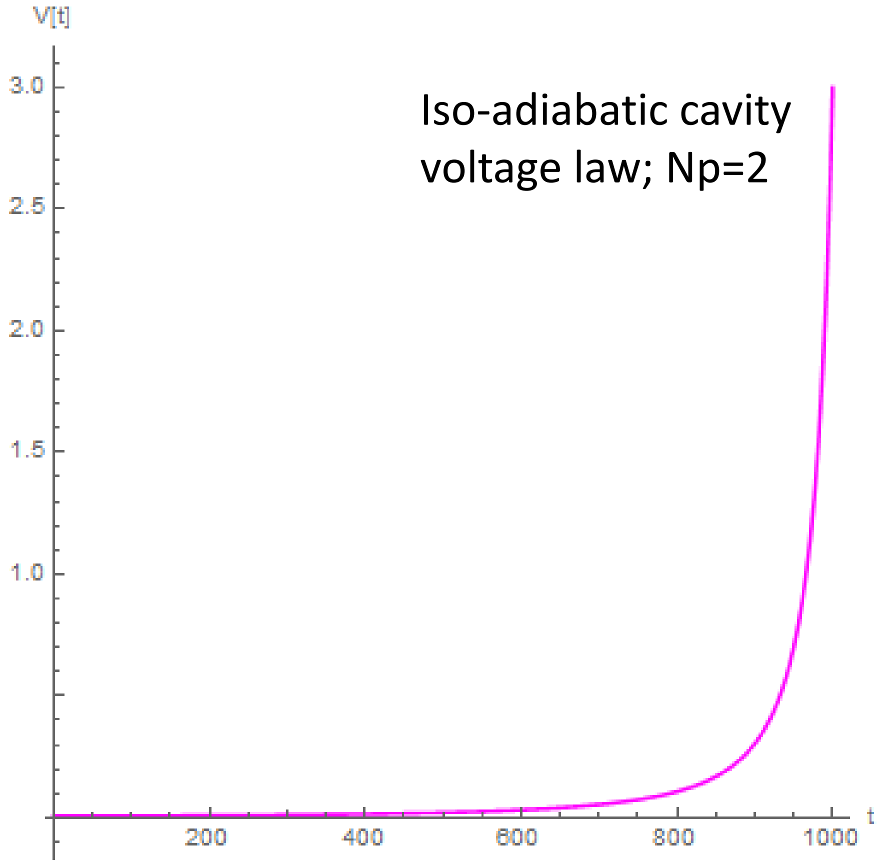
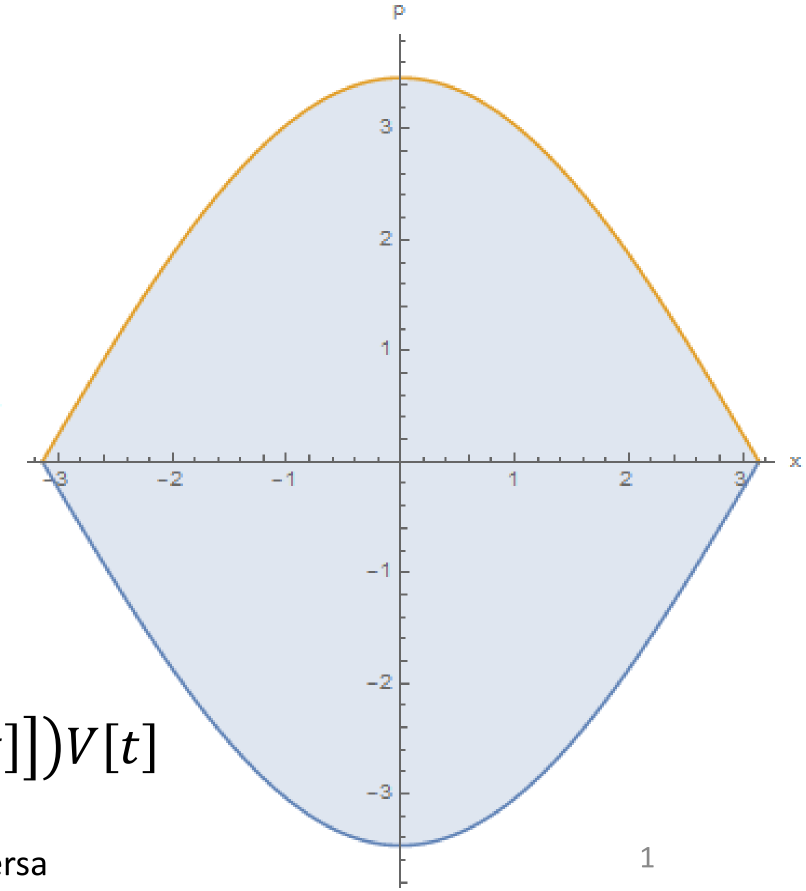
ADIABATIC CAPTURE OF LONGITUDINAL PHASE SPACE IN RISING VOLTAGE RF BUCKET

