Single-Particle Linear Dynamics
Lecture 4

Transfer Maps for "Linear" Elements

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April 2022

In the previous lecture, we derived a Hamiltonian for the motion of a particle through an accelerator.

This "accelerator Hamiltonian" can be applied for a general electromagnetic field, allows a curved reference trajectory, and uses dynamical variables that remain small for particles following a trajectory close to the reference trajectory.

We applied the accelerator Hamiltonian to the case of a dipole magnet. To obtain a linear transfer map, we made an approximation by expanding the Hamiltonian as a power series to second order in the dynamical variables.

There are several interesting effects arising from the Hamiltonian for a dipole magnet: these include dispersion (variation of the bending angle with the energy of the particle) and weak focusing.

Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

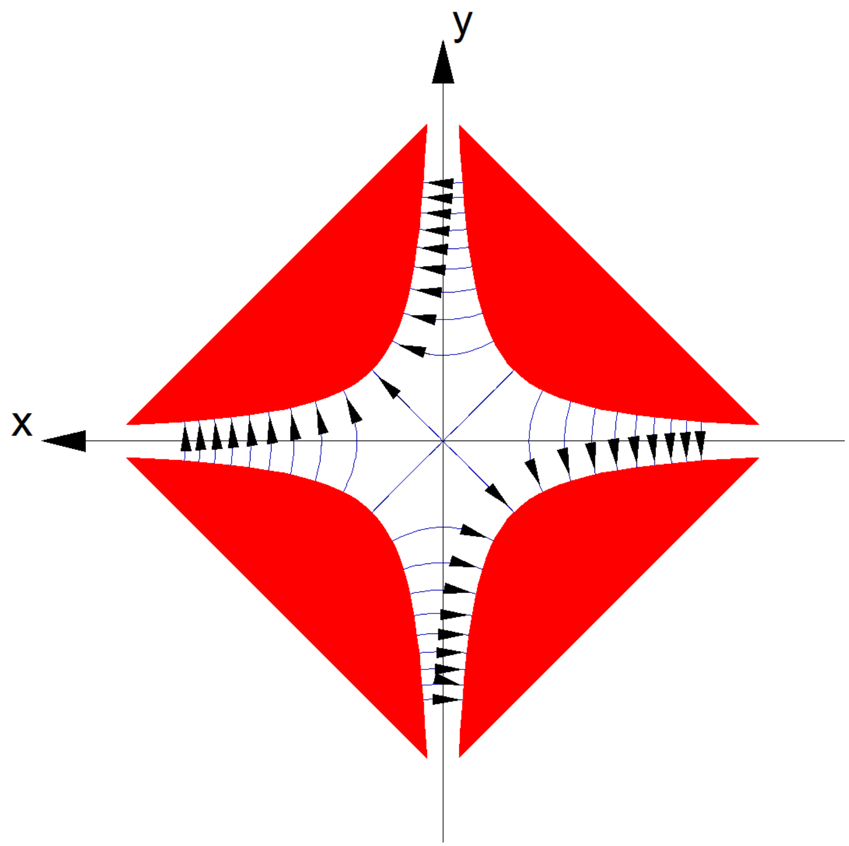
1. Review of Hamiltonian mechanics
2. The accelerator Hamiltonian in a straight coordinate system
3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates
4. Transfer maps for linear elements
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

In this lecture, we shall continue our derivation of transfer maps for "linear" beamline elements.

To the drift space and dipole, we shall add the quadrupole, the RF cavity, and the solenoid.

Note that all elements are in fact nonlinear. By "linear" elements, we mean those for which the principle effects on the beam may be obtained by expanding the Hamiltonian to second order in the dynamical variables. We shall make extensive use of this approximation - usually called the *paraxial approximation*.

Recall the magnetic field inside a normal quadrupole magnet:



$$B_x = b_2 \frac{y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}.$$

The field inside a normal quadrupole magnet in Cartesian coordinates may be written:

$$B_x = b_2 \frac{y}{r_0} \qquad B_y = b_2 \frac{x}{r_0} \qquad B_s = 0 \tag{1}$$

Note that on the axis of the quadrupole, the field strength is zero. Therefore, we can choose the reference trajectory to lie along the axis, in which case there is no curvature: we can work in a straight coordinate system.

The above field may be derived from the potential:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = -\frac{1}{2}\frac{b_2}{r_0}\left(x^2 - y^2\right) \tag{2}$$

The Hamiltonian describing the motion inside a quadrupole, using the usual accelerator variables, is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - a_s \tag{3}$$

The longitudinal component a_s of the normalised vector potential is:

$$a_s = q\frac{A_s}{P_0} = -\frac{1}{2}\frac{q}{P_0}\frac{b_2}{r_0}\left(x^2 - y^2\right) \tag{4}$$

where q is the charge on the particle, and P_0 is the reference momentum.

