The Cockcroft Institute of Accelerator Science and Technology

UNIVERSITY OF LIVERPOOL

Single-Particle Linear Dynamics
Lecture 3

The Accelerator Hamiltonian in a Curved Coordinate System

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In the previous lecture, we derived a Hamiltonian for the motion of a particle through an electromagnetic field, with dynamical variables appropriate for a particle accelerator.

The dynamical variables are defined so that for particles close to the reference trajectory, and with energy close to the reference energy, the values of the dynamical variables should remain small as the particle moves through the accelerator.

Since the dynamical variables take small values, we can make approximations to the Hamiltonian to construct linear maps.

We saw how the technique of approximating the Hamiltonian could be applied to find a symplectic map for a field-free region (a *drift space*).

So far we have assumed that the reference trajectory is a straight line.

Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

1. Review of Hamiltonian mechanics
2. The accelerator Hamiltonian in a straight coordinate system
3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates
4. Transfer maps for linear elements
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

In this lecture, we shall see how to modify the Hamiltonian to deal with cases where the reference trajectory is curved.

This will allow us to deal with dipole magnets, in which all charged particles follow curved paths.

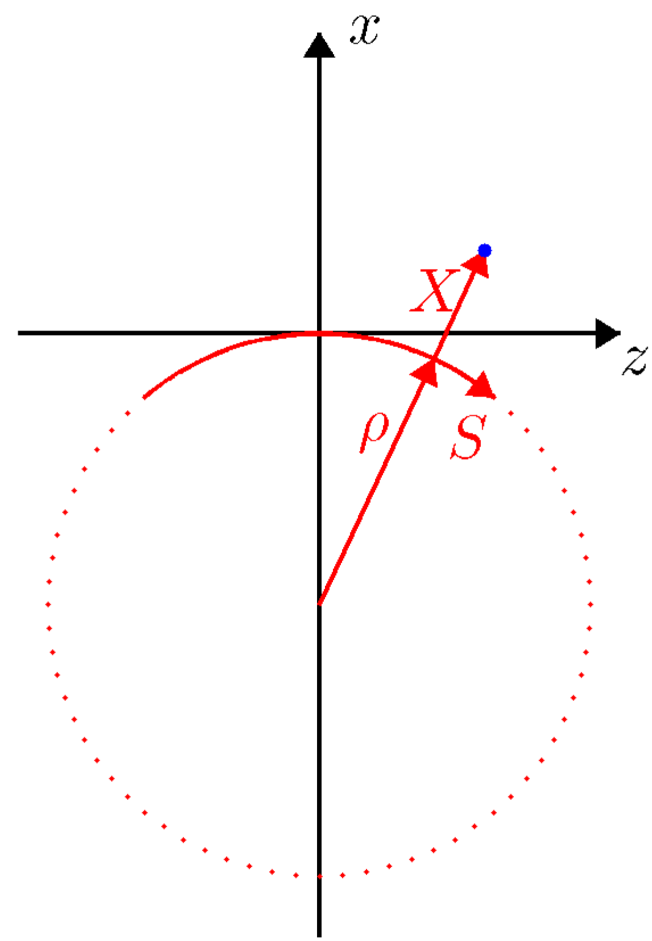
Using a curved reference trajectory in dipole magnets allows us to maintain small values for the dynamical variables, even where the deflection from the dipole is large. This means we can continue to use series expansion approximations for the Hamiltonian in such cases.

Ultimately, we shall derive the linear transfer map (transfer matrix) for a dipole.

The first step is to define some appropriate curvilinear coordinates.

The original (Cartesian) coordinates (x, y, z) are related to the new coordinates (X, Y, S) as follows:

$$
\begin{aligned}
x &= (\rho + X)\cos\left(\frac{S}{\rho}\right) - \rho && (1) \\
y &= Y && (2) \\
z &= (\rho + X)\sin\left(\frac{S}{\rho}\right) && (3)
\end{aligned}
$$



The required transformation of coordinates can be derived from a mixed-variable generating function.

Using the same mixed-variable generating function to derive the conjugate momenta in the new coordinate system ensures that the transformation is canonical.

An appropriate mixed-variable generating function is:

$$F_3(X, p_x, Y, p_y, S, p_z) = -\left[(\rho + X)\cos\left(\frac{S}{\rho}\right) - \rho\right] p_x - Y p_y - \left[(\rho + X)\sin\left(\frac{S}{\rho}\right)\right] p_z \tag{4}$$

The old and new coordinates and momenta are related by:

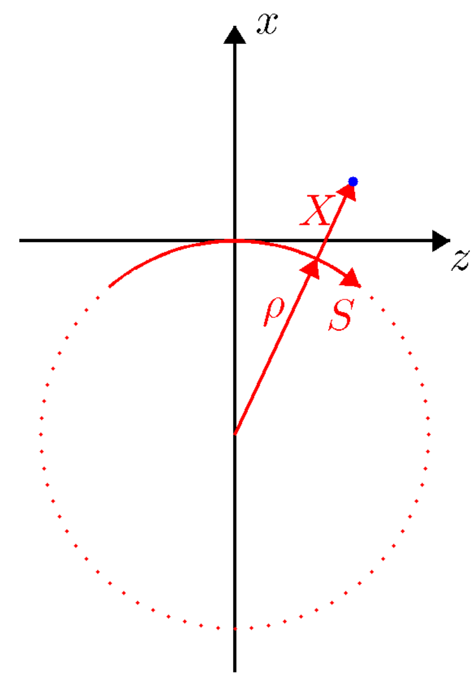
$$x_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial X_i} \tag{5}$$

The coordinates transform as required:

$$x = (\rho + X)\cos\left(\frac{S}{\rho}\right) - \rho \qquad (6)$$

$$y = Y \qquad (7)$$

$$z = (\rho + X)\sin\left(\frac{S}{\rho}\right) \qquad (8)$$



The new transverse momenta are given by:

$$P_X = p_x \cos\left(\frac{S}{\rho}\right) + p_z \sin\left(\frac{S}{\rho}\right) \qquad (9)$$

$$P_Y = p_y \qquad (10)$$

The curvature of the trajectory has a surprising effect on the longitudinal component of the momentum: P_S is *not* just the tangential component of the momentum in Cartesian coordinates!

