Single-Particle Linear Dynamics
Lecture 2

The Accelerator Hamiltonian in a Straight Coordinate System

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In the previous lecture, we saw how the dynamics of a conservative system could be derived from an appropriate Hamiltonian.

The Hamiltonian is an expression containing the coordinates and conjugate momenta (the canonical dynamical variables).

Using the Hamiltonian in Hamilton's equations gives the equations of motion for the system. These are first-order simultaneous differential equations that one must solve to find explicit expressions for the coordinates and momenta as functions of the independent variable (usually, the time t).

We looked at a number of examples, including the Hamiltonian for a non-relativistic particle moving through an electromagnetic field.

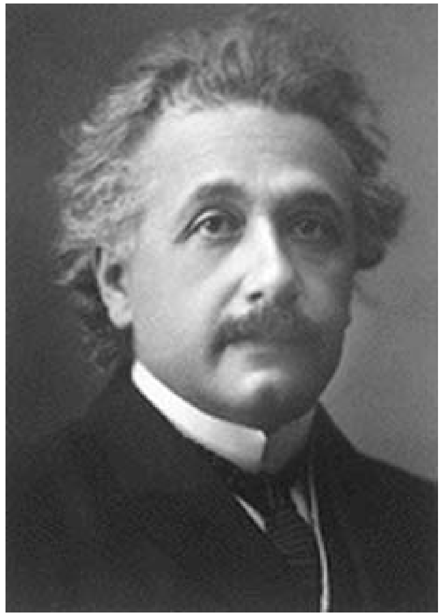
Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

1. Review of Hamiltonian mechanics
2. The accelerator Hamiltonian in a straight coordinate system
3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates
4. Transfer maps for linear elements
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

Goals of This Lecture

In this lecture, we study the Hamiltonian for a relativistic particle moving through an electromagnetic field in a straight coordinate system.

We shall use canonical transformations to express the Hamiltonian in terms of dynamical variables that are convenient for accelerator physics.



Albert Einstein, 1879-1955

Einstein's equation in Special Relativity relating the energy E and momentum $\bar{\mathbf{p}}$ of a particle is:

$$E^2 = \bar{\mathbf{p}}^2 c^2 + m^2 c^4 \qquad (1)$$

where m is the rest mass. Note that $\bar{\mathbf{p}}$ in this equation is the *mechanical* momentum (indicated by the bar), not the conjugate (canonical) momentum.

We saw in Lecture 1 that the Hamiltonian often took the form:

$$H = T + V \tag{2}$$

where T is the kinetic energy, and V is the potential energy; i.e. the Hamiltonian is often the total energy of the system, expressed in canonical variables.

Therefore, using Einstein's equation (1), we write down for our relativistic Hamiltonian:

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{3}$$

where, in the absence of an electromagnetic field, the conjugate momentum $\mathbf{p}$ is equal to the mechanical momentum $\bar{\mathbf{p}}$.

What equations of motion does the Hamiltonian (3) lead to? Using Hamilton's equations, we obtain:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{cp_i}{\sqrt{\mathbf{p}^2 + m^2c^2}} \tag{4}$$

and:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = 0 \tag{5}$$

Equation (5) simply expresses the conservation of momentum: there are no forces acting on the particle, because we have not yet introduced any electromagnetic field.

Equation (4) is equally interesting. Rearranging, we find:

$$\mathbf{p} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2/c^2}} \tag{6}$$

where, as usual, $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$.

To summarise, the Hamiltonian (3):

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{7}$$

leads to the conservation of momentum (4):

$$\dot{\mathbf{p}} = 0 \tag{8}$$

and an expression for the relativistic momentum (6):

$$\mathbf{p} = \boldsymbol{\beta} \gamma m c \tag{9}$$

where:

$$\boldsymbol{\beta} = \frac{\dot{\mathbf{x}}}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{10}$$

Finally, substituting (9) and (10) back into the expression for the Hamiltonian (7), and identifying the energy E of the particle with the Hamiltonian, we find:

$$E = \gamma m c^2 \tag{11}$$

Eqs. (9) and (11) are as expected from Special Relativity.

