Single Particle Dynamics for Particle Accelerators, Lecture 6

Linear Optics in Periodic, Uncoupled Beamlines.

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The Story So Far...

We have derived dynamical maps for charged particles in commonly-used linear beamline elements, including drift spaces, bending magnets, normal and skew quadrupoles, solenoids and RF accelerating cavities. We have seen how the first-order maps can be written as *transfer matrices*.

The dynamical maps were derived from first principles of classical (Hamiltonian) mechanics, special relativity, and electromagnetism (Maxwell's equations). To obtain linear, symplectic maps, we approximated the exact Hamiltonian for each type of element, by making a series expansion to second-order in the dynamical variables.

Having obtained the transfer matrices for the common linear beamline elements, we are now able to calculate the trajectory of a charged particle along a beamline constructed from these elements, in the linear approximation, from given initial conditions.

Part II (Lectures 6 – 10): Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines

7. Including longitudinal dynamics

8. Bunches of many particles

9. Coupled optics

10. Effects of linear imperfections

The detailed description of the motion of a charged particle in an electromagnetic field may be regarded as the study of the particle "dynamics". While this is essential for accelerator physics, it is also rather cumbersome. It is convenient to have a "high level" description of a beamline that allows one readily to understand important general features of the motion of a particle through the beamline, without worrying about the detailed motion of individual particles. Such a description may be regarded as the study of the beamline "optics".

Our goal in the remainder of this course will be to develop and understand the subject of beamline optics. Critical for this will be the fact that the transfer matrices are *symplectic*. This is why we went to so much trouble with Hamiltonian mechanics: exact solutions to Hamilton's equations are guaranteed to produce symplectic maps (Liouville's theorem).

Tackling beamline optics for a general beamline is too big a step all at once. To provide a "gentler" introduction, we will begin by considering a linear beamline with two important properties.

1. The beamline is periodic: it consists of a repeated unit, or "cell", itself consisting of a given set of elements.

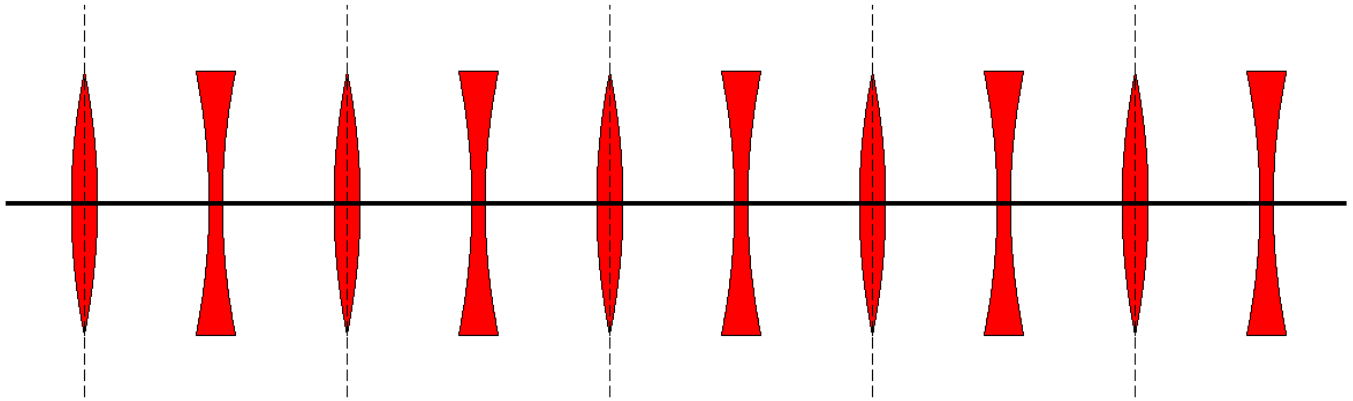
2. The beamline is uncoupled: the transfer matrices for each individual element are block-diagonal.

After developing this special (but very important) case, we shall return to the more general case.

Example of a Periodic, Uncoupled, Linear Beamline

The *FODO* lattice is an important example of a periodic, uncoupled, linear beamline.

A single FODO cell consists of one (horizontally) focusing and one (horizontally) defocusing normal quadrupole.



If we understand the dynamics in one cell of this beamline, we understand the optics in the entire beamline. Let's calculate the transfer matrix for a single cell.

We shall use the "thin lens" approximation for the quadrupoles. That is, we shall take the limit:

$$L \to 0 \qquad k_1 L \to \frac{1}{f} \tag{1}$$

where L is the length of the quadrupole, k_1 is the normalized quadrupole gradient, and f is a constant (the "focal length" of the quadrupole).

In the thin lens limit (1), the transfer matrix for a quadrupole becomes:

$$R_Q(f) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/f & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Recall the transfer matrix for a drift space of length L:

$$R_D(L) = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Using the transfer matrices (2) and (3), we can find the transfer matrix for a whole FODO cell. For the horizontally focusing quadrupole, we write:

$$f = 2f_0 \tag{4}$$

since the cell starts half-way through a horizontally focusing quadrupole; for half a quadrupole, the focal length is twice that for a full quadrupole. For the vertically focusing quadrupole:

$$f = -f_0 \tag{5}$$

Note that as we move to the *right* along the beamline, the matrices get added to the *left* of the matrix product: the matrices therefore appear in the reverse order of the elements in the beamline (though in this case, the product is symmetric). The total transfer matrix is:

$$R = R_Q\left(2f_0\right) \cdot R_D(L) \cdot R_Q\left(-f_0\right) \cdot R_D(L) \cdot R_Q\left(2f_0\right) \tag{6}$$

Performing the matrix multiplications in (6), we find the transfer matrix for a full FODO cell:

