Cockcroft Lectures: Single Particle Dynamics for Particle Accelerators

Problem Set 6

A. Wolski & B, Muratori

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1. (a) Find a constraint on the values of α, β and γ, so that the matrix:

$$R = I\,\cos\mu + S\cdot A\,\sin\mu \qquad (1)$$

is symplectic for any value of μ, where I is the 2×2 identity matrix, and the matrices S and A are given by:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \qquad (2)$$

Show that, with this constraint, A is symplectic.

(b) Find a constraint on the values of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$, so that the matrix:

$$\tilde{R} = I\,\cosh\mu + S\cdot\tilde{A}\,\sinh\mu \qquad (3)$$

is symplectic for any value of μ, where:

$$\tilde{A} = \begin{pmatrix} \tilde{\gamma} & \tilde{\alpha} \\ \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \qquad (4)$$

Show that, with this constraint, $\tilde{A}$ satisfies:

$$\tilde{A}^{\mathrm{T}}\cdot S\cdot\tilde{A} = -S \qquad (5)$$

(c) Show that J, defined by:

$$J = \frac{1}{2}\begin{pmatrix} x & p_x \end{pmatrix}\cdot A\cdot\begin{pmatrix} x \\ p_x \end{pmatrix} \qquad (6)$$

is an invariant under the transformation:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to R\cdot\begin{pmatrix} x \\ p_x \end{pmatrix} \qquad (7)$$

and that $\tilde{J}$, defined by:

$$\tilde{J} = \frac{1}{2} \begin{pmatrix} x & p_x \end{pmatrix} \cdot \tilde{A} \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{8}$$

is an invariant under:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to \tilde{R} \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{9}$$

(d) Using the appropriate invariants of a symplectic transfer matrix R, sketch the phase-space diagrams produced by applying the transfer matrix repeatedly to a particle in the cases (i) $\mathrm{Tr}\, R < 2$, and (ii) $\mathrm{Tr}\, R > 2$, where $\mathrm{Tr}\, R$ is the trace of R. Comment on the stability of the motion in each case.

2. Using the generating function:

$$F_1 = F_1(x, \phi_x) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x) \tag{10}$$

relate the "new" canonical variables (ϕ_x, J_x) to the "old" canonical variables (x, p_x).

3. Find a relationship between the quadrupole focal length f_0 and the drift length L in a FODO cell with phase advance 90°. Find the maximum and minimum values of the beta functions in such a cell, as functions of the focal length of the quadrupoles.