What about the electromagnetic field? For the nonrelativistic case, we found that the Lorentz force equation followed from the Hamiltonian if the potential energy was:

$$V = q\phi \tag{12}$$

and the *conjugate* or *canonical* momentum was:

$$\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A} \tag{13}$$

so that the non-relativistic Hamiltonian took the form:

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi \tag{14}$$

This suggests that for the relativistic case, the Hamiltonian should be:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{15}$$

Our Hamiltonian for relativistic particles in an electromagnetic field is (15):

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{16}$$

What are the equations of motion that follow from this Hamiltonian?

Hamilton's first equation gives:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{c\,(p_x - qA_x)}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \tag{17}$$

Rearranging gives:

$$\mathbf{p} - q\mathbf{A} = \boldsymbol{\beta}\gamma mc \tag{18}$$

In other words, the *canonical* momentum is given by:

$$\mathbf{p} = \boldsymbol{\beta}\gamma mc + q\mathbf{A} \tag{19}$$

The Hamiltonian (15) is:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2c^4} + q\phi \tag{20}$$

Hamilton's second equation gives:

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} \tag{21}$$

$$\begin{aligned} &= \frac{qc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \times \\ &\quad \left[(p_x - qA_x) \frac{\partial A_x}{\partial x} + (p_y - qA_y) \frac{\partial A_y}{\partial x} + (p_z - qA_z) \frac{\partial A_z}{\partial x} \right] \\ &\quad -q\frac{\partial \phi}{\partial x} \end{aligned} \tag{22}$$

This looks a bit frightening, but with the help of the expression (18) for the canonical momentum, we find that:

$$\frac{dp_x}{dt} = q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) - q\frac{\partial \phi}{\partial x} \tag{23}$$

The equation of motion for the canonical momentum (23) has exactly the same form as for the non-relativistic case (see Lecture 1, Appendix A).

Using the results from the previous lecture, we can write down the solution:

$$\frac{d}{dt}(\mathbf{p} - q\mathbf{A}) = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \tag{24}$$

where the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are defined in the usual way:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{25}$$

Recalling the expression for the canonical momentum (18) in the relativistic case, we have:

$$\frac{d}{dt}\boldsymbol{\beta}\gamma mc = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \tag{26}$$

We now have a Hamiltonian (15) that describes the motion of a relativistic charged particle in a general magnetic field:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{27}$$

Now consider a straight accelerator beamline: we can choose coordinates so that the magnets, RF cavities and other components are at defined locations along the z axis.

As a particle travels along the beamline, we know the longitudinal coordinate z at which a particle arrives at a particular component, but we don't necessarily know the time at which it arrives.

This means it is more convenient to work with the coordinate z as the independent variable, rather than the time t.

A change in independent variable from time t to the z coordinate may be accomplished with recourse to the principle of least action, that we saw in Lecture 1.

The details of the transformation are given in Appendix A.

As a result of the change in independent variable from t to z, the canonical variables (z, p_z) are replaced by the variables $(-t, E)$:

- the new longitudinal coordinate is $-t$, where t is the time at which the particle crosses a plane perpendicular to the reference trajectory at a distance z along the reference trajectory,
- the new longitudinal conjugate “momentum” is the total energy E of the particle.

Path Length as the Independent Variable

Changing the independent variable from time t to co-ordinate z means that we express the dynamical variables as functions of z, rather than as functions of time t.

The transverse coordinates and momenta are then:

$$x = x(z), \quad p_x = p_x(z), \quad y = y(z), \quad p_y = p_y(z) \tag{28}$$

The londitudinal coordinate and momenum are:

$$-t = -t(z), \quad E = E(z) \tag{29}$$

The Hamiltonian (from which we obtain the equations of motion) is derived in Appendix A:

$$H_1 = -\sqrt{\frac{(E - q\phi)^2}{c^2} - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} - qA_z \tag{30}$$

Note that this Hamiltonian is still a function of z.

It is convenient to work with variables whose values remain small as the particle moves through the accelerator: this enables us to make some useful approximations.