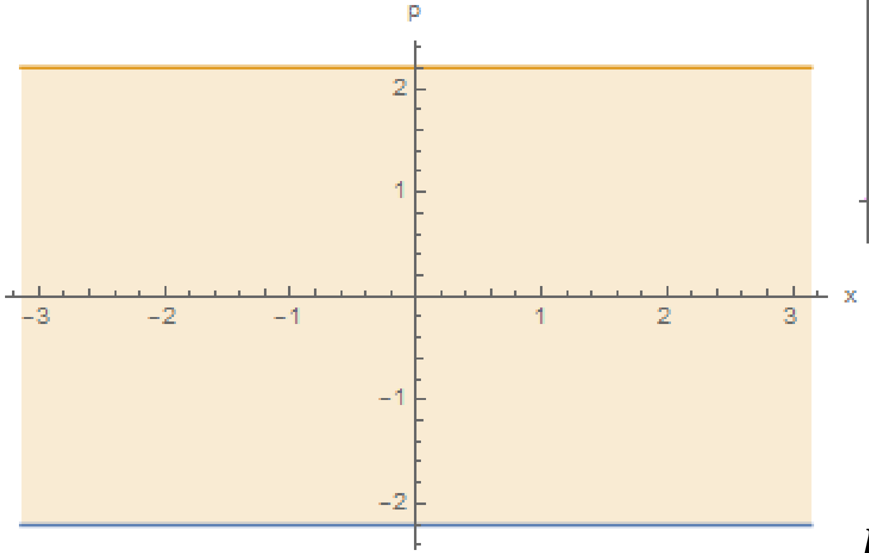
Shane R. Koscielniak, 2022 Sept 1st



Iso-adiabatic cavity voltage law; Np=2

Final phase space





Initial phase space

Hamiltonian

$$H[x,p,t] == \frac{1}{2}Ap[t]^2 + (1 - \mathrm{Cos}[x[t]])V[t]$$

H[t] is calculated from x[t] & p[t], not visa versa

Definitions

```
In[ ]:= H[x, p, t] = A p[t]^2 / 2 + V[t] (1 - Cos[x[t]])
```

$$Out[\,]= \frac{1}{2} A\, p[t]^2 + (1 - \mathrm{Cos}[x[t]])\, V[t]$$

```
In[ ]:= xdot = D[H[x, p, t], p[t]]
Out[ ]= A p[t]

In[ ]:= pdot = -D[H[x, p, t], x[t]] // Expand // Simplify
Out[ ]= -Sin[x[t]] V[t]

In[ ]:= Hdot = D[H[x, p, t], t] /. p'[t] → pdot /. x'[t] → xdot // Expand
Out[ ]= V'[t] - Cos[x[t]] V'[t]
```

$p[t] == 2\pi R_s\, \delta p == 2\pi\, \delta E / \omega_s;$

x[t] == particle RF phase;

$$In[\,]:= A == -\frac{h\, \eta\, \omega_s}{2\pi\, p_s\, R_s};$$

V[t] == e V

Rs = synchronous radius;

ps = synchronous relativistic momentum;

ω = small amplitude synchrotron frequency;

δp = relativistic momentum deviation;

δE = relativistic energy deviation;

h = harmonic number;

η = longitudinal slip factor;

e=electron charge;

V= peak value of cavity gap voltage per turn around the ring

ADIABATIC CAPTURE OF LONGITUDINAL PHASE SPACE IN RISING VOLTAGE RF BUCKET

Where: Circular, charged-particle accelerator; constant magnetic field; fixed-frequency RF electric fields

What process: take an un-bunched, coasting, particle beam and capture into a rising-voltage RF bucket while controlling emittance growth

Theory outputs: Universal trapping law; Bunch momentum spectrum; Bunch longitudinal profile; Optimized voltage law (exponential); Prediction of r.m.s. spread of Hamiltonian, etc.

