in the beamline, which means calculating the Twiss parameters using equation (1) for every point in the beamline. Fortunately, there are computer codes that will do this for us.

Part II (Lectures 6 – 10): Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines

7. Including longitudinal dynamics

8. Bunches of many particles

9. Coupled optics

10. Effects of linear imperfections

In practice, being able to track individual particles is important, but not good enough. This is because particles in accelerators are usually collected together in bunches that may contain more than 10^{10} particles. Describing the behaviour of a bunch by writing down the optics of each individual particle is cumbersome, to say the least! However, we do need to know things like the variation in the size of the bunch along a beamline. In this lecture, we will use particle optics to describe the behaviour of a bunch consisting of a large number of particles.

Note that we do not consider interactions between the particles in the bunch: we assume that the particles move independently, and we consider only the effects of external electromagnetic fields. In this sense, we are not doing true "multiparticle dynamics", which is a topic for another course.

Consider a bunch consisting of a large number of particles. We use the notation $\langle \cdot \rangle$ to describe the average of some quantity (usually, a dynamical variable) over all particles in the bunch. For example, $\langle x \rangle$ is the average horizontal coordinate of the particles in the bunch. If all particles have the same charge, this is the value that would be recorded by a *horizontal beam position monitor* (horizontal BPM) as the bunch goes past.

The quantities $\langle x_i \rangle$, where x_i $(1 < i < 6)$ is one of the dynamical variables $(x, p_x, y, p_y, z, \delta)$ are usually called the *first order moments* of the bunch distribution. Collectively, the first order moments of the bunch distribution are also called the bunch *centroid*.

The quantities:

$$\Sigma_{ij} = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle \tag{5}$$

are called the *second order moments* of the bunch distribution. While the *first order moments* describe the *position* of the bunch, the *second order moments* describe the *size* of the bunch.

The second order moments Σ_{ij} form a symmetrix matrix Σ, usually referred to as the *sigma matrix*. The diagonal components of the sigma matrix are the variances of the dynamical variables over all the particles in the bunch, for example:

$$\Sigma_{11} = \langle x^2 \rangle - \langle x \rangle^2 = \sigma_x^2 \tag{6}$$

If we know the transfer matrix $R = R(s_0, s_1)$ from a point s_0 along the reference trajectory to a point s_1 along the reference trajectory, we can write down the transformation of the sigma matrix. First, we observe that the first order moments transform from s_0 to s_1 as:

$$\langle x_i \rangle_{s_1} = \sum_{i'=1}^{6} R_{ii'} \langle x_{i'} \rangle_{s_0} \tag{7}$$

Similarly, the second order moments transform as:

$$\Big(\langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle \Big)_{s_1} = \sum_{i',j'=1}^{6} R_{ii'} R_{jj'} \Big(\langle x_{i'} x_{j'} \rangle - \langle x_{i'} \rangle \langle x_{j'} \rangle \Big)_{s_0} \tag{8}$$

In matrix form, this can be written:

$$\Sigma(s_1) = R \cdot \Sigma(s_0) \cdot R^{\mathsf{T}} \tag{9}$$

An *uncoupled* bunch distribution is one for which all the components of the sigma matrix outside the 2×2 block diagonals are zero. If the sigma matrix has non-zero components outside the 2×2 block diagonals, then the distribution is said to be coupled.

As we shall see, an uncoupled distribution travelling down an uncoupled beamline will stay uncoupled. However, the effect of coupling in a beamline is to introduce coupling into a (previously uncoupled) bunch distribution. Coupling in beamlines and coupling in bunch distributions are therefore related.

For example, consider the dispersion terms in a transfer matrix. Such terms will introduce a *correlation* between transverse position and energy deviation, so that $\Sigma_{16} = \langle x\delta \rangle \neq 0$.

From the sigma matrix Σ, we can construct the matrix $\Sigma \cdot S$, where:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{10}$$

Note that S is the matrix used to define a *symplectic* matrix M; that is, M is symplectic if it satisfies the equation:

$$M^{\mathsf{T}} \cdot S \cdot M = S \tag{11}$$

In the first part of this course, we went to some trouble to derive the transfer matrices in such a way that the transfer matrix for any linear element (and therefore for any linear beamline) is symplectic.