To construct appropriate variables, we introduce the *reference momentum* P_0. In principle, P_0 can be chosen to have any value you wish; but you would be wise to choose a value close to the nominal momentum of particles in your accelerator.

It is easy to see that if we make the substitutions:

$$p_i \mapsto \tilde{p}_i = \frac{p_i}{P_0} \tag{31}$$

then Hamilton's equations remain unchanged as long as we simultaneously make the substitution:

$$H_1 \mapsto \tilde{H} = \frac{H_1}{P_0} \tag{32}$$

In terms of the *normalised momenta* (31), the Hamiltonian is:

$$\tilde{H} = -\sqrt{\frac{(E - q\phi)^2}{P_0^2 c^2} - \frac{m^2 c^2}{P_0^2} - (\tilde{p}_x - a_x)^2 - (\tilde{p}_y - a_y)^2} - a_z \quad (33)$$

where the normalised vector potential is defined by:

$$\mathbf{a} = \frac{q}{P_0}\mathbf{A} \quad (34)$$

The transverse normalised momenta $\tilde{p}_x$ and $\tilde{p}_y$ should now be small, but the longitudinal normalised momentum E/P_0 will in general be close to the speed of light, c.

To ensure that the longitudinal variables take small values, we make one final transformation...

To express the dynamics in terms of a suitable longitudinal variables (with values that remain small), we can make a canonical transformation using a mixed-variable generating function: the details are given in Appendix B.

The new longitudinal co-ordinate is:

$$Z = \frac{z}{\beta_0} - ct \tag{35}$$

where β_0 is the (scaled) velocity of a particle with the reference momentum P_0.

The new longitudinal momentum δ is given by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{36}$$

For a relativistic particle with the reference momentum P_0, δ will be zero. δ is generally called the "energy deviation".

After all the transformations, let us clean up the notation.

First of all, we write the independent variable (distance along the reference trajectory) as s, rather than z.

Then, we write the dynamical variables in the transverse plane as (x, p_x) and (y, p_y). The canonical momenta are given by:

$$p_x = \frac{\gamma m v_x + qA_x}{P_0} \quad \text{and} \quad p_y = \frac{\gamma m v_y + qA_y}{P_0} \tag{37}$$

where v_x and v_y are the components of the velocity along the x and y axes.

Finally, in the longitudinal plane we write the coordinate Z as z, so the variables are (z, δ):

$$z = \frac{s}{\beta_0} - ct \quad \text{and} \quad \delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{38}$$

The final Hamiltonian, for a relativistic particle in an electromagnetic field, is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{m^2c^2}{P_0^2}} - a_z \tag{39}$$

Since $\frac{mc}{P_0} = 1/\gamma_0\beta_0$, where $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ we can write:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - a_z \tag{40}$$

Let us remind ourselves of a few definitions.

The following are physical constants:

- q is the charge of the particle;
- m is the rest mass of the particle;
- c is the speed of light.

The electromagnetic potential functions are:

$$\phi(x, y, z; s), \qquad \mathbf{a}(x, y, z; s) = \frac{q}{P_0}\mathbf{A}(x, y, z; s) \tag{41}$$

The *reference momentum* P_0 can be chosen freely, but should have a value close to the nominal momentum of particles in the accelerator.

β_0 is the normalised velocity of a particle moving with the reference momentum.

The dynamical variables are:

$$(x, p_x), \qquad (y, p_y), \qquad (z, \delta) \tag{42}$$

The transverse coordinates x and y give the position of the particle in a plane perpendicular to the reference trajectory (using a Cartesian coordinate system).

Formally, p_x and p_y are the canonical momenta conjugate to x and y: their physical significance is best understood in terms of the equations of motion, which we shall consider shortly.

The coordinate z describes the longitudinal position of the particle relative to the reference particle, i.e. the distance by which the particle is ahead or behind the reference particle, measured along the reference trajectory.

The energy deviation is given by (36):

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{43}$$

Using Hamilton's equations with the Hamiltonian (40), we can derive the equation of motion for the longitudinal coordinate z.