Family of Voltage Laws, Np>0

$$V[t] == \left(1 - \frac{\left(1 - \mathrm{C0}^{-1/\mathrm{Np}}\right)t}{T}\right)^{-\mathrm{Np}} \frac{\mathrm{V0}}{\mathrm{C0}}$$

Start: t=0; Stop t=T

$$V[0] == \frac{V[T]}{\mathrm{C0}}$$

Ideally C0 >>100

Are the solutions of

$$V[t]V''[t] == \left(1 + \frac{1}{\mathrm{Np}}\right)V'[t]^2$$

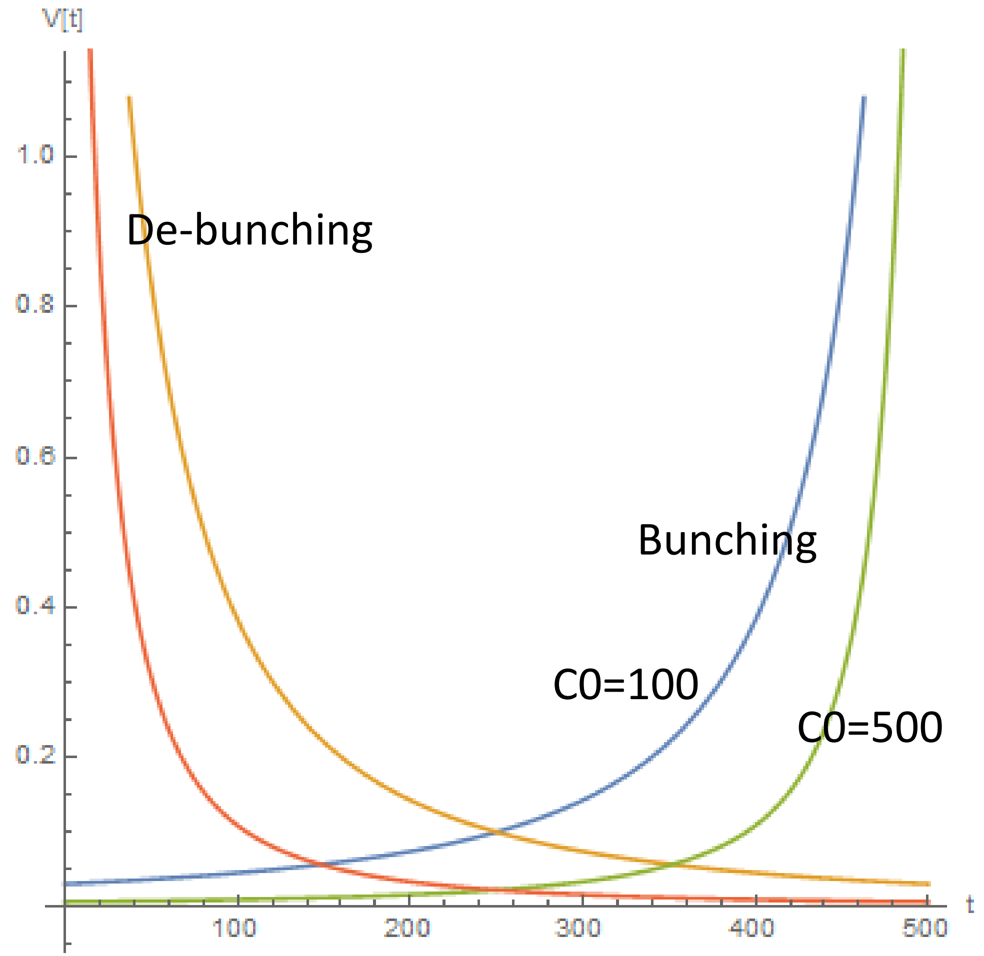
Np=2 gives so-called iso-adiabatic capture law

- Lilliequist & Symon, MURA-491 (1959); https://lss.fnal.gov/lists/fermilab-reports-mura.html
- U. Bigliani, CERN-SI-Int-EL-68-2 (1968)

Hamiltonian

$$H[x,p,t] == \frac{1}{2}Ap[t]^2 + \left(1 - \mathrm{Cos}\left[x[t]\right]\right)V[t]$$

Note: this Hamiltonian does not change the nature of the fixed points (x,p)=(0,0) and (x,p)=(π,0); and so the Jacobi elliptic functions can be used as short-term approximate solutions.