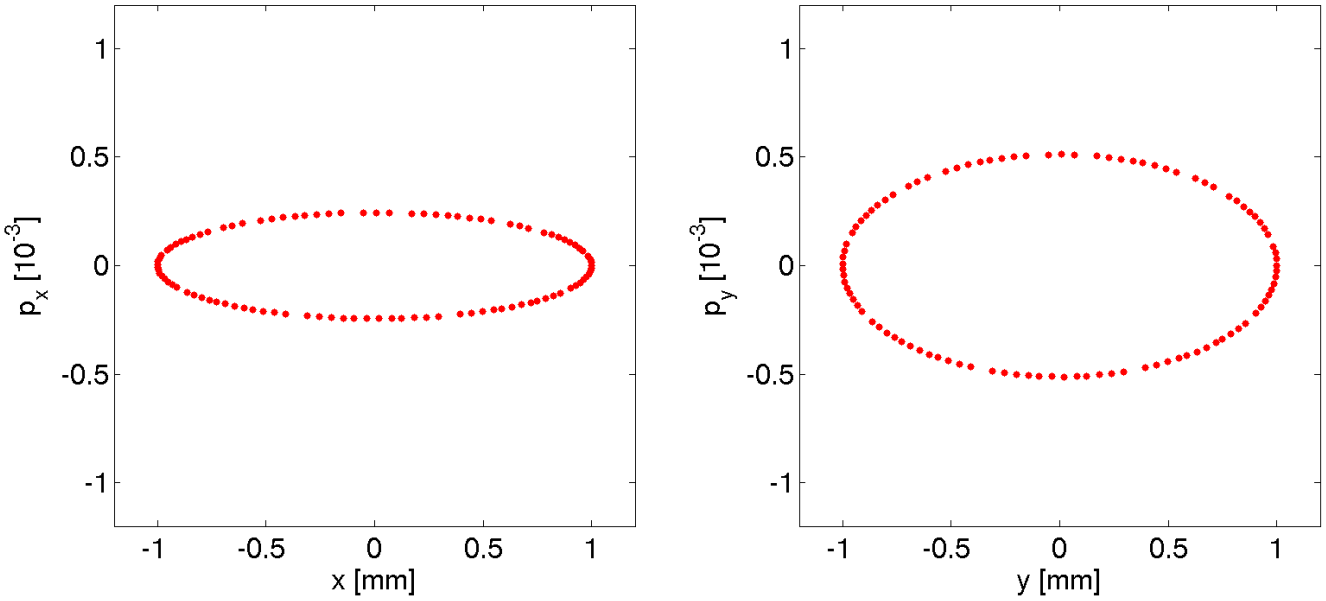
$$R = \begin{pmatrix} 1-\frac{L^2}{2f_0^2} & \frac{L}{f_0}(L+2f_0) & 0 & 0 & 0 & 0 \\ \frac{L}{4f_0^3}(L-2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f_0^2} & -\frac{L}{f_0}(L-2f_0) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f_0^3}(L+2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

This already looks fairly complicated, even with the simplifying approximation of "thin" quadrupoles. What does it mean for the beamline optics?

Let us consider the case $L = 1\,\mathrm{m}$, $f_0 = \sqrt{2}\,\mathrm{m}$. Take a particle with initial coordinates at the start of a FODO cell:

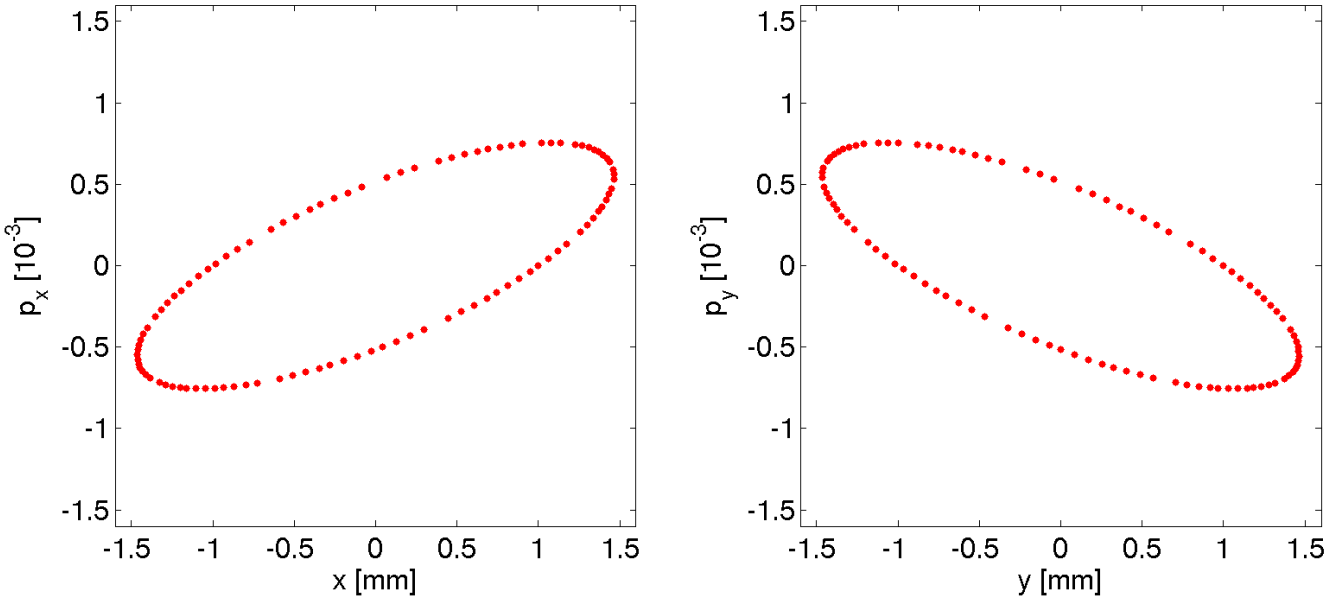
$$x = 1\,\mathrm{mm}, \quad p_x = 0, \quad y = 1\,\mathrm{mm}, \quad p_y = 0 \tag{8}$$

Now track the particle through 100 FODO cells by applying the transfer matrix (7) to the vector constructed from the coordinates, and plot p_x *vs* x, and p_y *vs* y:



The optics appear quite simple: the particle position in phase space at the exit of each FODO cells describes an ellipse in each (transverse) plane in phase space.