$$P_S = p_z \left(1 + \frac{X}{\rho}\right) \cos\left(\frac{S}{\rho}\right) - p_x \left(1 + \frac{X}{\rho}\right) \sin\left(\frac{S}{\rho}\right) \tag{11}$$

To complete the transformation, we also need to express the components of the vector potential in the new coordinate system:

$$A_X = A_x \cos\left(\frac{S}{\rho}\right) - A_z \sin\left(\frac{S}{\rho}\right) \tag{12}$$

$$A_Y = A_y \tag{13}$$

$$A_S = A_z \cos\left(\frac{S}{\rho}\right) + A_x \sin\left(\frac{S}{\rho}\right) \tag{14}$$

What form does the Hamiltonian take in the new coordinate system?

Recall the general expression for the Hamiltonian for a relativistic particle in Cartesian coordinates, in an electromagnetic field:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{15}$$

The transformation into "accelerator variables" in a curvilinear coordinate system follows exactly the same lines as the transformations in a straight coordinate system...

...the only difference is that when we change the independent variable from t to s (and switch the Hamiltonian from H to $-P_S$), we pick up a factor $1 + x/\rho$ from equation (11).

The result – our final "Accelerator Hamiltonian" – is:

$$H = -(1+hx)\sqrt{\left(\frac{1}{\beta_0}+\delta-\frac{q\phi}{P_0c}\right)^2-(p_x-a_x)^2-(p_y-a_y)^2-\frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)\,a_s+\frac{\delta}{\beta_0} \tag{16}$$

where we have (as usual) renamed our variables so as to tidy up the notation; and we have defined the "curvature":

$$h = \frac{1}{\rho} \tag{17}$$

Note that, from the figures shown in the previous slides, the curvature h is *positive* for a bend moving towards the *negative* x direction. This is simply a convention.

We can now write down the equations of motion, with a curved reference trajectory, for a relativistic particle moving through any field for which we know the potentials ϕ and $\mathbf{a}$.

Before writing down and solving the equations of motion for a particle travelling through various kinds of magnet, RF cavity etc., we should know something about the fields generated by these devices.

Recall that the fields are the derivatives of the potentials:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{18}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{19}$$

Allowed physical fields must be solutions of Maxwell's equations...

James Clerk Maxwell, 1831-1879

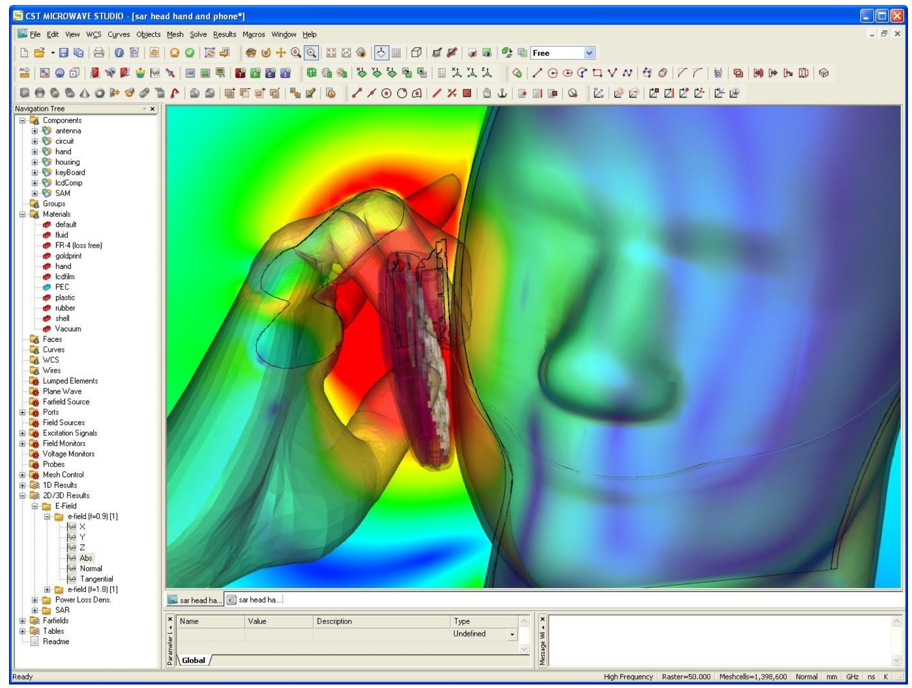
$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{20}$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

Finding solutions to Maxwell's equations for a given set of boundary conditions is in general no easy task.

Significant effort has been devoted to developing computer codes to solve this problem accurately and efficiently.



Such codes are often developed for commercial applications, but also have important applications in accelerator physics.

Fortunately, for linear beam dynamics, we are interested in a few simple cases: for understanding the basic properties of common accelerator components, we don't need to use "Maxwell solvers".

In particular, we note that we can write the field in a "long straight multipole" magnet as:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \tag{21}$$

where b_n and a_n are arbitrary coefficients (chosen to give the correct field map), and r_0 is an arbitrary "reference radius".

It is readily shown that the field of (21) satisfies Maxwell's equations (20).