DEBUNCHING VOLTAGE LAW

$$V[T] == \left(1 - \frac{\left(1 - C0^{1/\mathrm{Np}}\right)t}{T}\right)^{-\mathrm{Np}} \mathrm{V0}$$

REQUIRED FULL BUCKET CAPTURE VOLTAGE V[T]

The un-bunched particle beam has Hamiltonian

$$\mathrm{H0}[x,p] == \frac{1}{2} A p[t]^2$$

With H0 running from 0 to H0max

Phase Space Area = (momentum)x(displacement)

Equating the initial (100%) area of the beam, 4π pmax, to the area of the final RF-bucket, 16 Sqrt[V[T]/A],

we find the required capture voltage:

$$\left\{\mathrm{V[T]} \rightarrow \frac{1}{16} A\pi^2 \mathrm{Pmax}^2\right\} \qquad \left\{\mathrm{V[T]} \rightarrow \frac{\mathrm{H0max}\ \pi^2}{8}\right\}$$

Solutions of the pendulum oscillator Hamiltonian are the Jacobi elliptic functions with amplitude parameter “m”
These functions are defined both inside (m<1) and outside (m>1) the separatrix (m=1)

Here the small-amplitude synchrotron frequency *w* is set at unity. To restore *w* make the substitutions $t \to wt$ and $\tau \to w\tau$

Libration (bounded motion, m<1) $\text{Period} = \text{Tau} == 4\text{EllipticK}[m]$

(1) $x[t] == 2\text{ArcSin}\left[\sqrt{m}\text{JacobiSN}[t,m]\right]$ $p[t] == 2\sqrt{m}\text{JacobiCN}[t,m]$

(2) $x[t] == 2\text{ArcSin}\left[\sqrt{m}\text{JacobiCD}[t,m]\right]$ $p[t] == -2\sqrt{(1-m)m}\text{JacobiSD}[t,m]$

Rotation (unbounded motion, m>1) $\text{Two}-\text{periods} = \text{Tau} == 4\sqrt{\frac{1}{m}}\text{EllipticK}\left[\frac{1}{m}\right]$

(3) $x[t] == 2\text{ArcSin}\left[\text{JacobiSN}\left[\sqrt{m}t,\frac{1}{m}\right]\right]$ $p[t] == 2\sqrt{m}\text{JacobiDN}\left[\sqrt{m}t,\frac{1}{m}\right]$

(4) $x[t] == 2\text{ArcSin}\left[\text{JacobiCD}\left[\sqrt{m}t,\frac{1}{m}\right]\right]$ $p[t] == -2\sqrt{-1+m}\text{JacobiND}\left[\sqrt{m}t,\frac{1}{m}\right]$

Parameter “m” is presumed to be a constant (chosen for each trajectory)

Instantaneous Hamiltonian H=2mV
Capture condition: H(t)<2V(t)
Final, bounding Hamiltonian H=2V[T]

CHANGE OF HAMILTONIAN IN RISING VOLTAGE RF BUCKET

At t=0, largest value of $m == \frac{\text{H0max}}{2V[0]} == \frac{4\text{C0}}{\pi^2} >> 1$

At t=T, all captured particles have m<1

But if V[t] varies slowly enough, we may hope that the Jacobi functions with m[t] are still valid solutions for at least one 1/2-period: that is one up & down the confining potential (or one down & up)

The rate of change of the Hamiltonian is $H'[t] == (1 - \text{Cos}[x[t]])V'[t]$

Inside the bucket we may substitute Jacobi solutions (1) or (2) for x[t]

For example, $H'[t] == 2m\text{JacobiSN}[(t - \text{t0})w, m]^2 V'[t]$

But if V[t] varies slowly enough compared with the oscillation period, we may replace the instantaneous Jacobi function with its average effect over one ½-period ; by integrating from –τ/4 to +τ/4, and dividing by τ/2. Essentially this is the work done by the changing potential during a ½-period.

$$\overline{H'[t]} == 2\left(1 - \frac{\text{EllipticE}[m]}{\text{EllipticK}[m]}\right)V'[t]$$

Inside the bucket, for most trajectories most of the time m<<1, so we may substitute the Taylor series expansion

$$\overline{H'[t]} == \left(m + \frac{m^2}{8} + \frac{m^3}{16} + \cdots\right) V'[t]$$

We retain only the first order term and substitute m=H(t)/(2V(t))

$$H'[t] == \frac{H[t]V'[t]}{2V[t]}$$

The solution is $H[t > tc] == \frac{H[\text{tc}]\sqrt{V[t]}}{\sqrt{V[\text{tc}]}}$ Where *tc* is the time of capture

So, inside the bucket, the Hamiltonian grows (approximately) as Sqrt[time]

Outside the bucket we may substitute Jacobi solutions (3) or (4) for x[t] into the expression for dH/dt

For example, $H'[t] == 2\text{JacobiCD}\left[\sqrt{m}(-\text{t0} + u)w, \frac{1}{m}\right]^2 V'[t]$

But if V[t] varies slowly enough compared with the oscillation period, we may replace the instantaneous Jacobi function with its average effect over one ½-period ; by integrating from –τ/4 to +τ/4, and dividing by τ/2. Essentially this is the work done by the changing potential during a ½-period.