What happens if we repeat the exercise, but starting the FODO cell at the center of the drift before the (horizontally) defocusing quadrupole? Again, we plot ellipses, but this time, they are tilted:



Why do we get ellipses in phase space when we plot the coordinates of the particle after each FODO cell?

To understand this, start by recalling that the transfer matrices are all symplectic. So the transfer matrix for the FODO cell obeys the symplectic constraint:

$$R^{\mathsf{T}} \cdot S \cdot R = S \tag{9}$$

where S is the antisymmetric matrix:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{10}$$

Since S is block diagonal, and (for the special cases we are considering) the transfer matrix is also block diagonal, we can consider just one of the degrees of freedom.

The 2×2 transfer matrix for the horizontal motion can be written, quite generally:

$$R_2 = I_2 \cos \mu_x + S_2 \cdot A_2 \sin \mu_x \tag{11}$$

where I_2 is the 2×2 identity matrix, μ_x is a parameter, S_2 is one of the 2×2 block diagonals of S, and A_2 is a 2×2 symmetric matrix.

Note that the right-hand side of (11) has four degrees of freedom, corresponding to the three independent components of the symmetric matrix A_2, and the parameter μ_x. The values of the components of A_2 and the parameter μ_x can be determined from the values of the four components of R_2.

Considering motion in the horizontal plane, let us write:

$$A_2 = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix} \tag{12}$$

The symplectic constraint:

$$R_2^{\mathsf{T}} \cdot S_2 \cdot R_2 = S_2 \tag{13}$$

becomes, applied to the right hand side of (11):

$$\beta_x \gamma_x - \alpha_x^2 = 1 \tag{14}$$

If R_2 is symplectic, we find from (14):

$$R_2^{\mathsf{T}} \cdot A_2 \cdot R_2 = A_2 \tag{15}$$

In other words, the matrix A_2 is invariant under a transformation representing the transport through a FODO cell.

Invariants of transformations are interesting and useful. In this case, it follows immediately from (15) that if we construct the quantity J_x:

$$J_x = \frac{1}{2}\begin{pmatrix} x & p_x \end{pmatrix} \cdot A_2 \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{16}$$

then under a transformation representing transport through a FODO cell:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto R_2 \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{17}$$

and:

$$J_x \mapsto \frac{1}{2}\begin{pmatrix} x & p_x \end{pmatrix} \cdot R_2^{\mathsf{T}} \cdot A_2 \cdot R_2 \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} = J_x \tag{18}$$

In other words, J_x is also invariant under transport through the FODO cell. Notice that J_x can be written:

$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2\right) \tag{19}$$

Recall that the components of A_2 are functions only of the components of the transfer matrix for a complete periodic cell. In other words, the parameters α_x, β_x and γ_x characterise the *beamline* rather than the dynamics of a particular particle.

α_x, β_x and γ_x are usually called the *Twiss parameters*. Of course, there are corresponding Twiss parameters for the vertical and longitudinal motion.

The Twiss parameters can be calculated at any point in a periodic, uncoupled linear beamline from the transfer matrix for one period, starting at the chosen point.

From the Twiss parameters and the phase space coordinates of a particle at a given point, we can calculate the value of J_x (and corresponding vertical and longitudinal quantities) for the particle.

Note that the *motion* of a particle when expressed in cartesian coordinates (x, p_x) is in general *not* periodic with the lattice; in other words, after each cell, (x, p_x) are different from their values at the start of the cell.

However, the Twiss parameters *are* periodic with the lattice. Their values are derived from the transfer matrix for a single cell, and they are uniquely defined at any point in the cell. The Twiss parameters have the same values at the corresponding point in each of the cells in the beamline.

Recall equation (19):