Since the transfer matrix $R = R(s_0, s_1)$ is symplectic, $\Sigma \cdot S$ transforms as follows:

$$\Sigma \cdot S \mapsto R \cdot \Sigma \cdot R^{\mathsf{T}} \cdot S = R \cdot \Sigma \cdot S \cdot R^{-1} \tag{12}$$

where in the last step, we have used the definition (11) of a symplectic matrix.

Now, we make the observation that for any matrices M and N, the *eigenvalues* of $N \cdot M \cdot N^{-1}$ are the *same* as the eigenvalues of M. Therefore, *the eigenvalues of $\Sigma \cdot S$ are conserved under symplectic, linear transport represented by a symplectic transfer matrix R.*

For single-particle transport, the derivation of a conserved quantity (the action) simplified the description of the optics. We can similarly use the conserved quantities for a beam distribution (the eigenvalues of $\Sigma \cdot S$) to our advantage, as we shall see shortly.

Consider the special case of an *uncoupled* bunch distribution with first order moments $\langle x_i \rangle$ all equal to zero. The horizontal part of the sigma matrix may be written:

$$\Sigma_2 = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} \tag{13}$$

It follows that the eigenvalues of $\Sigma_2 \cdot S_2$ are $\lambda_\pm$ given by:

$$\lambda_\pm = \pm i \epsilon_x \tag{14}$$

where:

$$\epsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2} \tag{15}$$

The quantity ϵ_x defined by (15) is known as the *horizontal emittance* of the bunch. Similar expressions may be written for the vertical and longitudinal emittances.

Since the eigenvalues of the matrix $\Sigma \cdot S$ are conserved under linear symplectic transport, the beam emittances for an uncoupled distribution, defined by (15):

$$\epsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2} \tag{16}$$

and similar expressions for the vertical and longitudinal degrees of freedom, are also conserved. Remember that the emittances defined in this way are only conserved for *uncoupled* bunches.

It is possible to generalise the definition of the emittances to coupled distributions. In general, the 6×6 matrix $\Sigma \cdot S$ has *six* eigenvalues, $\pm i\epsilon_k$ with $k = \mathrm{I}, \mathrm{II}, \mathrm{III}$, corresponding to the horizontal, vertical and longitudinal degrees of freedom, respectively. The quantities ϵ_k are the three emittances, which are conserved under linear symplectic transport.

For a periodic beamline, any distribution for which the sigma matrix remains invariant under transport through one periodic cell is known as a *matched distribution*.

In other words, if R is the transfer matrix for one periodic cell, and the sigma matrix Σ satisfies:

$$\Sigma = R \cdot \Sigma \cdot R^{\mathsf{T}} \tag{17}$$

then Σ is a matched distribution.

In an uncoupled periodic beamline, the Twiss parameters are defined by the transfer matrix for a single periodic cell, using equation (1). Given the Twiss parameters, we can transform to action-angle variables, using equation (3):

$$\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2J_x}\cos\phi_x \\ -\sqrt{2J_x}\sin\phi_x \end{pmatrix} \tag{18}$$

Let us assume that the angle variables of all particles in the bunch are uniformly distributed, i.e.:

$$\langle \sin\phi_x \rangle = \langle \cos\phi_x \rangle = 0, \qquad \langle \sin^2\phi_x \rangle = \langle \cos^2\phi_x \rangle = \frac{1}{2} \tag{19}$$

With this assumption, and using equations (3) and (2), we find that:

$$\langle x^2 \rangle = \beta_x \langle J_x \rangle \tag{20}$$

$$\langle x p_x \rangle = -\alpha_x \langle J_x \rangle \tag{21}$$

$$\langle p_x^2 \rangle = \gamma_x \langle J_x \rangle \tag{22}$$

From equations (20) - (22), and (2), we find:

$$\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 = \langle J_x \rangle^2 \tag{23}$$

Then, from comparison with equation (15) we can write:

$$\epsilon_x = \langle J_x \rangle \tag{24}$$

In other words, the horizontal emittance of a bunch is simply the average of the horizontal actions over every particle in the bunch. This is consistent with the fact that (as we already know) the emittance and the action are each conserved under linear symplectic transport.