$$\overline{H'[t]} == 2m\left(1 - \frac{\text{EllipticE}\left[\frac{1}{m}\right]}{\text{EllipticK}\left[\frac{1}{m}\right]}\right) V'[t]$$

Outside the bucket, for most trajectories most of the time m>>1, so we may substitute the series expansion

$$\overline{H'[t]} == \left(1 + \frac{41}{1024m^3} + \frac{1}{16m^2} + \frac{1}{8m} + \cdots\right) V'[t]$$

We retain only the first order term

$H'[t] == V'[t]$ The solution is $\mathrm{H}[t] == \mathrm{H}[0] - V[0] + V[t]$

TO SUMMARISE: Adiabatic capture is a two-step process.

Step 1) almost linear growth of Hamiltonian (from t=0) until capture at t=tc

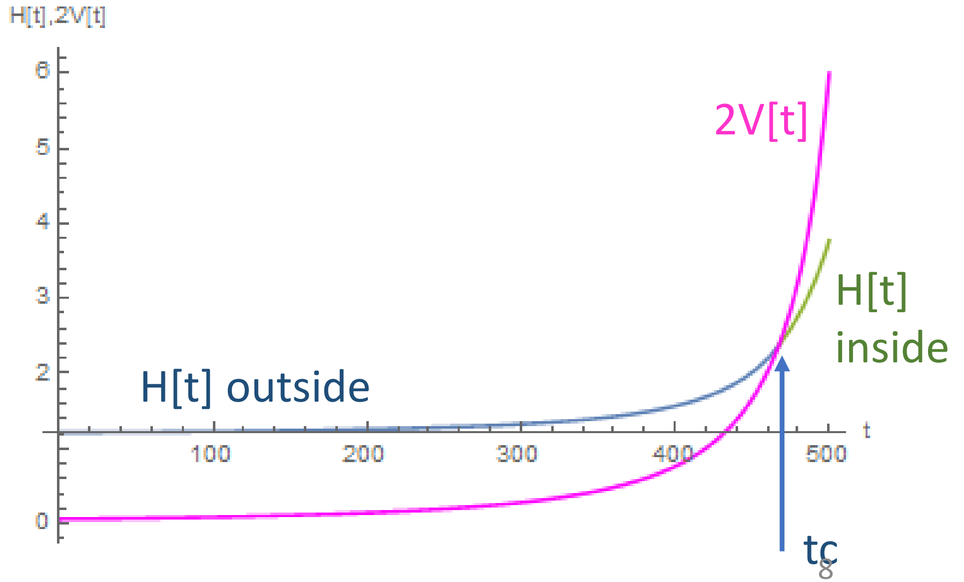
At capture H[t]=2V[t] @ t=tc

Step 2) Sqrt[t] growth from capture until the voltage ramp is complete at t=T

Cascading these two steps leads to a final energy capture law H[T]=f(H[0]) that is independent of the voltage law V(t) provided only that it is "adiabatic".

[Note, the time of capture (*tc*) depends on the voltage law; but the Hamiltonian value at *tc* is independent of the voltage law.]