In a field-free region:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \tag{44}$$

It follows from Hamilton's equations:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}} \tag{45}$$

For the special case $p_x = p_y = 0$, and using:

$$\frac{1}{\beta_0} + \delta = \frac{E}{P_0 c} = \frac{\gamma}{\gamma_0 \beta_0} \tag{46}$$

we find:

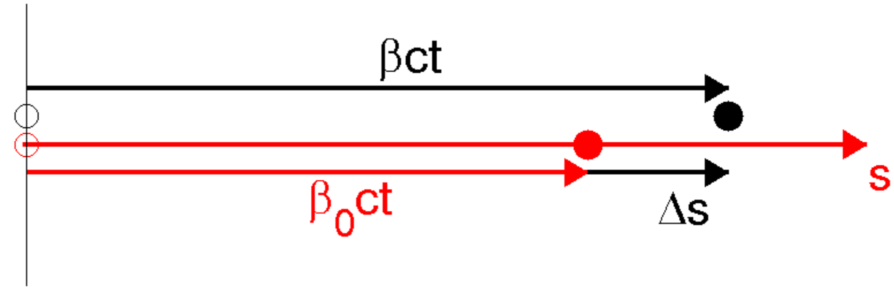
$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{\beta} \tag{47}$$

Therefore:

$$\frac{d}{ds}\beta z = \frac{\beta}{\beta_0} - 1 \tag{48}$$

If β is constant (i.e. in the absence of an electric field), this becomes:

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{\beta} \tag{49}$$



Now consider two particles moving along the reference trajectory; one (the reference particle), with speed $\beta_0 c$, and the other with speed βc.

The rate of change of distance Δs between them is:

$$\frac{d}{ds}\Delta s = \frac{\beta ct - \beta_0 ct}{\beta_0 ct} = \frac{\beta}{\beta_0} - 1 \tag{50}$$

Comparing (48) and (50), we see that in a field-free region, for a particle moving along the reference trajectory, the rate of change of βz is equal to the rate of change of the distance of the particle from the reference particle.

The particle *leads* the reference particle by distance βz.

Staying in a field-free region, from the Hamiltonian (44) we use Hamilton's equations:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} \qquad \frac{dy}{ds} = \frac{\partial H}{\partial p_y} \tag{51}$$

to find:

$$p_x = D\frac{x'}{\sqrt{1 + x'^2 + y'^2}} \approx x' \quad \text{and} \quad p_y = D\frac{y'}{\sqrt{1 + x'^2 + y'^2}} \approx y' \tag{52}$$

where:

$$D = \sqrt{1 + \frac{2\delta}{\beta_0} + \delta^2} \approx 1 + \frac{\delta}{\beta_0} \tag{53}$$

and the approximations are valid for $x'^2 + y'^2 \ll 1$, and $|\delta| \ll 1$. The prime indicates the derivative with respect to the path length s:

$$x' \equiv \frac{dx}{ds} \quad \text{and} \quad y' \equiv \frac{dy}{ds} \tag{54}$$

Finally, let us consider the evolution of the dynamical variables in a drift space (field-free region) of length L.

The Hamiltonian is (44):

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \tag{55}$$

Since there is no dependence on the coordinates, the momenta are constant:

$$\Delta p_x = 0, \qquad \Delta p_y = 0, \qquad \Delta\delta = 0 \tag{56}$$

The transverse coordinates change as follows:

$$\frac{\Delta x}{L} = \frac{p_x}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \tag{57}$$

$$\frac{\Delta y}{L} = \frac{p_y}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \tag{58}$$

From (45), we have the change in the longitudinal coordinate:

$$\frac{\Delta z}{L} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \tag{59}$$

Equations (56), (57), (58) and (59) constitute the transfer map for a drift space: they tell us how to calculate the values of the dynamical variables at the exit of the drift space, given the values at the entrance.

Note that the map is nonlinear: the changes in the variables have a nonlinear dependence on the initial values of the variables.

However, we can make a linear approximation to the transfer map by using Taylor expansions for the changes in the coordinates, (57), (58) and (59).

For small values of the canonical momenta, first-order expansions provide reasonable accuracy for most applications. We can then write the *transfer map* as a matrix...