Note that equation (23) was derived with the assumption that the beam centroid was on the reference trajectory (equivalent to the assumption that the angle variables of the particles in the bunch are uniformly distributed).

Finally, we can write for a *matched* distribution in the case of an *uncoupled* bunch:

$$\langle x^2 \rangle = \beta_x \epsilon_x \tag{25}$$

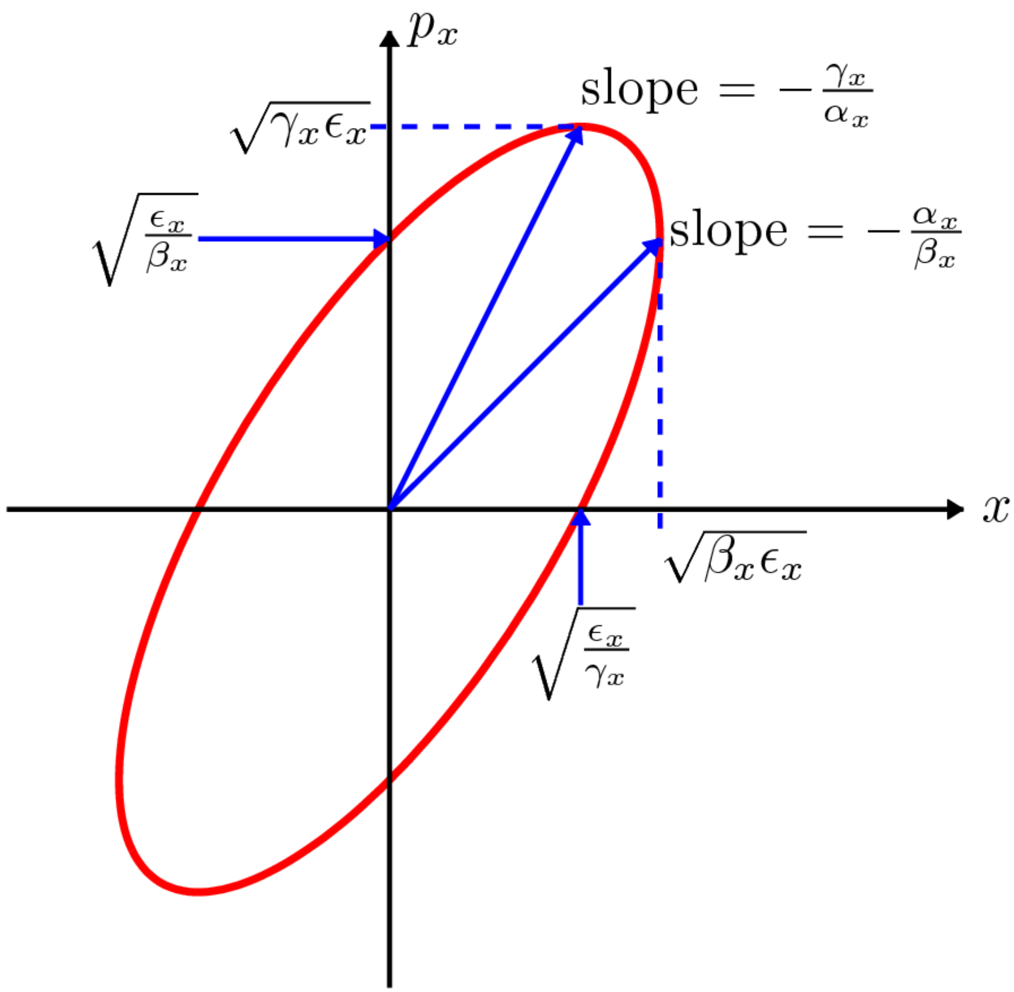
$$\langle x p_x \rangle = -\alpha_x \epsilon_x \tag{26}$$

$$\langle p_x^2 \rangle = \gamma_x \epsilon_x \tag{27}$$

In other words, the *size* of the bunch is given by the Twiss parameters combined with the emittance.

In the case of single-particle dynamics, we interpreted the Twiss parameters as giving the local variation in the amplitude of oscillation in a periodic beamline: the action gave the *invariant* amplitude of oscillation. Now, in the case of a bunch of particles, we have another interpretation of the Twiss parameters: they give the local variation in the size of the bunch, with the emittance representing an *invariant* size of the bunch.