$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2\right) \tag{20}$$

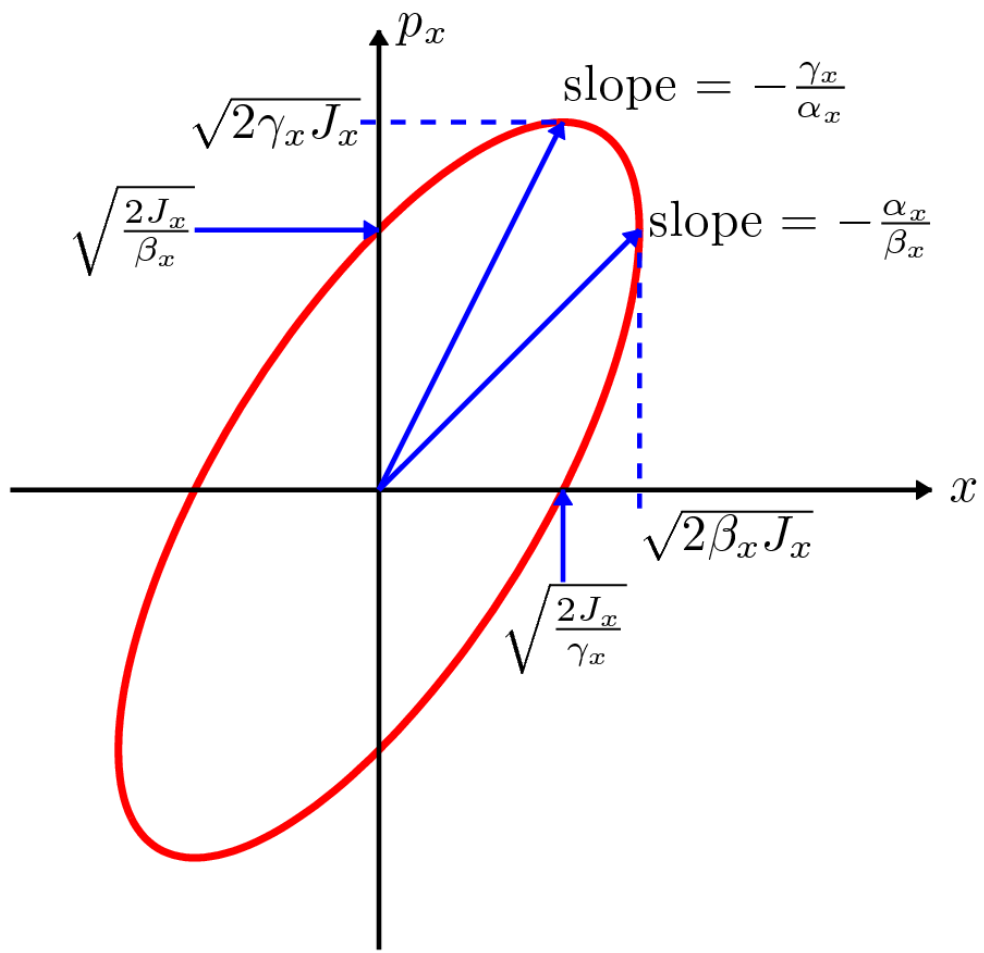
J_x defined in this way is usually called the *action* of a particle. If β_x and γ_x are both positive (or both negative), then the equation:

$$J_x = \text{constant} \tag{21}$$

defines an ellipse in phase space with area $2\pi J_x$. Since the action J_x for a given particle is the same at the exit of every periodic cell, the phase space coordinates must lie on an ellipse with area $2\pi J_x$. This explains the ellipses that we saw earlier, when we tracked through a FODO lattice.

The *shape* of the ellipse is determined by the values of the Twiss parameters...

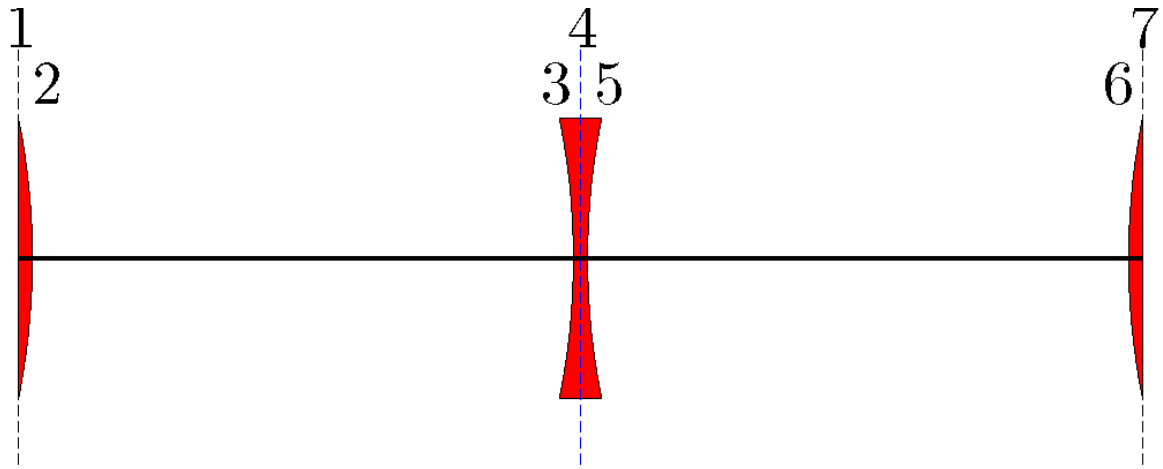
$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2\right) \qquad \text{Area} = 2\pi J_x \qquad (22)$$

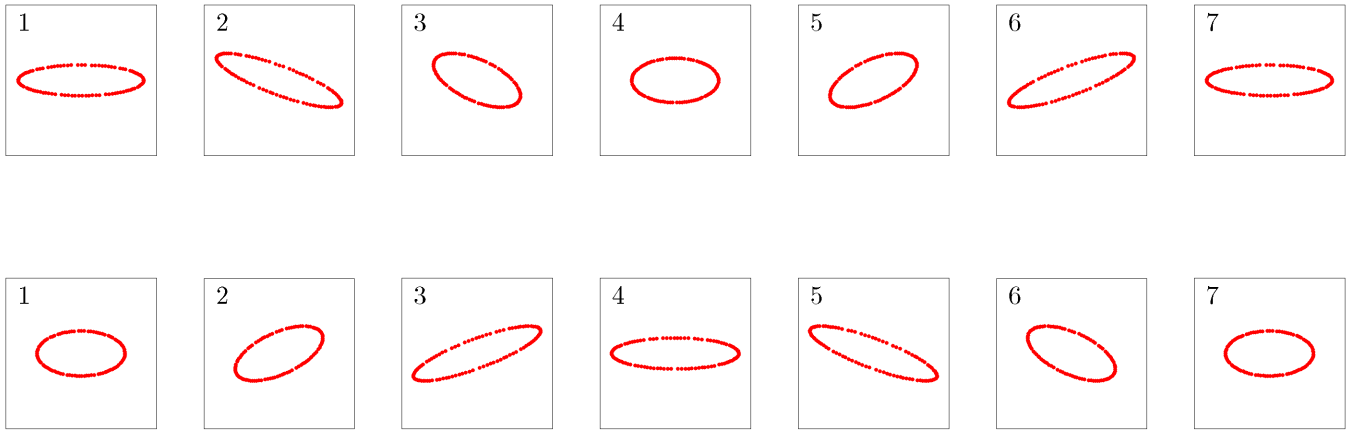


Consider a set of particles distributed around a phase space ellipse with a given action. As we move along the beamline, the shape of the phase space ellipse changes with the same periodicity as the beamline. At any given point in the beamline, the shape of the ellipse is given by the appropriate Twiss parameters.