For a drift space, we can write:

$$\vec{x}(s = L) \approx R\,\vec{x}(s = 0) \tag{60}$$

where:

$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad R = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{61}$$

This approximation is valid only for $|\delta| \ll 1$, $|p_x| \ll 1$, $|p_y| \ll 1$, and $\gamma_0 \gg 1$.

Note that by making a linear approximation to the exact solutions to the equations of motion there is a danger that we lose symplecticity.

In fact, in the case of a drift space, we are safe and the linear approximation is still symplectic.

However, there is an alternative way of constructing an approximate solution, that ensures we retain symplecticity.

Instead of making a Taylor expansion of the solutions to the equations of motion, we can expand the Hamiltonian to second order in the dynamical variables.

We can then solve the new Hamiltonian exactly to produce a linear map.

In other words, we approximate the Hamiltonian, rather than the equations of motion: an exact solution to *any* Hamiltonian is symplectic.

Expanding the Hamiltonian (55) to second order in the dynamical variables (and dropping constant terms that make no contribution to the equations of motion), we construct the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\frac{\delta^2}{\beta_0^2\gamma_0^2} \qquad (62)$$

This is much simpler than Hamiltonians we have recently looked at!

Solving the equations of motion is very easy, and we find once again that the transfer matrix for a drift of length L is given by:

$$R = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{63}$$

For describing particle motion in high-energy accelerators, we use a relativistic Hamiltonian with momentum variables normalised to a *reference momentum*.

The beamline is generally designed for particles with momenta close to the reference momentum.

The fields (and hence the Hamiltonian) change continually along the beamline. This means it is more convenient to work with the *path length* as the independent variable, rather than the time.

We can construct *linear maps* for accelerator components by expanding the appropriate relativistic Hamiltonian to second order in the dynamical variables.

The advantage of approximating the Hamiltonian (rather than finding approximate solutions to the exact Hamiltonian) is that the map produced in this way is guaranteed to be symplectic.

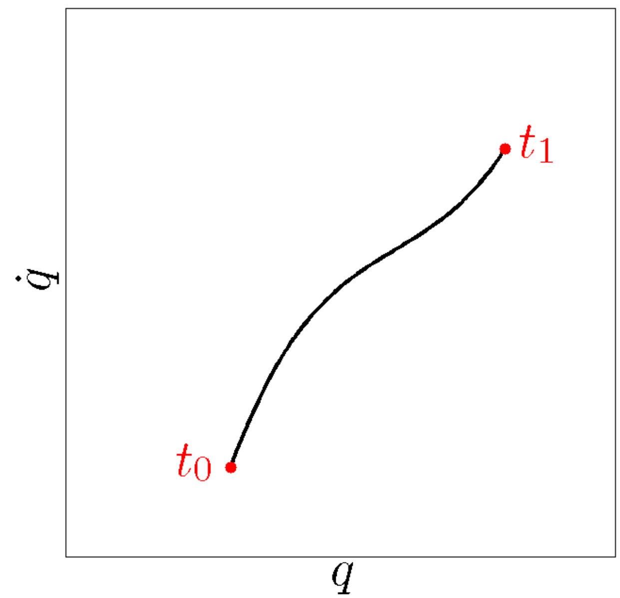
For an expansion of the Hamiltonian (to second order) to be valid, the values of the dynamical variables must be small.

In this appendix, we show how to make a change in independent variable, from time t to longitudinal coordinate z.

The procedure for changing the independent variable starts from the Principle of Least Action...

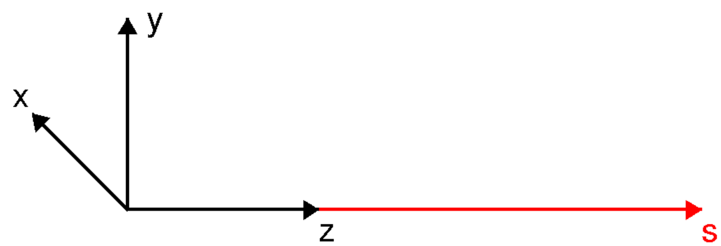
A particle will follow a path in phase space (a plot of velocity $\dot{q}$ vs coordinate q) for which the action S is a minimum:

$$\delta S = \delta \left[\int_{t_0}^{t_1} L \, dt \right] = 0 \qquad (64)$$



We can write the action in terms of the Hamiltonian:

$$S = \int_{t_0}^{t_1} (p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - H)\, dt \tag{65}$$



Let us choose our coordinates so that the z axis lies along the reference trajectory (with distance along the reference trajectory measured by the variable s).