For convenience, we define the *normalised quadrupole gradient*:

$$k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \tag{5}$$

In terms of the normalised quadrupole gradient (5) the Hamiltonian can be written:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{2}k_1\left(x^2 - y^2\right) \qquad (6)$$

Expanding the Hamiltonian (6) to second order in the dynamical variables (making the paraxial approximation) we construct the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_1 x^2 - \frac{1}{2}k_1 y^2 + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \qquad (7)$$

Note that this looks very much like the harmonic oscillator equation: for $k_1 > 0$ we have a "focusing" potential in x, and a "defocusing" potential in y. In z there is no focusing.

Solving the equations of motion for the Hamiltonian (7) we find the transfer matrix for a quadrupole of length L ($k_1 > 0$):

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

where

$$\omega = \sqrt{k_1} \tag{9}$$

Note that the field, if *focusing* in x is *defocusing* in y, and vice-versa. This is a direct consequence of the constraints on the magnetic field from Maxwell's equations: it is not possible to build a quadrupole that focuses or defocuses in both transverse planes simultaneously.

A skew quadrupole is obtained from a normal quadrupole by rotating the magnet 90° about the magnetic axis.

The skew multipole field components are given by the c_n coefficients in the multipole expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \tag{10}$$

For a skew quadrupole, all coefficients are zero except for a_2:

$$B_x = a_2 \frac{x}{r_0} \qquad B_y = -a_2 \frac{y}{r_0} \tag{11}$$

The magnetic vector potential is given by:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = a_2 xy \tag{12}$$

If we define:

$$k_{1s} = -\frac{q}{P_0}\frac{a_2}{r_0} \tag{13}$$

where P_0 is the reference momentum, and r_0 is the reference radius of the magnet, then the normalised vector potential is:

$$a_s = -k_{1s}xy \tag{14}$$

and the Hamiltonian for a skew quadrupole is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + k_{1s}xy \tag{15}$$

Making the paraxial approximation, we find the second-order Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + k_{1s}xy + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \tag{16}$$

Note the term in xy: this leads to *coupling* of the horizontal and vertical motion. The skew quadrupole gives a horizontal kick proportional to the vertical offset of the particle, and vice-versa.

Hamilton's equations with the second-order skew quadrupole Hamiltonian (16) may be solved as for the normal quadrupole.

The resulting map is linear, and so it may be written as a transfer matrix, R (for $k_{1s} > 0$):

$$\begin{pmatrix} \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & 0 & 0 \\ -\frac{1}{2}\omega(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & -\frac{1}{2}\omega(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & 0 & 0 \\ \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & 0 & 0 \\ -\frac{\omega}{2}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & -\frac{\omega}{2}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

where

$$\omega = \sqrt{k_{1s}} \tag{18}$$

Now we know how to focus the beam horizontally (dipole, or quadrupole with $k_1 > 0$) and vertically (quadrupole with $k_1 < 0$).

But nothing we have seen so far produces any longitudinal focusing.

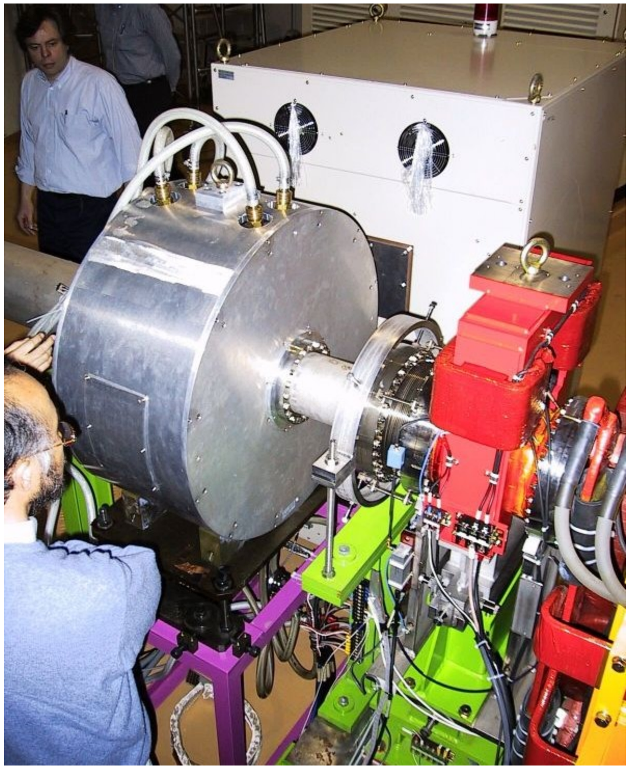
If we want to control the bunch size in all three dimensions, some kind of longitudinal focusing will be necessary.