The Twiss parameters define the *shape* of the bunch distribution in phase space. The emittance gives the *area* $= \pi\epsilon_x$ of the *rms ellipse*:



The action and the emittance are both conserved quantities: the action for an individual particle, and the emittance for a bunch consisting of a large number of particles.

Since the emittance is just the average of the actions of all particles in the bunch, we can understand the meaning of a matched distribution.

In transport through one periodic cell of the beamline, all particles move round phase space ellipses: the ellipses have the same shape (determined by the Twiss parameters), but different areas (corresponding to the different actions of the particles).

A "snapshot" of the bunch in phase space, if not able to resolve individual particles, will look identical from one periodic cell to the next.

If the bunch is not properly matched to the lattice, then the ellipse corresponding to the bunch distribution will appear to "rotate" in phase space from one periodic cell to the next.

The distribution may still be described by Twiss parameters, but the values of the Twiss parameters for the bunch distribution will not match the values of the Twiss parameters derived from the periodic cell of the beamline.

This observation tells us how to define the Twiss parameters for a non-periodic beamline...

If the beam distribution has first order moments equal to zero and is uncoupled, then the sigma matrix can *always* be written in the form:

$$\Sigma_2 = \begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle \\ \langle xp_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \tag{28}$$

where $\pm i\epsilon_x$ are the eigenvalues of $\Sigma_2 \cdot S_2$. No assumption is made about the periodicity (or lack of periodicity) of the beamline.

To evolve the distribution from one point (s_0) of the beamline to another (s_1) we simply apply the transfer matrix $R = R(s_0, s_1)$ between the two points:

$$\Sigma(s_1) = R \cdot \Sigma(s_0) \cdot R^{\mathsf{T}} \tag{29}$$

If R is symplectic (as it ought to be!) then the emittance ϵ_x is conserved. We can calculate the Twiss parameters at any point along the beamline, and these Twiss parameters describe how the sigma matrix evolves along the beamline.

High-energy accelerator beamlines generally start with some particle source; for example, a laser pulse striking a cathode in an electromagnetic field to produce a bunch of electrons. Modelling the source is usually a fairly complicated business. But if we know the properties of the laser, the cathode, and the electromagnetic field in which it sits, we can generally determine the sigma matrix for the bunch of particles that comes out of the source. From that sigma matrix, we can determine the Twiss parameters: this determines the values of the Twiss parameters at the start of the beamline.

Often, we want to inject the bunch into a periodic beamline. Getting the sigma matrix at the start of the periodic beamline correctly *matched* into the beamline means making the Twiss parameters for the distribution equal to the Twiss parameters for the periodic beamline: this can generally be done with some combination of quadrupoles.

Finally, let us consider what happens to a bunch of particles in a linear accelerator, consisting of a string of accelerating cavities that increase the energy of each particle in the bunch significantly. The effect of each cavity is to change the energy deviation δ of each particle. After some number of cavities, the energy deviation of a typical particle becomes large enough that we no longer trust the linear approximations that we made in deriving the transport matrices (which relied on series expansions in the dynamical variables). At that point we start to think about redefining the reference momentum P_0; a good choice would be one that sets the average energy deviation of the bunch $\langle\delta\rangle$ to zero (or very close to it).

What are the effects of changing the reference momentum?

Let us consider just the transverse variables. The coordinates x and y are independent of the reference momentum. However, the conjugate momenta p_x is defined as:

$$p_x = \frac{\gamma m \dot{x} + q A_x}{P_0} \tag{30}$$

where $\dot{x}$ is the transverse velocity, γ the relativistic factor for the particle, m the rest mass, q the electric charge, A_x the horizontal component of the magnetic vector potential, and P_0 the reference momentum.

Under a change of reference momentum, we therefore have the transformations:

$$P_0 \mapsto P_0', \qquad x \mapsto x, \qquad p_x \mapsto \frac{P_0}{P_0'} p_x \tag{31}$$

The transformations of the dynamical variables in equation (31) are *not* symplectic. Therefore, the emittances of a bunch will not be conserved under a change in reference momentum.