Changing the variable of integration from time t to path length z, the action becomes:

$$S = \int_{z_0}^{z_1} \left(p_x x' + p_y y' + p_z - Ht' \right) dz \tag{66}$$

where the prime denotes the derivative with respect to z.

The action with time as the independent variable is (65):

$$S = \int_{t_0}^{t_1} (p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - H)\, dt \tag{67}$$

and the action with path length as the independent variable is (66):

$$S = \int_{z_0}^{z_1} \left(p_x x' + p_y y' - H t' + p_z \right) dz \tag{68}$$

Comparing equations (67) and (68), we see that to describe the motion in Hamiltonian mechanics with path length z as the independent variable, we should take as our canonical variables:

$$(x, p_x)\,, \quad (y, p_y)\,, \quad (-t, H) \tag{69}$$

and use for the Hamiltonian:

$$H_1 = -p_z \tag{70}$$

Identifying the Hamiltonian H with the energy E, we can rearrange equation (15) to express p_z as:

$$p_z = \sqrt{\frac{(E - q\phi)^2}{c^2} - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} + qA_z \quad (71)$$

Therefore, in the new variables, our Hamiltonian is:

$$H_1 = -\sqrt{\frac{(E - q\phi)^2}{c^2} - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} - qA_z \quad (72)$$

where E is the total energy of the particle.

In the mathematical formalism of Hamiltonian mechanics, E is (in this case) a canonical momentum variable conjugate to $-t$.

In this appendix, we show how to make a canonical transformation, so that the longitudinal momentum of a particle in an accelerator is described by a variable with a small value.

We start with the Hamiltonian (33), in which the longitudinal coordinate is $-t$, and the canonical momentum is the total energy of the particle E.

We make a canonical transformation to new longitudinal variables, using a mixed-variable generating function of the second kind:

$$F_2(x, P_x, y, P_y, -t, \delta, z) = xP_x + yP_y + \left(\frac{z}{\beta_0} - ct\right)\left(\frac{1}{\beta_0} + \delta\right) \quad (73)$$

where P_x, P_y and δ are our new momentum variables, and β_0 is the normalised velocity of a particle with the reference momentum P_0.

The equations for a transformation defined by a mixed-variable generating function of the second kind are:

$$\tilde{p}_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = \tilde{H} + \frac{\partial F_2}{\partial z} \tag{74}$$

Using these equations, we find that the transverse variables are unchanged:

$$\tilde{p}_x = P_x, \qquad X = x \tag{75}$$

$$\tilde{p}_y = P_y, \qquad Y = y \tag{76}$$

The old and new longitudinal variables are related by:

$$\frac{E}{P_0} = c\left(\frac{1}{\beta_0} + \delta\right), \qquad Z = \frac{z}{\beta_0} - ct \tag{77}$$

Finally, the new Hamiltonian (dropping a constant term) is:

$$K = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (P_x - a_x)^2 - (P_y - a_y)^2 - \frac{m^2c^2}{P_0^2}} - a_z \tag{78}$$

The new dynamical variable δ is given by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{79}$$

For a relativistic particle with the reference momentum P_0, δ will be zero. δ is generally called the "energy deviation".

We have made a series of transformations. Let us tidy up the notation, and rewrite:

$$K \mapsto H, \qquad P_i \mapsto p_i, \qquad z \mapsto s, \qquad Z \mapsto z \tag{80}$$

Then the Hamiltonian for a relativistic particle in an electromagnetic field, using the distance along a straight reference trajectory as the independent variable is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{m^2c^2}{P_0^2}} - a_z \tag{81}$$

Since $\frac{mc}{P_0} = 1/\gamma_0\beta_0$, where $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ we can write:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - a_z \tag{82}$$