The magnetic multipole field expansion is:

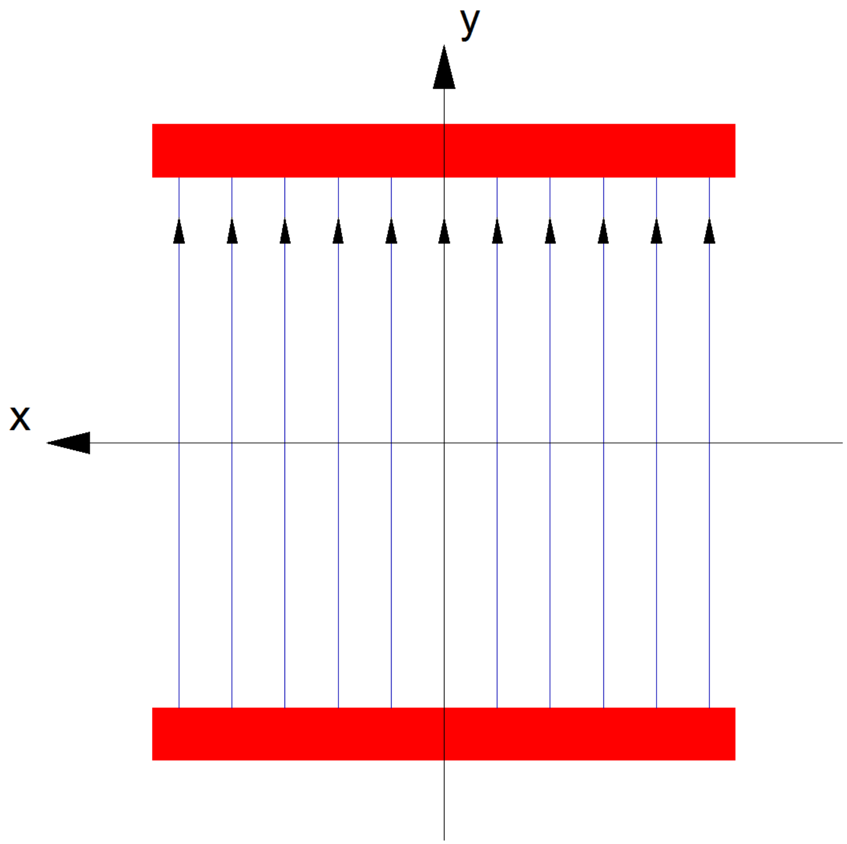
$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \tag{22}$$

The "multipole components" are indexed by the value of n: so $n = 1$ is a dipole; $n = 2$ is a quadrupole; $n = 3$ is a sextupole, etc.

An ideal multipole has coefficients a_n and b_n equal to zero, for all except one value of n.

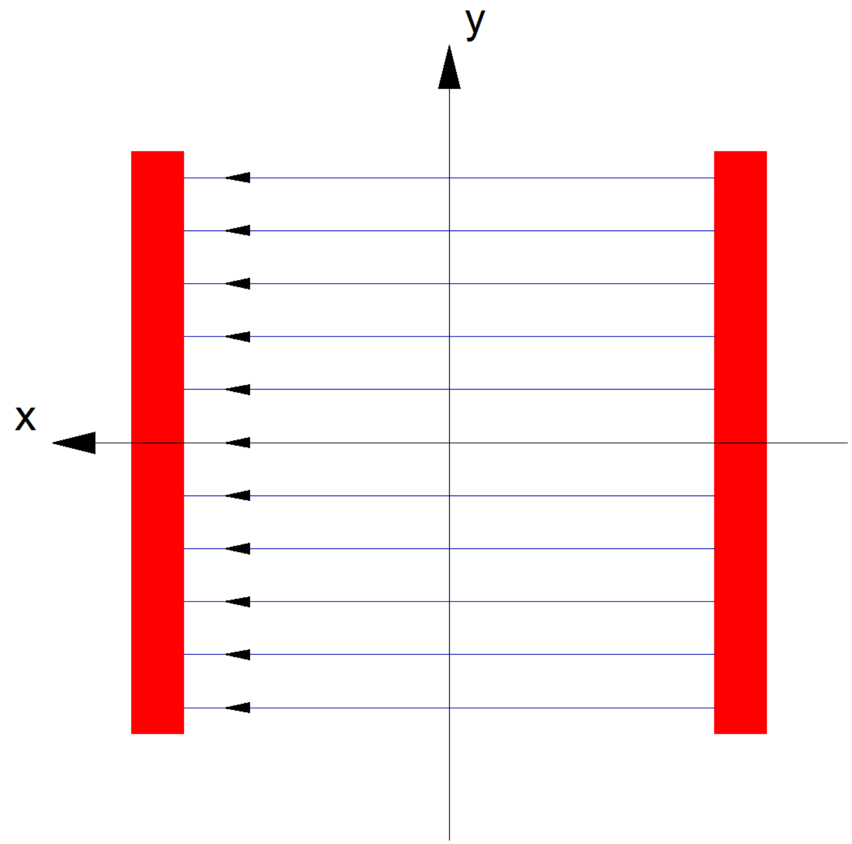
A "normal" multipole has $a_n = 0$ for all values of n.

A "skew" multipole has $b_n = 0$ for all values of n.