However, we can derive a quantity that *is* conserved under a change in reference momentum, as follows. From equation (31) and equation (15), it is clear that the horizontal emittance transforms as:

$$P_0 \mapsto P_0', \qquad \epsilon_x \mapsto \epsilon_x' = \frac{P_0}{P_0'}\epsilon_x \tag{32}$$

In other words:

$$P_0'\epsilon_x' = P_0\epsilon_x \tag{33}$$

Since $P_0 = \beta_0\gamma_0 mc$, we can write:

$$\beta_0'\gamma_0'\epsilon_x' = \beta_0\gamma_0\epsilon_x \tag{34}$$

The normalised emittance $\epsilon_{x,N}$ is defined by:

$$\epsilon_{x,N} = \beta_0 \gamma_0 \epsilon_x \qquad (35)$$

From equation (34), it follows that the normalised emittance is conserved under a change in reference momentum. It is often said that "the normalised emittance is conserved during acceleration". This can be slightly misleading, since the non-symplectic process that requires us to define the normalised emittance is the change in reference momentum, not the acceleration itself.

The normalised vertical emittance, defined in the same way as the normalised horizontal emittance, is similarly a conserved quantity under a change in reference momentum. The longitudinal emittance behaves slightly differently, but for ultrarelativistic particles ($\beta_0 \to 1$), the normalised longitudinal emittance is also conserved under a change in reference momentum.

To emphasise the difference between the normalised emittance $\epsilon_{x,N}$ defined by equation (35) and the emittance ϵ_x defined by equation (15), the emittance ϵ_x of equation (15) is often referred to as the *geometric emittance*.

Note that when a beam is accelerated in a linac, and the reference momentum is scaled with the average beam energy, the geometric emittance varies in inverse proportion to the beam energy, i.e.:

$$\epsilon_x \propto \frac{1}{\beta_0 \gamma_0} \tag{36}$$

This effect is known as *adiabatic damping*. The consequence of adiabatic damping is that, for fixed beta functions, the size of a bunch decreases as the energy of the bunch is increased.

The *sigma matrix* is constructed from the second order moments of the beam distribution (5), and describes the *size* of a bunch of particles.

The transformation of the sigma matrix along a beamline is given by the transfer matrix $R = R(s_0, s_1)$ (9):

$$\Sigma(s_1) = R \cdot \Sigma(s_0) \cdot R^{\mathsf{T}} \tag{37}$$

The emittances ϵ_k of a bunch are defined in terms of the eigenvalues λ_k of $\Sigma \cdot S$:

$$\lambda_k = \pm i\epsilon_k \qquad k = \mathrm{I}, \mathrm{II}, \mathrm{III} \tag{38}$$

where the three values of k correspond to the three degrees of freedom of particles in the bunch. The emittances are conserved under linear symplectic transport.

For an *uncoupled* bunch, the horizontal emittance is given by (15):

$$\epsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2} \tag{39}$$

Similar expressions can be used for the vertical and longitudinal emittances.

The emittance of a bunch is the average of the actions of all particles in the bunch (24):

$$\epsilon_x = \langle J_x \rangle \tag{40}$$

The shape of the bunch distribution in phase space can be given by the Twiss parameters (28):

$$\Sigma_2 = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \qquad (41)$$

For a distribution that is properly *matched* to a periodic beamline, the Twiss parameters for the distribution are equal to the Twiss parameters for the beamline. The sigma matrix from one periodic cell to the next is then invariant.

The Twiss parameters at the start of a non-periodic beamline (e.g. from a particle source) can be determined from the shape of the distribution in phase space of a bunch at the start of the beamline (e.g. a bunch that is produced by the source).

When the energy of a bunch is increased (e.g. in an accelerating RF cavity), we usually want to change the reference momentum P_0 so that the energy deviations of particles in the bunch remain small. A change in the reference momentum requires a *non-symplectic* transformation, and the emittances ϵ_k are no longer conserved.

However, we can define (at least in the transverse planes) *normalised emittances*, $\epsilon_{x,N}$ and $\epsilon_{y,N}$, that are conserved under a change in reference momentum:

$$\epsilon_{x,N} = \beta_0 \gamma_0 \epsilon_x \tag{42}$$

and similarly for y.

The reduction in the *geometric emittance* ϵ_x with increasing reference momentum is known as *adiabatic damping*.