This can be provided by an RF cavity.

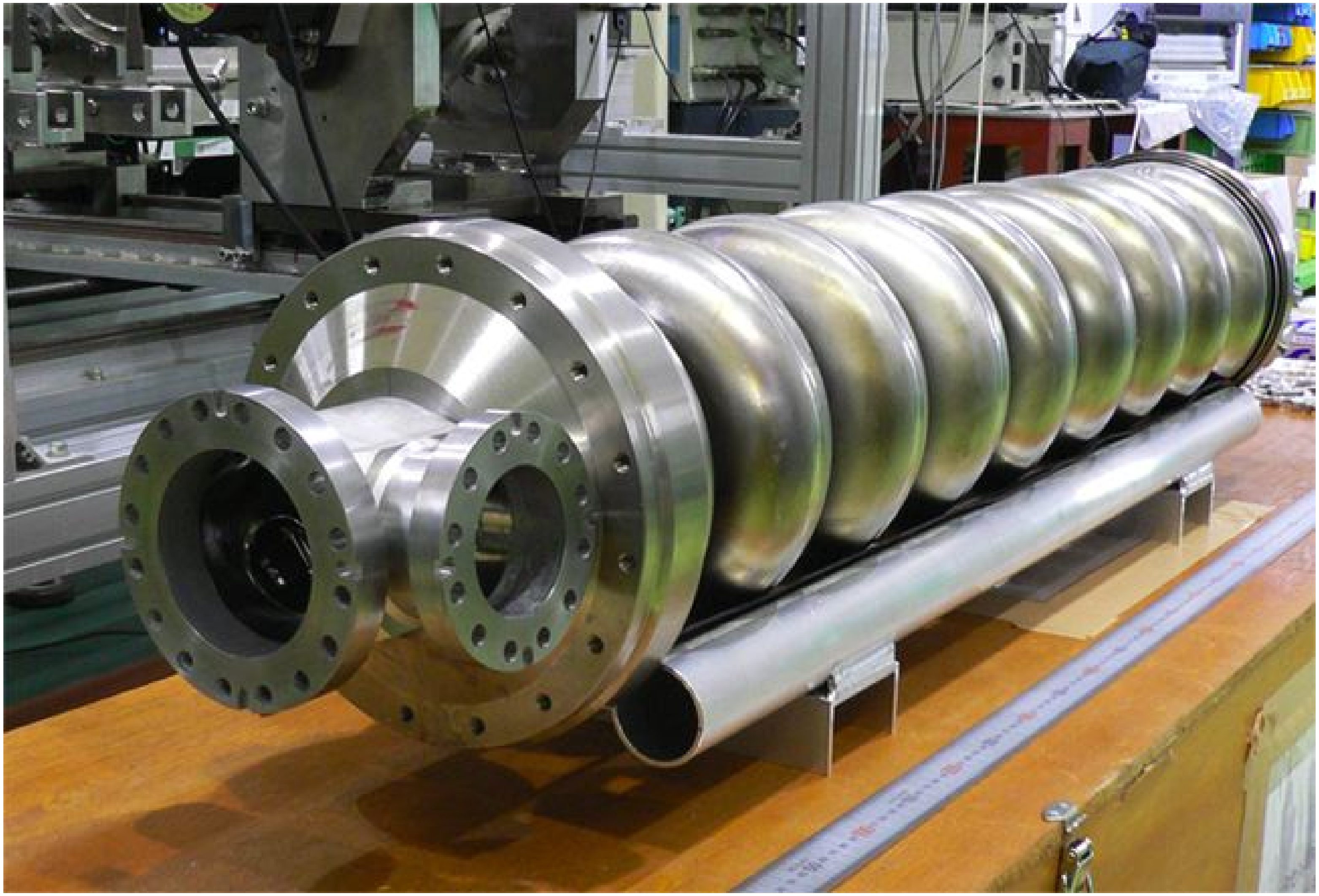
An RF cavity contains an electromagnetic field that has a sinusoidal dependence on time.

The dependence of the field strength on the spatial coordinates (x, y, s) is in general quite complicated, but in simple cases it can be broken down into a set of modes – just as the magnetic field in a multipole magnet can be broken down into a set of multipoles.