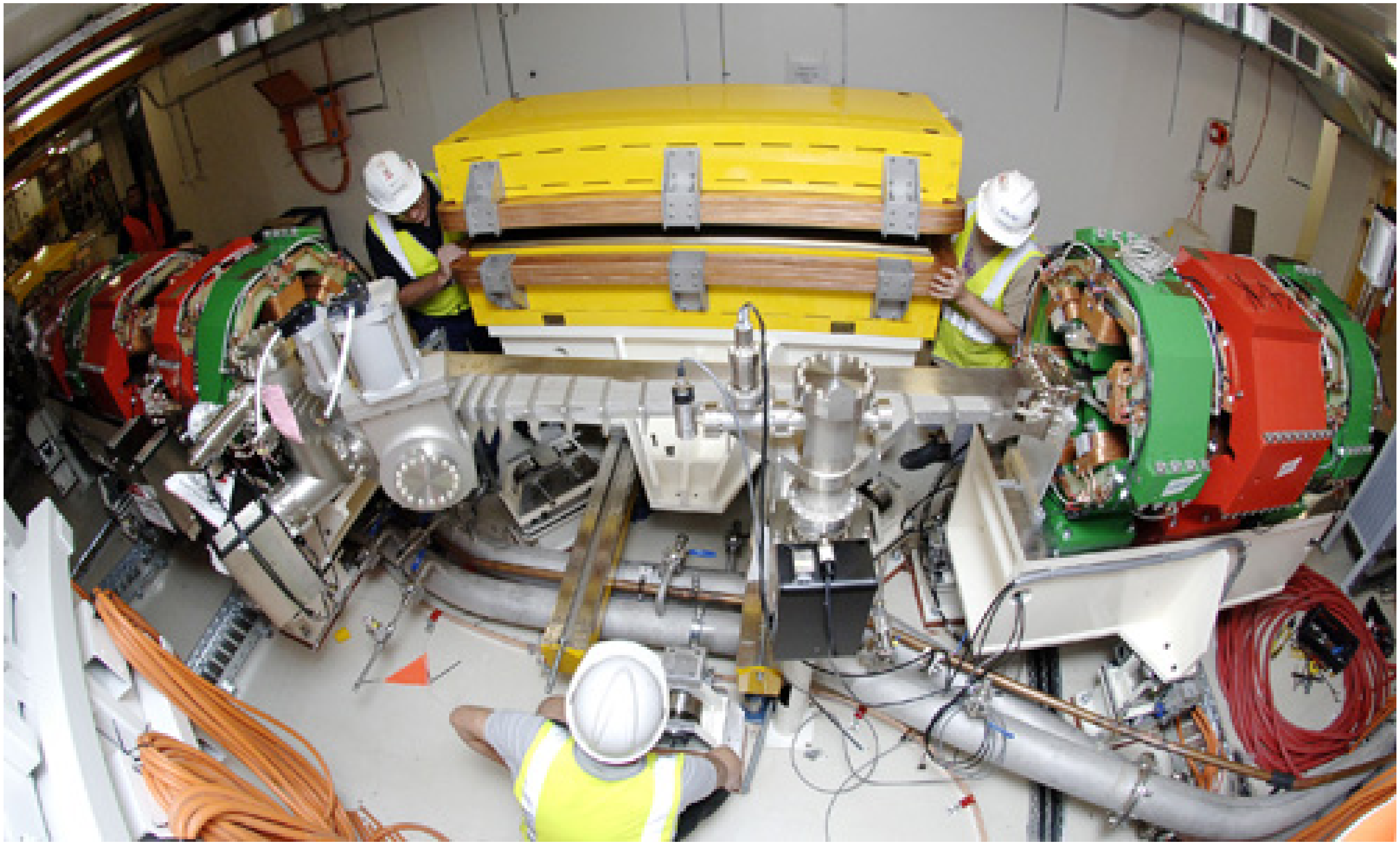
Normal dipole

$$B_x = 0, \quad B_y = b_1.$$

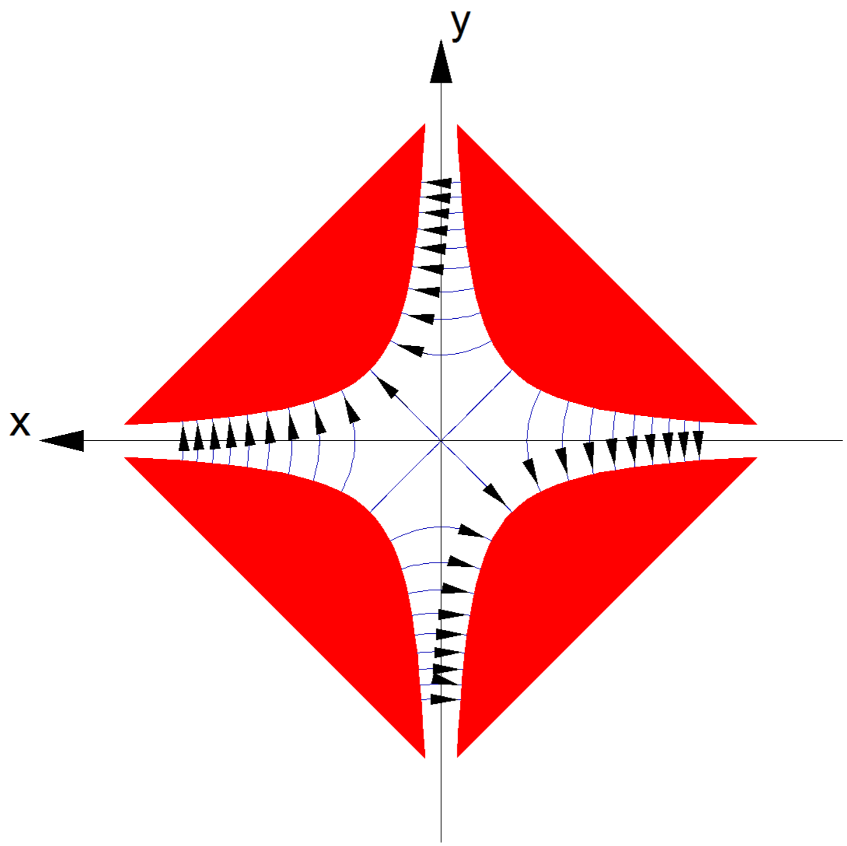


Skew dipole

$$B_x = a_1, \quad B_y = 0.$$

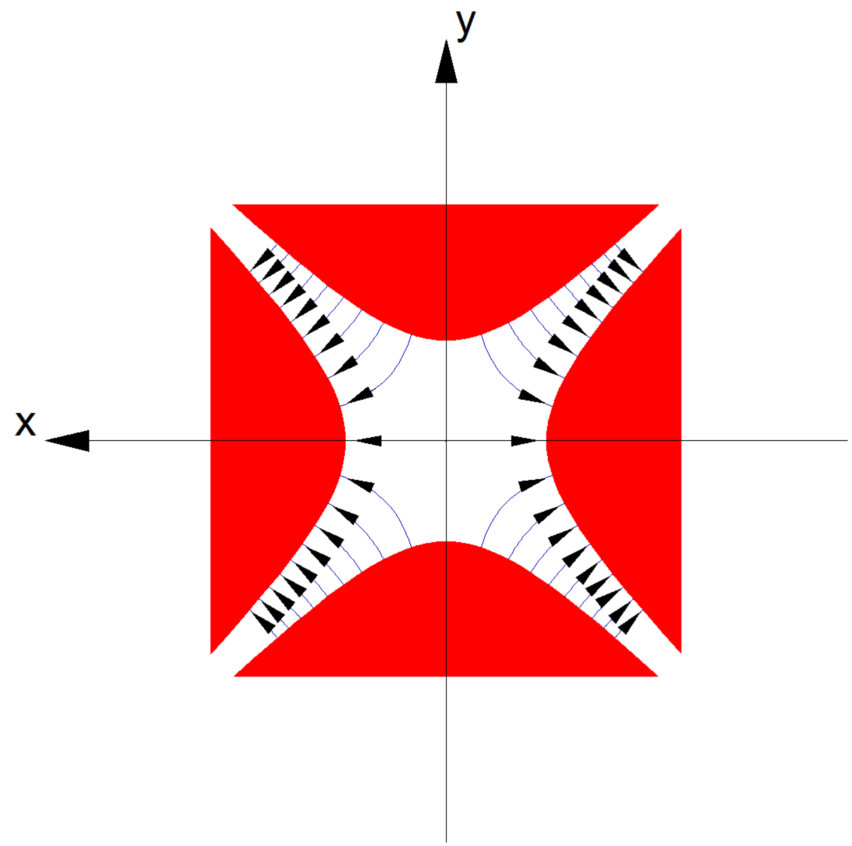


Dipole magnet being installed in the Australian synchrotron.