Since the transport is symplectic, Liouville's theorem applies, which tells us that the area of phase space bounded by the ellipse remains constant. But the action is proportional to this area: therefore, the action of any particle is conserved as the particle moves along a beamline.

Evolution of the Phase Space Ellipse in a FODO Cell





Horizontal (top) and vertical (bottom) phase space through a FODO cell.

Sometimes you see the quantity ϵ_x defined as:

$$\epsilon_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$

ϵ_x defined in this way is called the "Courant-Snyder invariant", or the "single-particle emittance"; it looks (to within a factor of two) just like the action. In my humble opinion, the phrase "single-particle emittance" should be banned: it is a recipe for confusion and ensuing disaster when we come to talk about the emittance of a bunch consisting of a large number of particles. Reference to the "Courant-Snyder invariant" is (just) acceptable, but is really not necessary since we have a perfectly well-defined quantity (the action, J_x) that, conceptually, is more meaningful: the action is a dynamical variable conjugate to the angle ϕ_x.

Any further mention of "single-particle emittance" in this course is absolutely prohibited.

The fact that the action J_x is constant gives us an easy way to work out how the Twiss parameters evolve along a beamline. Write the action at a point s_0 along the beamline:

$$J_x(s_0) = \frac{1}{2}\begin{pmatrix} x_0 & p_{x0} \end{pmatrix} \cdot A_2(s_0) \cdot \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} \tag{23}$$

and at some other point s_1:

$$J_x(s_1) = \frac{1}{2}\begin{pmatrix} x_1 & p_{x1} \end{pmatrix} \cdot A_2(s_1) \cdot \begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} \tag{24}$$

If the transfer matrix from s_0 to s_1 is $R_2(s_0, s_1)$, then

$$\begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} = R_2(s_0, s_1) \cdot \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} \tag{25}$$

Since the action is conserved, $J_x(s_1) = J_x(s_0)$, and it follows from (23), (24) and (25) that:

$$A_2(s_0) = R_2^{\mathsf{T}}(s_0, s_1) \cdot A_2(s_1) \cdot R_2(s_0, s_1) \tag{26}$$

Inverting equation (26) we find:

$$A_2^{-1}(s_1) = R_2(s_0, s_1) \cdot A_2^{-1}(s_0) \cdot R_2^{\mathsf{T}}(s_0, s_1) \tag{27}$$

where

$$A_2(s) = \begin{pmatrix} \gamma_x(s) & \alpha_x(s) \\ \alpha_x(s) & \beta_x(s) \end{pmatrix}, \qquad A_2^{-1}(s) = \begin{pmatrix} \beta_x(s) & -\alpha_x(s) \\ -\alpha_x(s) & \gamma_x(s) \end{pmatrix} \tag{28}$$

Equations (27) and (28) give us an easy way to propagate the Twiss parameters along a beamline, if we know the values of the Twiss parameters at the start of the beamline, $A_2^{-1}(s_0)$, and the transfer matrix $R_2(s_0, s_1)$ from the start to the point of interest.

To specify the location of a particle in phase space, we can give the coordinate x and the conjugate momentum p_x. Alternatively, we can give the action J_x and the position or *angle* around the phase space ellipse. The angle ϕ_x may be defined so that:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x \tag{29}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\phi_x + \alpha_x\cos\phi_x\right) \tag{30}$$

In other words:

$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2\right) \tag{31}$$

$$\tan\phi_x = -\alpha_x - \beta_x\frac{p_x}{x} \tag{32}$$

With the definitions (32), it turns out that the variables (ϕ_x, J_x) form a conjugate pair of canonical variables (with ϕ_x the coordinate, and J_x the conjugate momentum). This means that the equations of motion written in terms of action-angle variables must be derived from a Hamiltonian, and that a dynamical map written in terms of action-angle variables must be symplectic.

One way to see that (ϕ_x, J_x) form a canonical pair, is to derive them as new variables from the generating function:

$$F_1 = F_1(x, \phi_x) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x) \tag{33}$$

We have derived new variables (ϕ_x, J_x) for describing the dynamics; what does the Hamiltonian look like when expressed in the new action-angle variables?

First, we note that since J_x is constant:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x} = 0 \tag{34}$$

the Hamiltonian must be *independent* of ϕ_x.

Second, we recall that the Hamiltonian (in the paraxial approximation) is second-order in the variables (x, p_x); therefore, the Hamiltonian must be *linear* in J_x. Since the Hamiltonian generally involves a momentum term $p_x^2/2$, we deduce:

$$H = \frac{J_x}{\beta_x} \tag{35}$$

The Hamiltonian in action-angle variables takes a particularly simple form (35):

$$H = \frac{J_x}{\beta_x} \tag{36}$$

From Hamilton's equations, the equations of motion are:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x} = 0 \tag{37}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x} = \frac{1}{\beta_x} \tag{38}$$

Note that equation (38) tells us that the rate of change of the angle variable ϕ_x along the beamline is proportional to the reciprocal of the beta function β_x.

In an *uncoupled* beamline there is a relationship between the Twiss alpha and beta functions that is often useful. For such beamlines, we can write:

$$p_x = \frac{dx}{ds} \tag{39}$$

It then follows from (29), (30) and (38) that:

$$\alpha_x = -\frac{1}{2}\frac{d\beta_x}{ds} \tag{40}$$

In other words, the Twiss alpha function is related to the rate of change of the Twiss beta function along the beamline. In particular, we expect α_x to be zero at locations in the beamline where β_x reaches a local maximum or minimum.