For the simplest RF cavity, we only need consider a single mode – the TM_{010} mode.



RF cavity.



Superconducting 9-cell RF cavity.

Let us assume a simple cylindrical RF cavity, with radius ρ_0.

In the TM_{010} mode in cylindrical cavity, the electric field has components in cylindrical coordinates:

$$E_\rho = 0 \qquad E_\phi = 0 \qquad E_s = \hat{E}_s J_0(k\rho)\sin(\omega_{\mathrm{RF}} t + \phi_0) \tag{19}$$

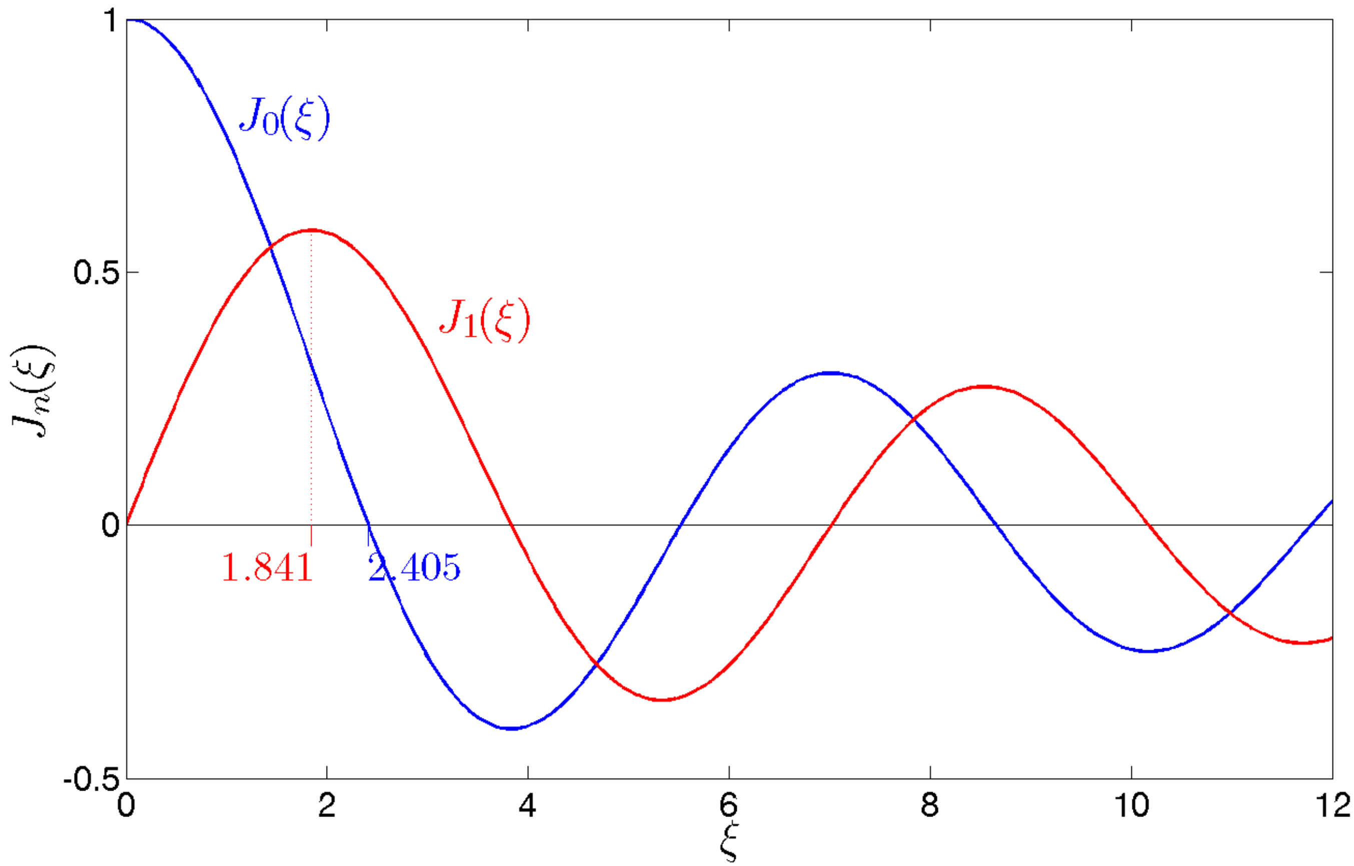
where $\rho = \sqrt{x^2 + y^2}$, ω_{RF} is the RF frequency, and ϕ_0 is an arbitrary phase, and J_0 is a Bessel function of the first kind.