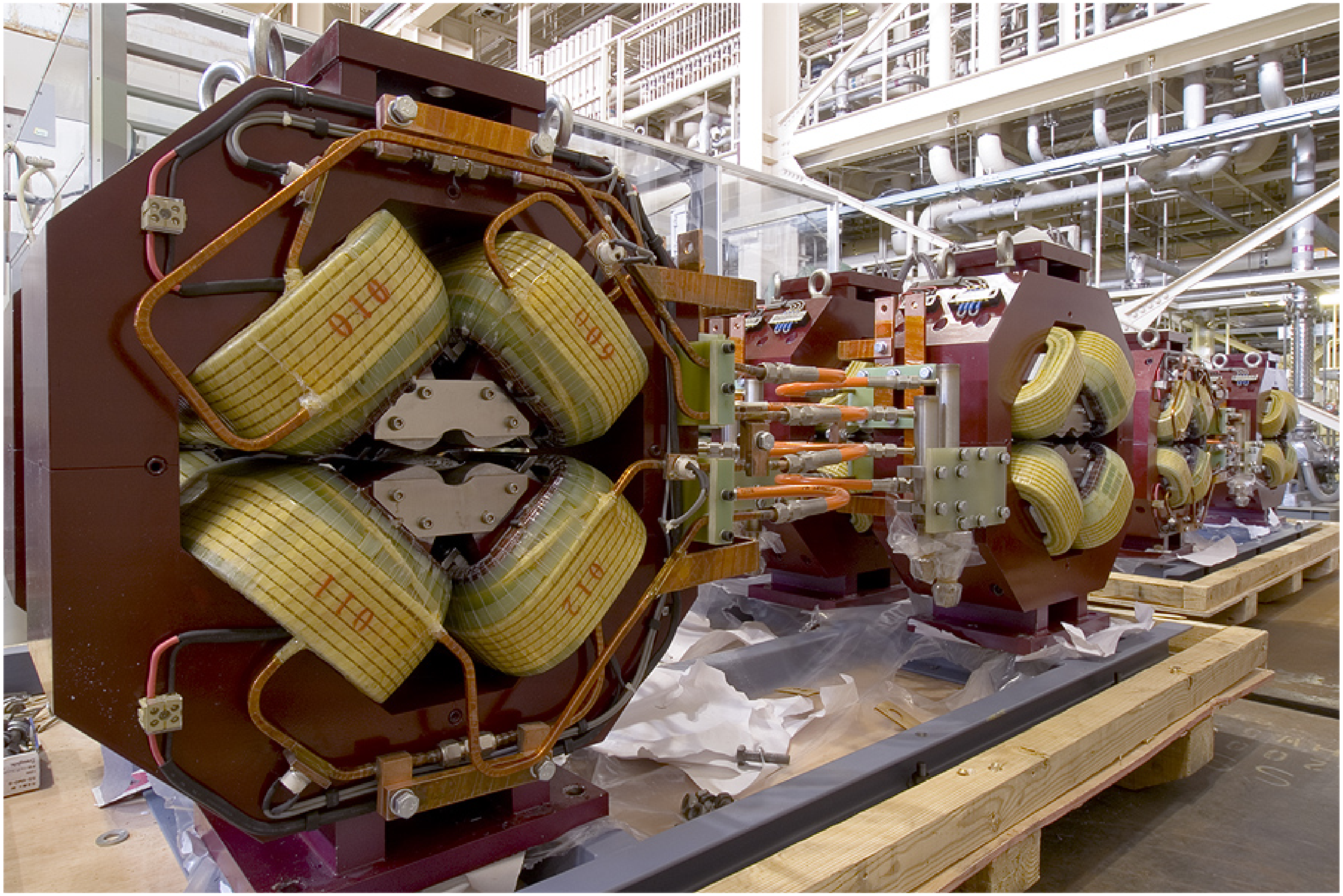
Normal quadrupole

$$B_x = b_2 \frac{y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}.$$

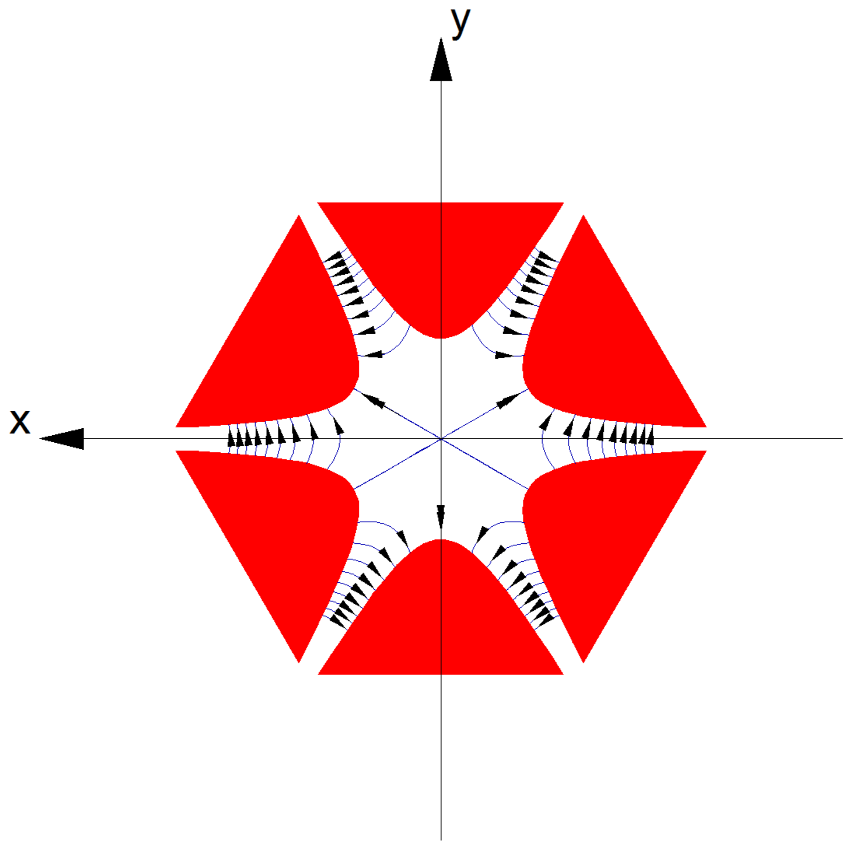


Skew quadrupole