The equations of motion look particularly simple in action-angle variables. It is important to remember that we haven't "thrown away" all the complex dynamics associated with the electromagnetic fields: but instead of appearing explicitly in the equations of motion, the fields are implicitly contained in the beta function β_x.

Recall that β_x is a function of position s along the beamline. Thus, the Hamiltonian (35) is explicitly dependent on the path length (independent variable) s. This is what we should expect: after all, the Hamiltonian in cartesian variables (x, p_x) is also a function of s, since it changes when we go from a drift space into a magnet etc.

The slightly surprising thing is that in action-angle variables, the Hamiltonian is a function of position even within an element of a single type, e.g. a drift space.

The *change* in the angle variable when we move from one point to another is sometimes called the *phase advance*. From the equation of motion (38), we see that the phase advance from a point s_1 to a point s_2 is given by:

$$\Delta\phi_x = \int_{s_1}^{s_2} \frac{1}{\beta_x} ds \tag{41}$$

Of course, there is also a vertical and a longitudinal phase advance.

In a periodic lattice, the beta function is uniquely defined by the periodic condition. Since the phase advance is an integral over the (reciprocal of the) beta function, we see that the phase advance over a single cell is the same, no matter what starting point we choose for the cell.

Strictly speaking, the above analysis applies only to linear beamlines, where there is no coupling between transverse and longitudinal motion. However, if we consider only particles that have zero energy deviation, then it may be applied also to beamlines that include bends (dipole magnets).

In a storage ring, the *betatron tunes* ν_x and ν_y are defined as the total phase advance in the horizontal and vertical planes respectively, for one complete pass round the ring, divided by 2π. Thus, the horizontal tune is given by:

$$\nu_x = \frac{\Delta\phi_x}{2\pi} = \frac{1}{2\pi}\int_0^{C_0} \frac{1}{\beta_x} ds \tag{42}$$

where C_0 is the path length for one complete turn around the ring, following the reference trajectory.

From equation (29), we can interpret the betatron tune as the number of betatron oscillations made in one complete turn around the ring. The betatron tunes are critical parameters for design and operation of storage rings, and, as we shall see later, there are practical constraints on the values the tunes should take.

The actual values of the betatron tunes for a storage ring are sometimes known as the *working point in tune space*.

Finally, we consider what the transfer matrix (11) looks like in action-angle variables. To begin with, we write the transformation between cartesian and action-angle variables (29) and (30) in the form:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} = N^{-1} \cdot \vec{J}_x \tag{43}$$

where:

$$\vec{J}_x = \begin{pmatrix} \sqrt{2J_x}\cos\phi_x \\ -\sqrt{2J_x}\sin\phi_x \end{pmatrix} \tag{44}$$

and:

$$N^{-1} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix}, \quad \text{or} \quad N = \begin{pmatrix} \frac{1}{\sqrt{\beta_x}} & 0 \\ \frac{\alpha_x}{\sqrt{\beta_x}} & \sqrt{\beta_x} \end{pmatrix} \tag{45}$$

Now, under the transformation:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto R_2 \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{46}$$

the vector $\vec{J}_x$ transforms as:

$$\vec{J}_x \mapsto N \cdot R_2 \cdot N^{-1} \vec{J}_x \tag{47}$$

From (11), (12) and (45) we find that:

$$N \cdot R_2 \cdot N^{-1} = \begin{pmatrix} \cos \mu_x & \sin \mu_x \\ -\sin \mu_x & \cos \mu_x \end{pmatrix} \tag{48}$$

In other words, in action-angle variables, the transfer matrix for one periodic cell simply appears as a rotation. Finally, we note:

$$\begin{pmatrix} \cos \mu_x & \sin \mu_x \\ -\sin \mu_x & \cos \mu_x \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2J_x} \cos \phi_x \\ -\sqrt{2J_x} \sin \phi_x \end{pmatrix} = \begin{pmatrix} \sqrt{2J_x} \cos(\phi_x + \mu_x) \\ -\sqrt{2J_x} \sin(\phi_x + \mu_x) \end{pmatrix} \tag{49}$$

Hence, the parameter μ_x in the transfer matrix (11) is simply the phase advance.

To finish, let us calculate the phase advance and Twiss parameters in a FODO cell. We shall take the start of the cell to be the center of the horizontally focusing quadrupole. The transfer matrix is written in general form, from (11) and (12):

$$R_2 = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \tag{50}$$

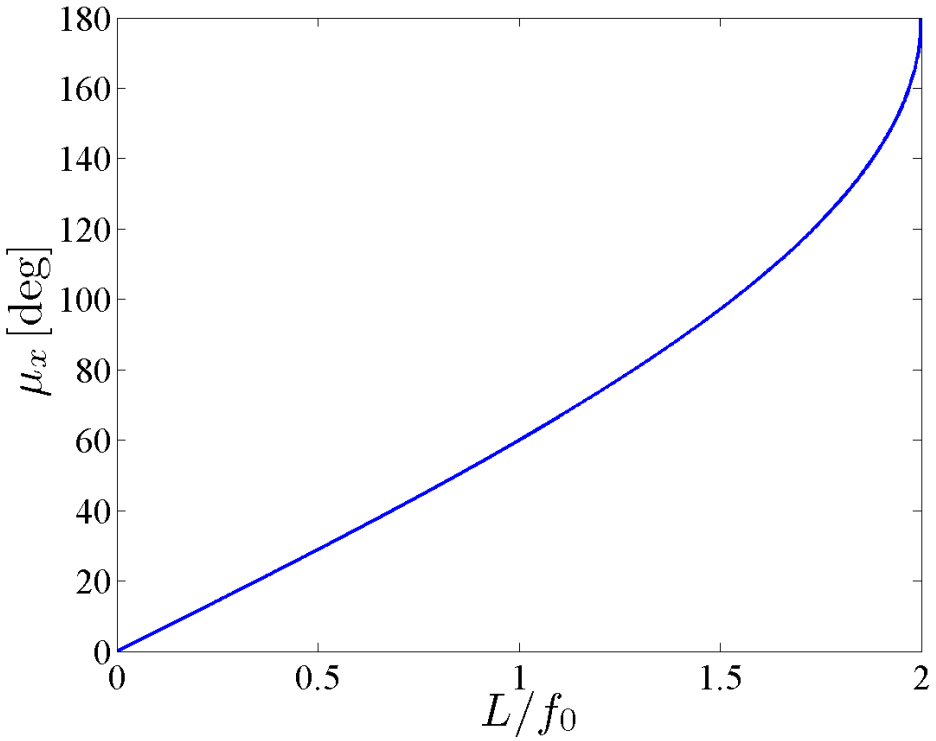
For a FODO cell, the horizontal part of the transfer matrix is, from (7):