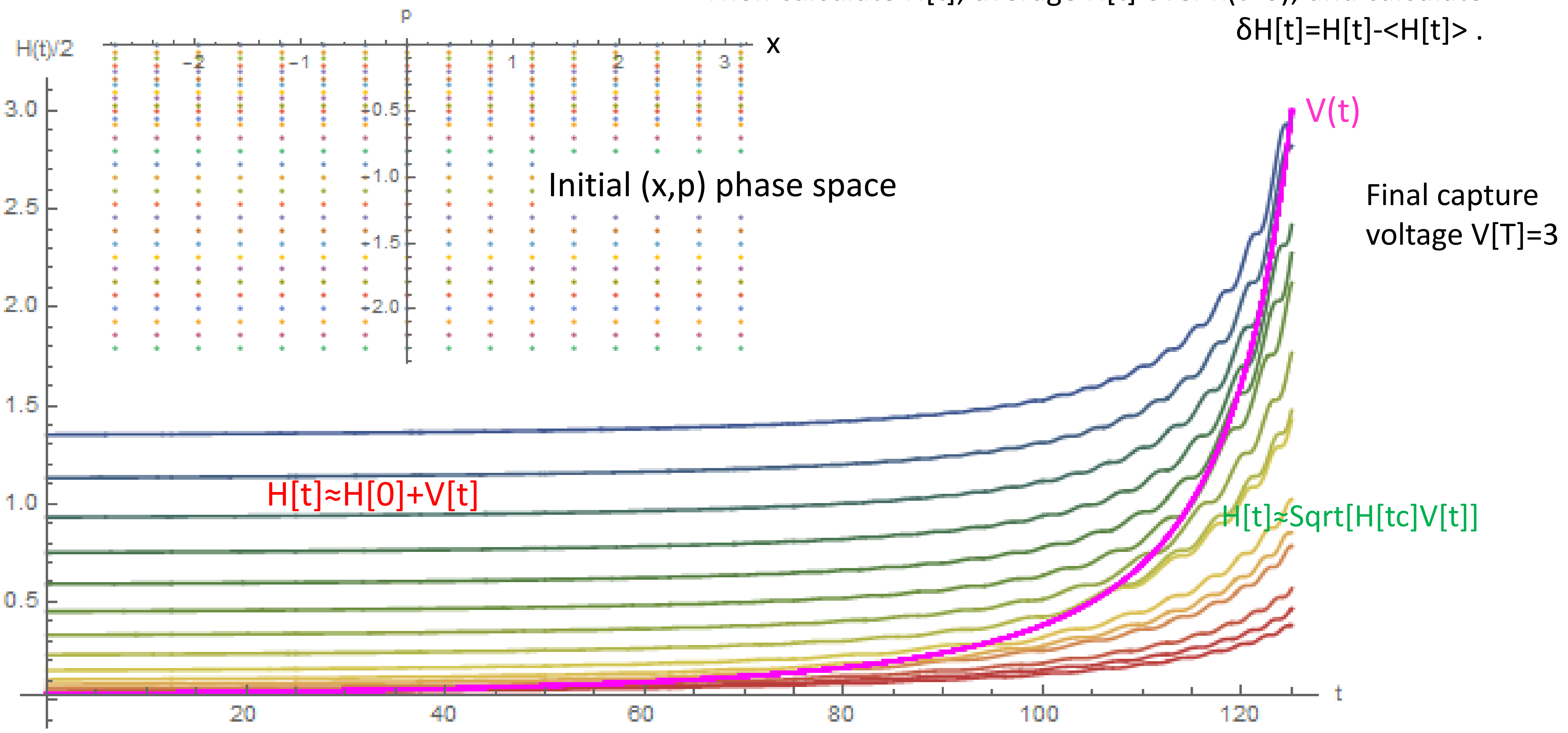
To lowest order $H[T] == \frac{\pi\sqrt{H[0]}\sqrt{V[T]}}{\sqrt{2}}$



CHANGE OF HAMILTONIAN IN RISING VOLTAGE RF BUCKET

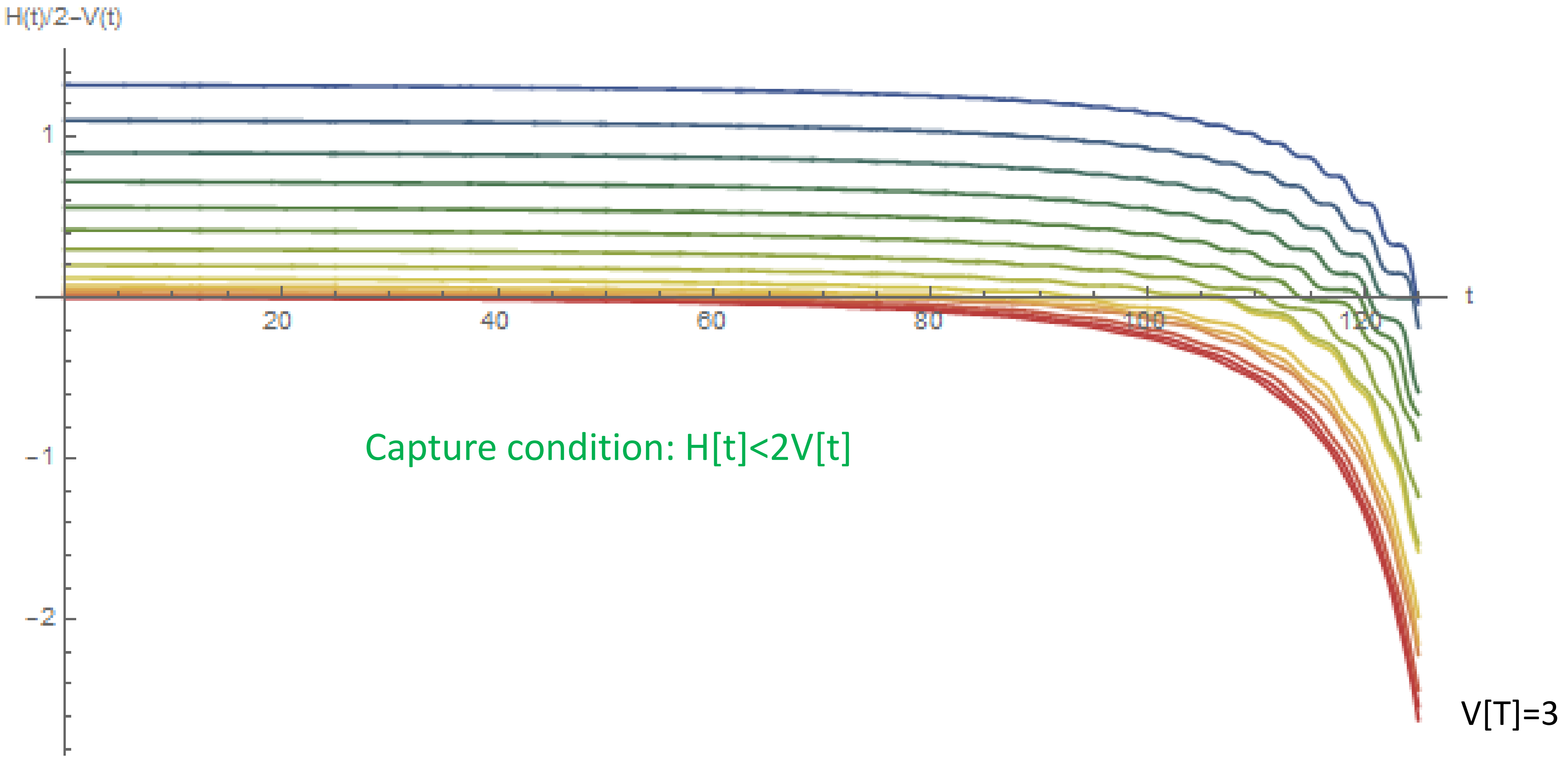
$$H0[x,p] == \frac{1}{2}Ap[t \leq 0]^2$$