The magnetic field is:

$$B_\rho = 0 \qquad B_\phi = \frac{k}{\omega}\hat{E}_s J_1(k\rho)\cos(\omega_{\mathrm{RF}} t + \phi_0) \qquad B_s = 0 \tag{20}$$

It can be shown that for $\omega_{\mathrm{RF}}/k = c$, the above fields satisfy Maxwell's equations, so they are valid electromagnetic fields.

We must also satsify the boundary conditions on the electric and magnetic fields at the walls of the cavity.



Bessel functions are solutions of the differential equation:

$$\xi^2 \frac{d^2 J_n}{d\xi^2} + \xi \frac{dJ_n}{d\xi} + (\xi^2 - n^2) J_n = 0 \tag{21}$$

for real n. Note that $J_0(\xi) = 0$ for $\xi \approx 2.405$.

If the cavity consists of a conducting cylinder of radius ρ_0 with axis along the reference trajectory, then the boundary conditions on the electric field require the longitudinal component E_s to vanish at $\rho = \rho_0$.

Hence, the frequency of the electromagnetic field in the cavity is determined by the cavity radius:

$$k\rho_0 \approx 2.405 \tag{22}$$

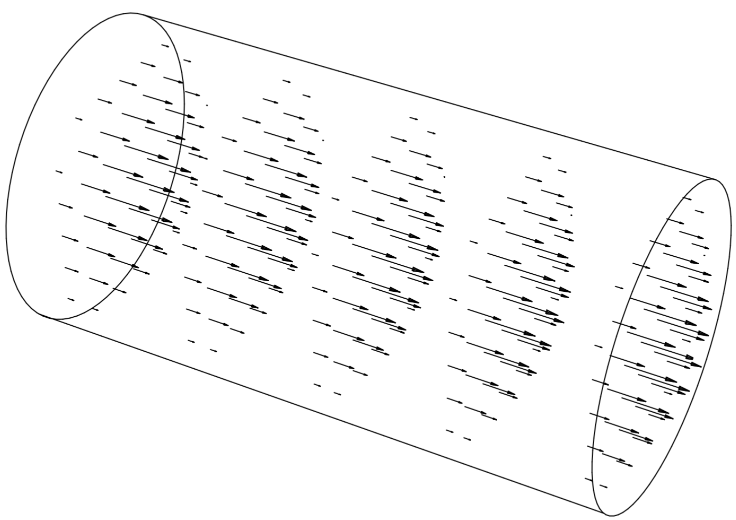
Since the function $J_0(\xi)$ has multiple zeroes, there are (infinitely) many other modes that may exist in the cavity.

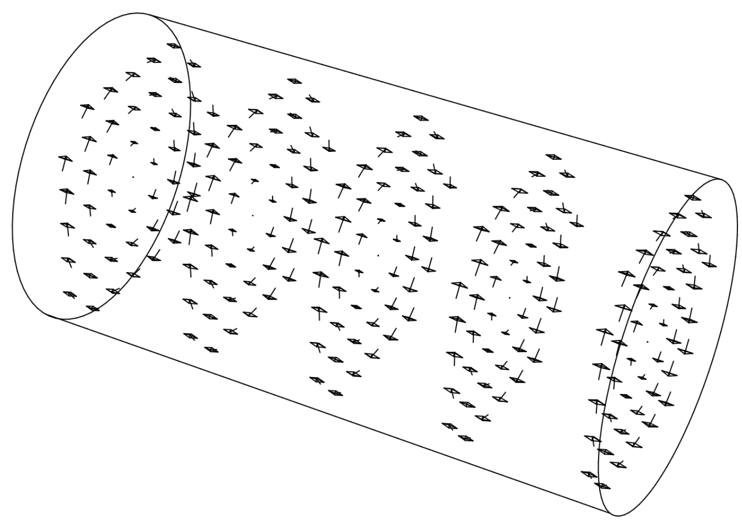
These *higher-order modes* have undesirable effects: significant efforts are made to suppress or "damp" higher-order modes in RF cavities in accelerators..

Note that if a particle is inside the cavity at $t = 0$ and the RF phase is $\phi_0 = 0$, then the particle is *accelerated* by the longitudinal electric field E_s.

The TM_{010} mode is sometimes called the *accelerating mode*.

Note also that only the magnetic field has a transverse component, but that the longitudinal component is zero: hence the name "TM" (for "transverse magnetic"). The mode numbers $(0, 1, 0)$ refer to the azimuthal, radial, and longitudinal directions, respectively.