$$R_2 = \begin{pmatrix} 1 - \frac{L^2}{2f_0^2} & \frac{L}{f_0}(L + 2f_0) \\ \frac{L}{4f_0^3}(L - 2f_0) & 1 - \frac{L^2}{2f_0^2} \end{pmatrix} \tag{51}$$

Equating the two forms of the transfer matrix (50) and (51), we find that the phase advance across the cell is given by:

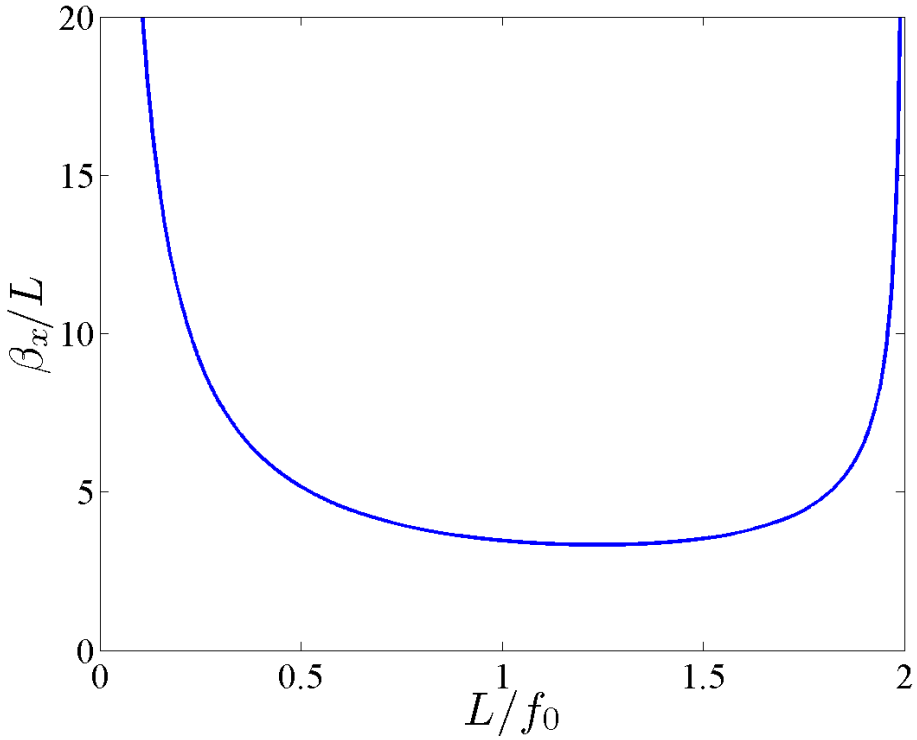
$$\cos\mu_x = 1 - \frac{L^2}{2f_0^2} \tag{52}$$

Note that if $L/f_0 > 2$, the cell is *unstable*.



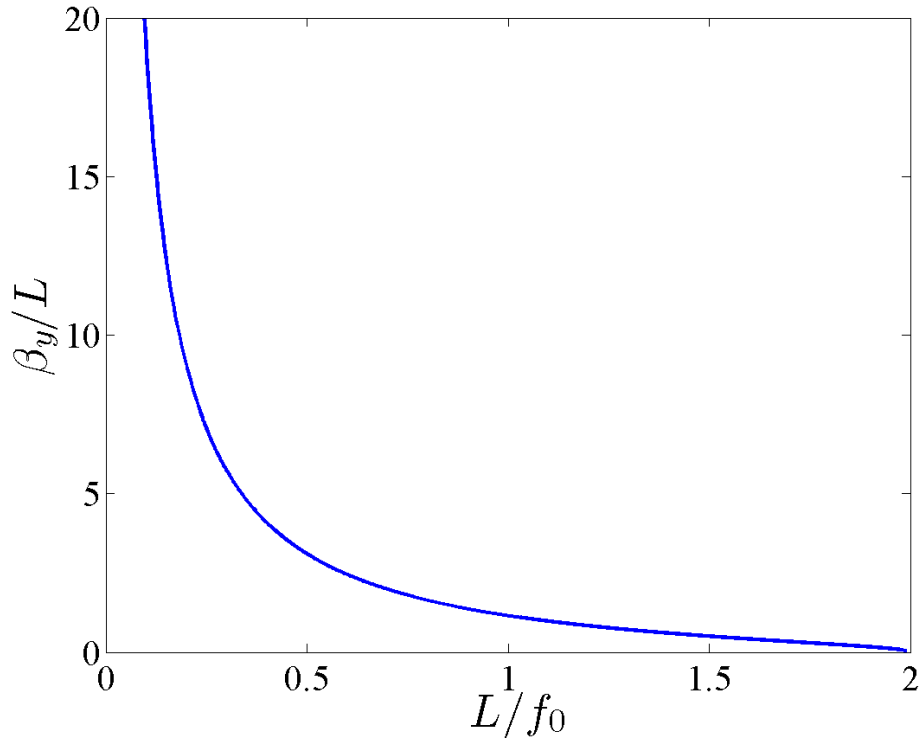
Again equating the two forms of the transfer matrix (50) and (51), we find that the horizontal alpha and beta functions at the start of the cell are given by:

$$\alpha_x = 0, \qquad \beta_x = \frac{2f_0(2f_0 + L)}{\sqrt{4f_0^2 - L^2}} \tag{53}$$