Numerical integration of the equations of motion for p[t] and x[t]. Then calculate H[t]; average H[t] over x(t=0); and calculate δH[t]=H[t]-<H[t]> .



H[t] averaged over initial RF phases x(t=0)

CHANGE OF HAMILTONIAN IN RISING VOLTAGE RF BUCKET
Numerical solution of Equations of motion for x,p



There are a variety of possible "adiabaticity parameters"

For example, the condition that dV/dt can be moved outside the integral is "change in V' during ½ period is very small" implies ε= ΔV'/V' =(V''τ/2)/V' =(V"/V')π/Sqrt[AV] <<1; as used by Lilliequist & Symon.

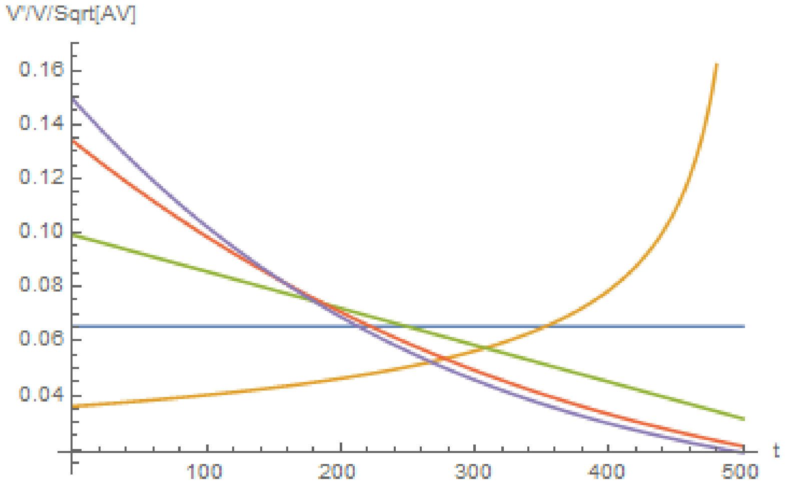
Other possible adiabaticity parameters are

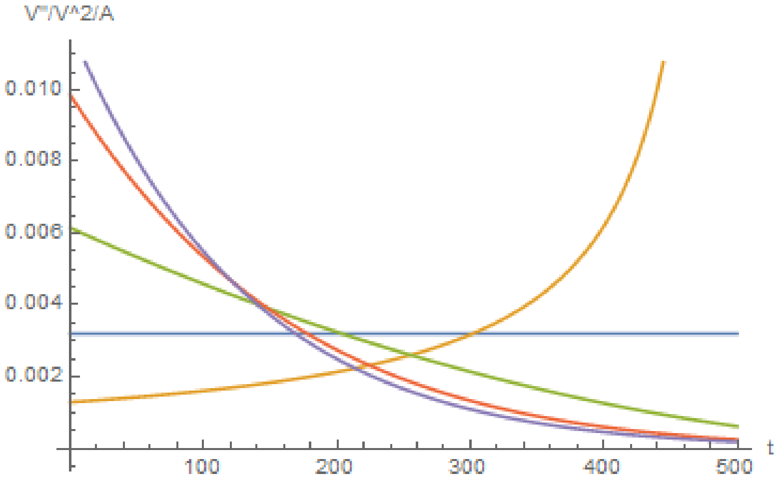
ΔV/V =(V'τ/2)/V = (V'/V)π/Sqrt[AV]