The TM_{010} mode fields may be derived from the time-dependent magnetic vector potential:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = \frac{\hat{E}_s}{\omega} J_0(k\rho) \cos\left(\omega_{\mathrm{RF}} t + \phi_0\right) \tag{23}$$

In the accelerator Hamiltonian, we use the path length s as the independent variable, rather than the time t. The relationship between the two involves the dynamical variable z:

$$ct = \frac{s}{\beta_0} - z \tag{24}$$

Therefore, we can write the Hamiltonian in the TM_{010} mode:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - \frac{q}{P_0} \frac{\hat{E}_s}{\omega} J_0(k\rho) \cos\left(\frac{k}{\beta_0} s - kz + \phi_0\right) \tag{25}$$

where (for the fields to satisfy Maxwell's equations) $\omega_{\mathrm{RF}}/k = c$.

The Hamiltonian (25) has an unpleasant feature that we have so far managed to avoid: it has an explicit dependence on the independent variable s.

This is allowed, but in this case makes the equations of motion very difficult to solve, and the paraxial approximation does not get us out of trouble.

To simplify the problem, we therefore *average* the Hamiltonian in s over the length of the cavity:

$$\langle H \rangle = \frac{1}{L} \int_{-L/2}^{L/2} H ds \tag{26}$$

where L is the length of the cavity.

The fields we have written down in (19) and (20) have no dependence on s, so we can in principle make the cavity any length we like.

However, for technical reasons, it is usual to make the cavity length $L = \pi/k$, i.e. half the wavelength of radiation of frequency ω_{RF}.

Using $L = \pi/k$, we can perform the integral in (26) and we find:

$$\langle H \rangle = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} - \frac{\alpha}{\pi} J_0(k\rho)\cos(\phi_0 - kz) \tag{27}$$

where

$$\alpha = \pi \frac{q}{P_0} \frac{\widehat{E}_s}{\omega_{\mathrm{RF}}} T = \frac{q\widehat{E}_s L}{P_0 c} T \qquad \text{with} \qquad T = \frac{2\beta_0}{\pi} \sin\left(\frac{\pi}{2\beta_0}\right) \tag{28}$$

T is called the *transit time factor*

Normally, we define the *cavity voltage*, $\widehat{V}$ such that:

$$\frac{\widehat{V}}{L} = \widehat{E}_s T \tag{29}$$

so:

$$\alpha = \frac{q\widehat{V}}{P_0 c} \tag{30}$$

Making the paraxial approximation, we find the Hamiltonian:

$$\begin{aligned} \langle H_2 \rangle \; = \; & \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{\alpha}{4\pi}\cos(\phi_0)k^2\left(x^2+y^2\right) - \\ & \frac{\alpha}{\pi}\sin(\phi_0)kz + \frac{\alpha}{2\pi}\cos(\phi_0)k^2z^2 + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \end{aligned} \qquad (31)$$

As usual, we can understand some important properties of the dynamics by inspection of the Hamiltonian.

Note first the transverse focusing term: the cavity is focusing in both the horizontal plane and the vertical plane simultaneously.

This is something we could not achieve by the use of static magnetic fields. In the RF cavity, it arises from the azimuthal component of the magnetic field in the TM_{010} mode.

To make use of the transverse focusing, we have to choose a phase ϕ_0 close to zero.

For an RF cavity, the Hamiltonian in the paraxial approximation is (31):

$$\begin{aligned}\langle H_2 \rangle &= \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{\alpha}{4\pi}\cos(\phi_0)k^2\left(x^2+y^2\right) - \\ &\quad \frac{\alpha}{\pi}\sin(\phi_0)kz + \frac{\alpha}{2\pi}\cos(\phi_0)k^2z^2 + \frac{\delta^2}{2\beta_0^2\gamma_0^2}\end{aligned}$$

Note next the appearance of a term linear in z: this will result in a change in the energy deviation independent of z, as long as the phase $\phi_0 \neq 0$ (and $\phi_0 \neq \pi$). This is the term that describes the acceleration of the particle.

Finally, note the term quadratic in z: this is the longitudinal focusing we were looking for.