Finally, we find that the vertical alpha and beta functions at the start of the cell (center of the horizontally focusing quadrupole) are given by:

$$\alpha_y = 0, \qquad \beta_y = \frac{2f_0(2f_0 - L)}{\sqrt{4f_0^2 - L^2}} \tag{54}$$

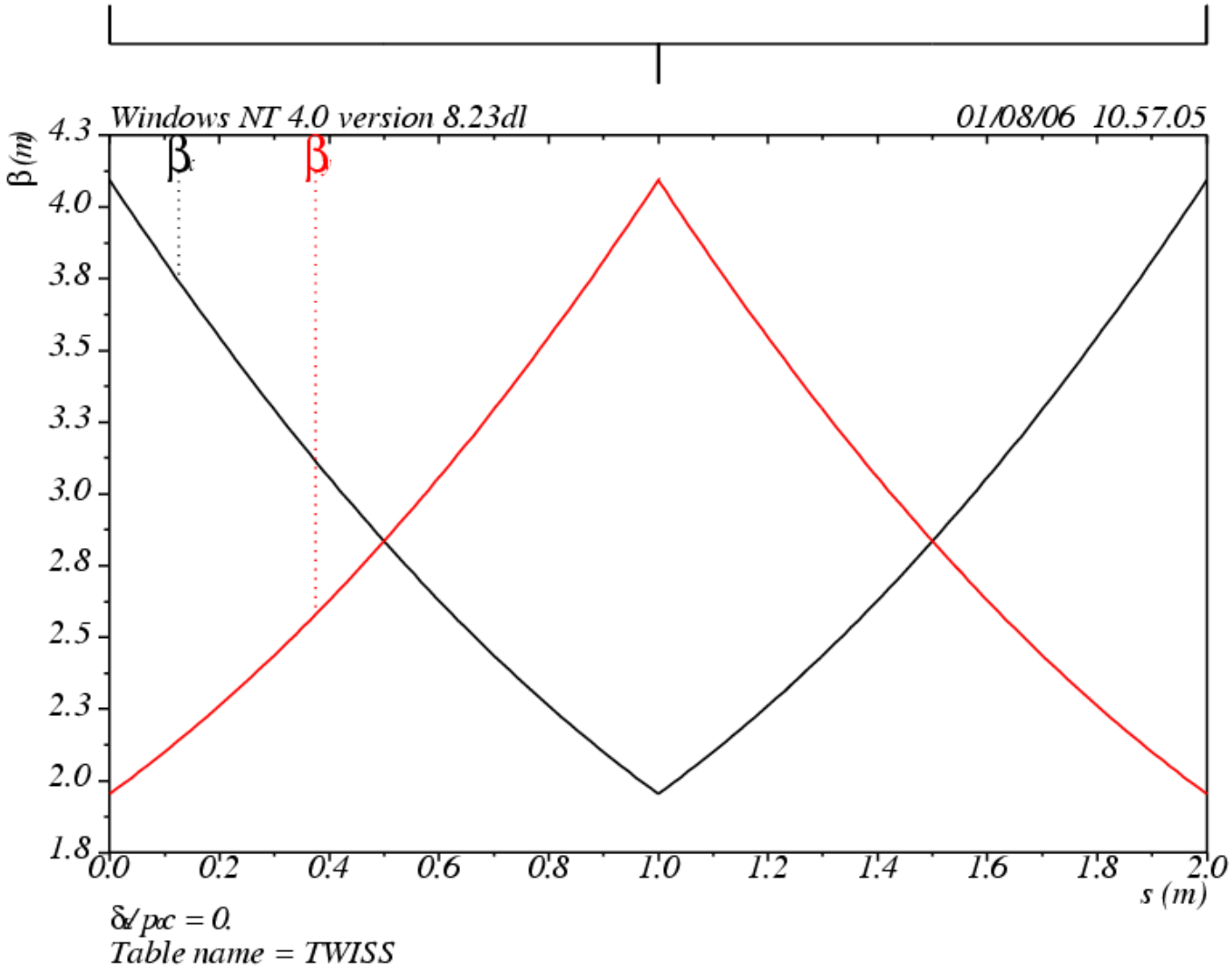


Of real interest is the variation in the Twiss parameters throughout a cell.

The Twiss parameters at the start of the cell can be calculated from the transfer matrix for the cell using equation (50), and can then be propagated along the cell using equations (27).

Generally, a lot of matrix multiplication is involved. Fortunately, there are many computer codes that will do this for us, for example, MAD (Methodical Accelerator Design)...

Twiss Parameters in a FODO Cell



A particle passing along a FODO beamline maps out an ellipse in phase space when observed at a given point in each successive FODO cell. The shape of the ellipse is described by the *Twiss parameters*, which change from point to point along the cell. The area of the ellipse is an invariant of the particle motion, equal to $2\pi J$, where J is the *action*.

The *beta function* (one of the Twiss parameters) describes the local amplitude of oscillation of the oscillations around the reference trajectory.

The angle through which the particle advances around the phase space ellipse in each FODO cell is called the *phase advance* of the cell.

The transfer matrix for a FODO cell may be expressed in terms of the Twiss parameters at the given point in the cell, and the phase advance.