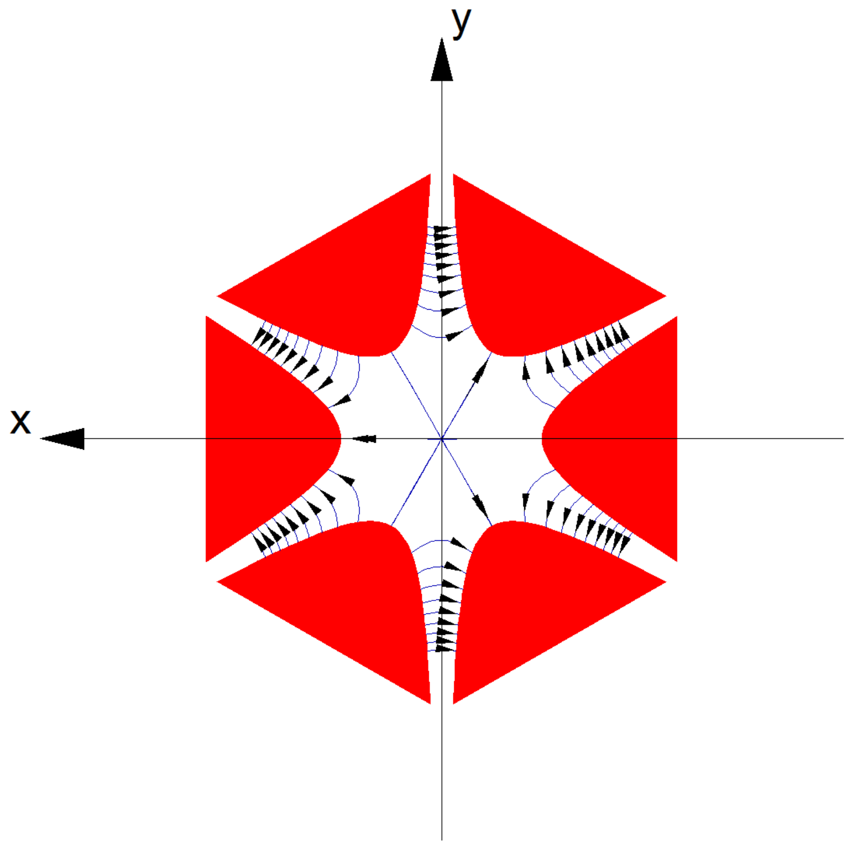
$$B_x = a_2 \frac{x}{r_0}, \qquad B_y = -a_2 \frac{y}{r_0}.$$



Quadrupole magnets (from IHEP, Beijing, China) awaiting installation in ATF2 (KEK, Tsukuba, Japan).