Solving the equations of motion in the transverse plane, we find that the solutions have zeroth-order as well as first-order terms:

$$\vec{x}(L) = R\vec{x}(0) + \vec{m} \tag{32}$$

The transfer matrix R is given by:

$$R = \begin{pmatrix} \cos(\psi_\perp) & \frac{L}{\psi_\perp}\sin(\psi_\perp) & 0 & 0 & 0 & 0 \\ -\frac{\psi_\perp}{L}\sin(\psi_\perp) & \cos(\psi_\perp) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\psi_\perp) & \frac{L}{\psi_\perp}\sin(\psi_\perp) & 0 & 0 \\ 0 & 0 & -\frac{\psi_\perp}{L}\sin(\psi_\perp) & \cos(\psi_\perp) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\psi_\parallel) & \frac{1}{\beta_0^2\gamma_0^2}\frac{L}{\psi_\parallel}\sin(\psi_\parallel) \\ 0 & 0 & 0 & 0 & -\beta_0^2\gamma_0^2\frac{\psi_\parallel}{L}\sin(\psi_\parallel) & \cos(\psi_\parallel) \end{pmatrix} \tag{33}$$

where:

$$\psi_\perp = \sqrt{\frac{\pi\alpha\cos(\phi_0)}{2}} \qquad \psi_\parallel = \frac{\sqrt{\pi\alpha\cos(\phi_0)}}{\gamma_0\beta_0} \tag{34}$$

The zeroth-order transverse terms in the solutions to the equations of motion are all identically zero. The zeroth-order longitudinal terms are:

$$m_z = \frac{2}{\pi} L \sin^2\left(\frac{\psi_\parallel}{2}\right) \tan(\phi_0) \tag{35}$$

$$m_\delta = \alpha \frac{\sin(\psi_\parallel)}{\psi_\parallel} \sin(\phi_0) \tag{36}$$

For small α (high energy particle in a cavity with a weak field), the map for the energy error δ becomes:

$$\Delta\delta \approx \frac{q\hat{V}}{P_0 c}\Big(\sin(\phi_0) - k z_0 \cos(\phi_0)\Big) \tag{37}$$

where $z_0 = z(0)$.

Solenoids are important components in accelerators. For example, detectors in colliders are often inside strong solenoids.

A solenoid has a uniform magnetic field in the longitudinal direction:

$$B_x = 0, \qquad B_y = 0, \qquad B_s = B_0. \tag{38}$$

It is not possible to derive this field from a vector potential having zero transverse components. A suitable potential is:

$$A_x = -\frac{1}{2}B_0 y, \qquad A_y = \frac{1}{2}B_0 x, \qquad A_s = 0. \tag{39}$$

This leads to the Hamiltonian:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - (p_x + k_s y)^2 - (p_y - k_s x)^2 - \frac{1}{\beta_0^2\gamma_0^2}} \tag{40}$$

where the normalised solenoid field strength k_s is given by:

$$k_s = \frac{1}{2}\frac{q}{P_0}B_0 \tag{41}$$

The fact that the vector potential has non-zero transverse components (unlike the other linear elements we have looked at) means that we have to be particularly careful with the meaning of the canonical momenta p_x and p_y.

But let us proceed with solving the equations of motion in the Hamiltonian (40), which we do by making the usual paraxial approximation:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_s^2 x^2 + \frac{1}{2}k_s^2 y^2 - \frac{1}{2}k_s x p_y + \frac{1}{2}k_s p_x y + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \quad (42)$$

Note the terms in x^2 and y^2: a solenoid provides horizontal and vertical focusing, rather than focusing in one plane and defocusing in the other.

Note also the coupling terms in xp_y and $p_x y$: motion lying initially in just one plane becomes (at least partially) transferred into the other plane.

We can solve the equations of motion derived from the Hamiltonian (42).

The resulting map can be expressed as a transfer matrix:

$$R = \begin{pmatrix} \cos^2(\omega L) & \frac{\sin(2\omega L)}{2\omega} & \frac{1}{2}\sin(2\omega L) & \frac{\sin^2(\omega L)}{\omega} & 0 & 0 \\ \frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) & -\omega\sin^2(\omega L) & \frac{1}{2}\sin(2\omega L) & 0 & 0 \\ -\frac{1}{2}\sin(2\omega L) & -\frac{\sin^2(\omega L)}{\omega} & \cos^2(\omega L) & \frac{\sin(2\omega L)}{2\omega} & 0 & 0 \\ \omega\sin^2(\omega L) & -\frac{1}{2}\sin(2\omega L) & -\frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

where:

$$\omega = k_s = \frac{1}{2}\frac{q}{P_0}B_0 \tag{44}$$

Multipole fields can be superposed on each other. This may help to reduce the length (and cost) of a beamline, but can also help to improve the dynamical properties of a lattice.

In the multipole field expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \tag{45}$$

superposed fields are described by having more than one non-zero coefficient b_n and/or a_n.

A magnet with superposed magnetic fields is generally called a "combined function" magnet.

Examples of combined function magnets widely used in accelerators are dipoles (bending magnet) with superposed quadrupole fields, and sextupoles with superposed skew quadrupole fields.