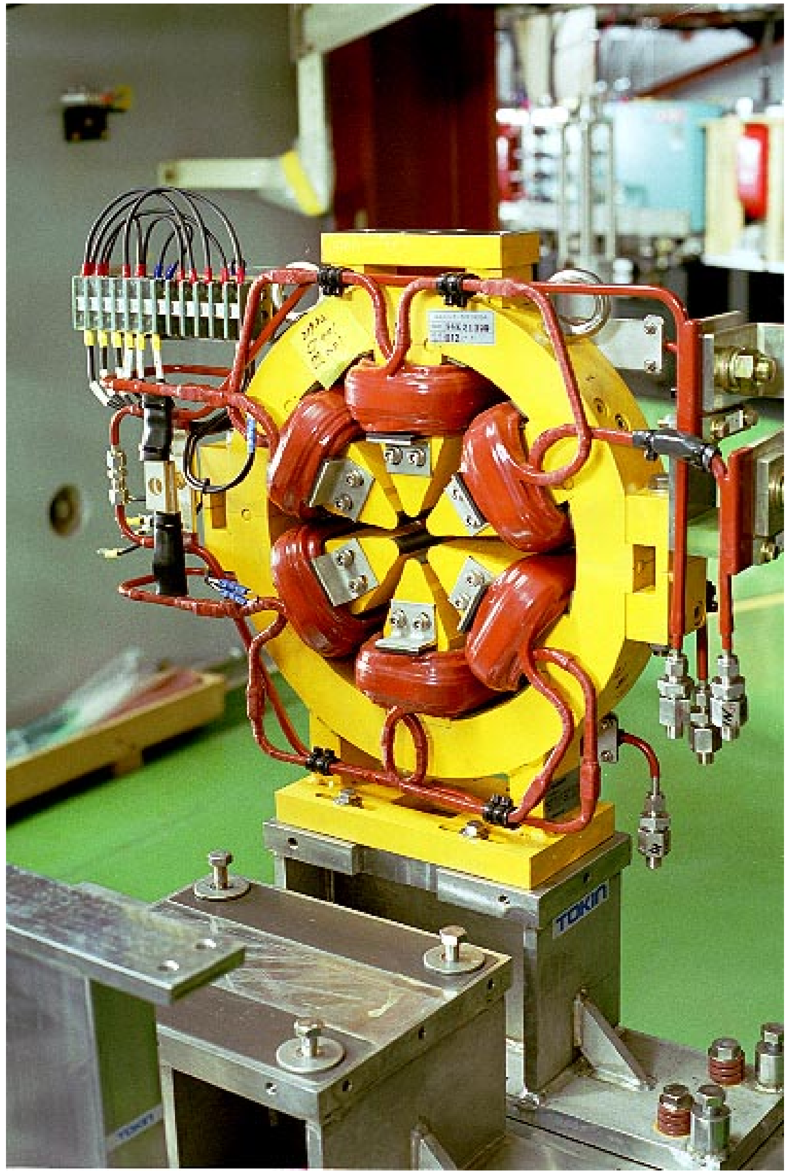


Normal sextupole



Skew sextupole

$$B_x = 2b_3\frac{xy}{r_0^2}, \quad B_y = b_3\frac{x^2-y^2}{r_0^2}. \qquad B_x = a_3\frac{x^2-y^2}{r_0^2}, \quad B_y = -2a_3\frac{xy}{r_0^2}.$$



Sextupole magnet from the ATF (KEK, Tsukuba, Japan).

Let us write down the magnetic vector potential:

$$A_x = 0, \qquad A_y = 0, \qquad A_z = -\Re \sum_{n=1}^{\infty} (b_n + i a_n) \frac{(x+iy)^n}{n r_0^{n-1}} \tag{23}$$

We find from the standard relation between the magnetic field and the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{24}$$

that the potential (23) gives the magnetic multipole field (22):

$$B_y + i B_x = -\frac{\partial A_z}{\partial x} + i \frac{\partial A_z}{\partial y} = \sum_{n=1}^{\infty} (b_n + i a_n) \left(\frac{x+iy}{r_0} \right)^{n-1} \tag{25}$$

Although there are many possible vector potentials that give the same field (25) (and all give the same equations of motion!) the particular choice (23), is convenient, because the transverse components are zero, and there is no dependence on the longitudinal coordinate.

Let us first consider the dipole field. This should be easy: the field is just a uniform field perpendicular to the reference trajectory...

...but there's a catch: a dipole field will lead to a curved physical trajectory for any charged particle moving through the field.

For that reason, we will need to use a curved reference trajectory.

When writing down the magnetic vector potential $\mathbf{A}$, we have to take into account the fact we are using curvilinear coordinates.

Our vector potential should satisfy:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{26}$$

with

$$B_x = 0, \qquad B_y = B_0, \qquad B_s = 0 \tag{27}$$

In general curvilinear coordinates (q_1, q_2, q_3), the curl of a vector field can be written:

$$[\nabla \times \mathbf{A}]_1 = \frac{1}{Q_2 Q_3}\left(\frac{\partial}{\partial q_2} Q_3 A_3 - \frac{\partial}{\partial q_3} Q_2 A_2\right) \tag{28}$$

$$[\nabla \times \mathbf{A}]_2 = \frac{1}{Q_3 Q_1}\left(\frac{\partial}{\partial q_3} Q_1 A_1 - \frac{\partial}{\partial q_1} Q_3 A_3\right) \tag{29}$$

$$[\nabla \times \mathbf{A}]_3 = \frac{1}{Q_1 Q_2}\left(\frac{\partial}{\partial q_1} Q_2 A_2 - \frac{\partial}{\partial q_2} Q_1 A_1\right) \tag{30}$$

where

$$Q_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 \tag{31}$$

In our coordinates (1), (2), (3), the curl of $\mathbf{A}$ is given by:

$$[\nabla \times \mathbf{A}]_x = \frac{\partial A_s}{\partial y} - \frac{1}{(1+hx)}\frac{\partial A_y}{\partial s} \tag{32}$$

$$[\nabla \times \mathbf{A}]_y = \frac{1}{(1+hx)}\frac{\partial A_x}{\partial s} - \frac{h}{(1+hx)}A_s - \frac{\partial A_s}{\partial x} \tag{33}$$

$$[\nabla \times \mathbf{A}]_s = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{34}$$

Using these expressions we find that the vector potential in our curvilinear coordinates:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) \tag{35}$$

gives the magnetic field:

$$B_x = 0 \qquad B_y = B_0 \qquad B_s = 0 \tag{36}$$

as desired.

Using the vector potential (35), and the general accelerator Hamiltonian (16) we construct the Hamiltonian for a dipole:

$$\begin{aligned} H \;=\; & \frac{\delta}{\beta_0} - (1+hx)\sqrt{\left(\frac{1}{\beta_0}+\delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\ & + (1+hx)\,k_0\left(x - \frac{hx^2}{2(1+hx)}\right) \end{aligned} \tag{37}$$

Note that the normalised dipole field strength is given by:

$$k_0 = \frac{q}{P_0}B_0 \tag{38}$$

where q is the charge of the reference particle, and P_0 is the reference momentum.

The full Hamiltonian for a dipole (37) looks rather intimidating.

We shall resort to the same technique we used to get a linear map for a drift space: we expand the Hamiltonian to second-order in the dynamical variables.

As before, this is valid as long as the dynamical variables remain small.

The second-order Hamiltonian is:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \qquad (39)$$

We can tell a good deal already just by looking at this Hamiltonian...

The second-order Hamiltonian is (39):

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{40}$$

Note the term $(k_0 - h)x$: a term in the Hamiltonian that is first order in one of the variables results in a zeroth-order term in the map for the conjugate variable.

In this case, we expect to see a horizontal deflection – a change in p_x. This happens if the curvature of the reference trajectory is not matched to the magnetic field of the dipole.