For linear dynamics, the most important combined function magnets are dipoles with superposed quadrupole fields.

In Cartesian coordinates, the field is:

$$B_y = b_1 + b_2 \frac{x}{r_0}, \quad B_x = b_2 \frac{y}{r_0}, \quad B_z = 0. \tag{46}$$

In bending magnets, we generally want to use a curved reference trajectory; however, using curvilinear coordinates complicates the description of the magnetic field in a combined function bend.

The magnetic field in a combined function bend may be derived from the vector potential:

$$A_x = 0 \tag{47}$$

$$A_y = 0 \tag{48}$$

$$\begin{aligned} A_s = & -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) \\ & -B_1\left(\frac{1}{2}\left(x^2 - y^2\right) - \frac{h}{6}x^3 + \frac{h^2}{24}\left(4x^4 - y^4\right) + \cdots\right) \end{aligned} \tag{49}$$

Note that the higher-order terms (x^3, x^4, y^4 etc.) arise from the curvature of the reference trajectory.

The higher-order terms are important for nonlinear dynamics, but do not contribute to the linear effects.

Using the vector potential (49) in the Hamiltonian, and making the paraxial approximation (expanding to second order) we have:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}(hk_0 + k_1)x^2 - \frac{1}{2}k_1 y^2 - \frac{h}{\beta_0}x\delta - \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{50}$$

where the normalised field strengths are defined as usual:

$$k_0 = \frac{q}{P_0}b_1, \qquad k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \tag{51}$$

The effect of the superposed gradient k_1 in the Hamiltonian is as expected: it simply provides additional transverse focusing.

Hamilton's equations with the Hamiltonian (50) can be solved.

In the horizontal plane, the solutions are:

$$x(s) = x(0)\cos(\omega_x s) + p_x(0)\frac{\sin(\omega_x s)}{\omega_x} + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{(1 - (\cos\omega_x s))}{\omega_x^2} \tag{52}$$

$$p_x(s) = -x(0)\omega_x \sin(\omega_x s) + p_x(0)\cos(\omega_x s) + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{\sin(\omega_x s)}{\omega_x} \tag{53}$$

where:

$$\omega_x = \sqrt{hk_0 + k_1} \tag{54}$$

In the vertical plane, the map for the combined function bend is:

$$y(s) = y(0)\cosh(\omega_y s) + p_y(0)\frac{\sinh(\omega_y s)}{\omega_y} \tag{55}$$

$$p_y(s) = y(0)\omega_y \sinh(\omega_y s) + p_y(0)\cosh(\omega_y s) \tag{56}$$

where

$$\omega_y = \sqrt{k_1} \tag{57}$$

The map in the vertical plane for a combined function bend is the same as for a quadrupole: the only focusing in the vertical plane comes from the quadrupole gradient.

In the longitudinal plane, the solutions are:

$$z(s) = z(0) - x(0)\frac{h}{\beta_0}\frac{\sin(\omega_x s)}{\omega_x} - p_x(0)\frac{h}{\beta_0}\frac{(1-\cos(\omega_x s))}{\omega_x^2} + \delta(0)\frac{s}{\beta_0^2\gamma_0^2}$$
$$- \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{h}{\beta_0}\frac{(\omega_x s - \sin(\omega_x s))}{\omega_x^3} \qquad (58)$$

$$\delta(s) = \delta(0) \qquad (59)$$

So far, we have only considered the dynamics of a particle *within* a given electromagnetic field: we have not thought about how to get particles in and out of the fields.

However, Maxwell's equations forbid abrupt changes in magnetic fields: there has to be some "transition region" within which there are non-zero fields that are not described by the usual multipole formulae.

The transition regions at either end of a magnet are usually called the "fringe fields".

Fringe fields have significant, and sometimes complicated, effects. For linear dynamics, the most important fringe fields are those at the ends of dipoles and solenoids.

Fringe fields at the ends of quadrupoles lead to (usually small) higher-order effects.

The precise effects of fringe fields depend on the design details of the magnet, e.g. the gap between the poles in a dipole.

To do things properly, one should construct the transfer map from a detailed field description. This often requires significant effort, and the techniques involved are beyond the scope of this course.

However, in many cases, we can make simple approximations that provide a good description of the gross effects.

These approximations are one of the topics covered in the next lecture.

We have now derived linear transfer maps for:

- separated and combined function dipoles
- solenoids
- normal and skew quadrupoles
- RF cavities

For each of these elements, we made the *paraxial approximation* by expanding the Hamiltonian to second order in the dynamical variables. This allowed us to find a linear map for each element. The linear map may be expressed as a transfer matrix.