If $k_0 = h$, then the curvature is properly matched, and the term $(k_0 - h)x$ vanishes: a particle initially on the reference trajectory and having the reference energy stays on the reference trajectory through the dipole.

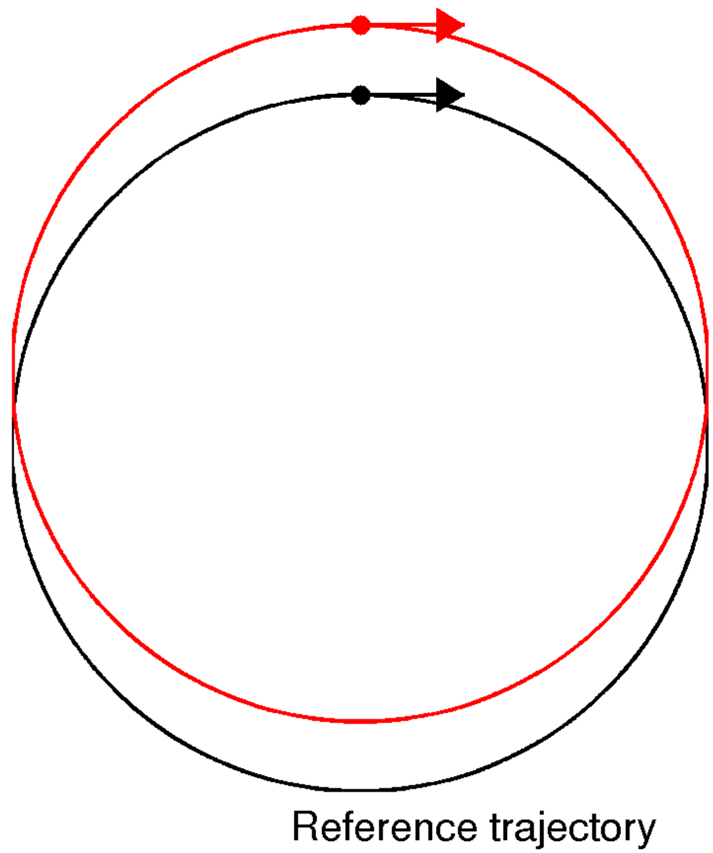
The second-order Hamiltonian is (39):

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{41}$$

Next, note the term $\frac{1}{2}hk_0x^2$: this looks like a "focusing term" – recall the potential energy term in the Hamiltonian for an harmonic oscillator.

It appears that in moving through a dipole, particles will *oscillate* about the reference trajectory. This is perhaps unexpected.

How do we understand this effect?



In a uniform magnetic field, the trajectories of two particles with some small initial offset will "oscillate" around each other.

The second-order Hamiltonian is (39):

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \qquad (42)$$

Finally, note the term $\frac{h}{\beta_0}x\delta$: this contains the product of two dynamical variables, the horizontal coordinate x, and the energy deviation δ.

The result of this term will be a *coupling* of the horizontal and longitudinal motion. For example, there will be a horizontal deflection depending on the energy of the particle.

This is called "dispersion", and is a consequence of the fact that for relativistic particles, the higher the particle's energy, the higher its mass, and the less effect there is on its trajectory from the Lorentz force.

Now we have the Hamiltonian for a dipole, and have considered some of the dynamics we are likely to expect from it. What are the solutions to the equations of motion?

Hamilton's equations following from the Hamiltonian (39) are essentially those for an harmonic oscillator. In the horizontal plane, the solutions are:

$$x(s) = x(0)\cos\omega s + p_x(0)\frac{\sin\omega s}{\omega} + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{(1-\cos\omega s)}{\omega^2} \tag{43}$$

$$p_x(s) = -x(0)\omega\sin\omega s + p_x(0)\cos\omega s + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{\sin\omega s}{\omega} \tag{44}$$

where:

$$\omega = \sqrt{hk_0} \tag{45}$$

In the vertical plane, the solutions are:

$$y(s) = y(0) + p_y(0)s \tag{46}$$

$$p_y(s) = p_y(0) \tag{47}$$

which is the same as for a drift space: there is no weak focusing in the vertical plane.

In the longitudinal plane, the solutions are:

$$\begin{aligned} z(s) &= z(0) - x(0)\frac{h}{\beta_0}\frac{\sin\omega s}{\omega} - p_x(0)\frac{h}{\beta_0}\frac{(1-\cos\omega s)}{\omega^2} + \delta(0)\frac{s}{\beta_0^2\gamma_0^2} \\ &\quad - \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{h}{\beta_0}\frac{(\omega s - \sin\omega s)}{\omega^3} \end{aligned} \tag{48}$$

$$\delta(s) = \delta(0) \tag{49}$$

Equations (43)–(49) constitute the transfer map for a dipole. Since the equations are linear, we can write them in the form of a transfer matrix, R.

Let us consider the case that the reference trajectory is matched to the dipole strength, i.e. $\omega = h = k_0$: this is the situation that we normally design in an accelerator.

In this case, the transfer matrix for a dipole of length L is:

$$R = \begin{pmatrix} \cos \omega L & \frac{\sin \omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos \omega L}{\omega \beta_0} \\ -\omega \sin \omega L & \cos \omega L & 0 & 0 & 0 & \frac{\sin \omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin \omega L}{\beta_0} & -\frac{1-\cos \omega L}{\omega \beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2 \gamma_0^2} - \frac{\omega L - \sin \omega L}{\omega \beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

Note that we have not yet included end effects – the edges of the dipole have their own dynamical effects on the beam!

To keep the values of the dynamical variables small in dipole magnets, we use a curved reference trajectory. Generally, we choose a reference trajectory that follows the path of a particle having the reference momentum.

We need to define the variables in the curved coordinate system carefully: this can be achieved using a canonical transformation.

The dynamics in dipoles displays some interesting features. These include *dispersion* (variation in trajectory with energy) and *weak focusing*.

The effect of weak focusing in a horizontal bending magnet is to keep the horizontal coordinate of a particle close to the reference trajectory: in the horizontal plane, particles oscillate around the reference trajectory with period equal to the period of the circular motion in the field of the magnet.