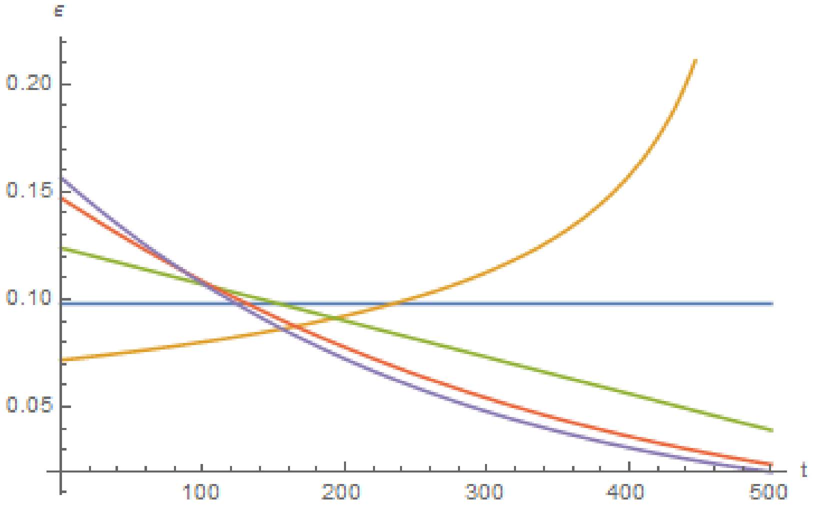
ΔV/V=(1/2) (V"/V)(τ/2)^2 = (π^2/2)V"/(A V^2), etc.

Because the voltage laws all satisfy $V[t]V''[t] == \left(1+\frac{1}{\mathrm{Np}}\right)V'[t]^2$

most of these choices produce similar ranking of cases as function of C0, T, Np

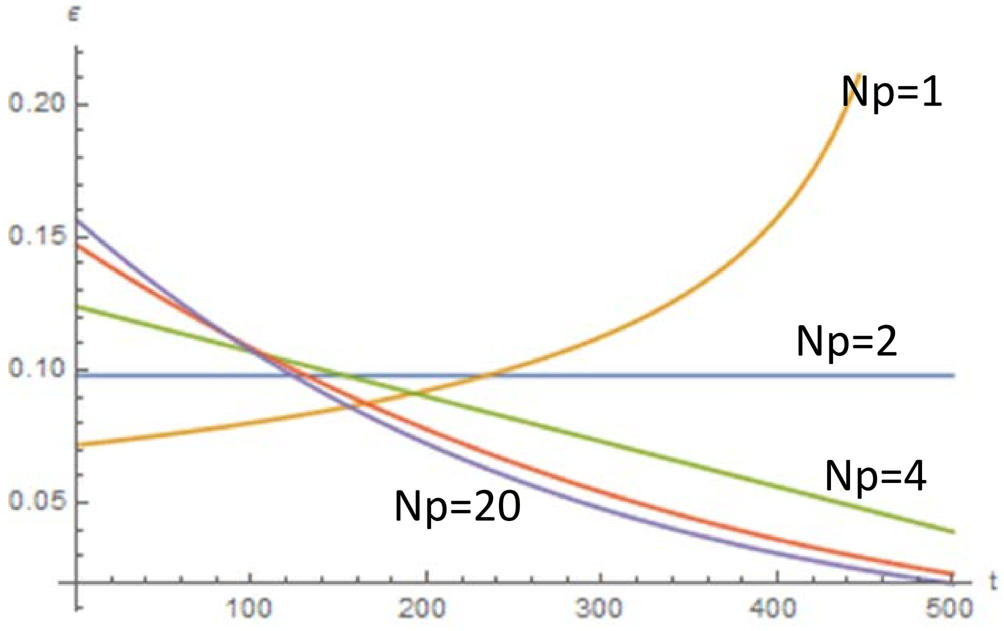






Blue=Np=2; gold=Np=1; green=Np=4; coral=Np=10; purple=Np=20

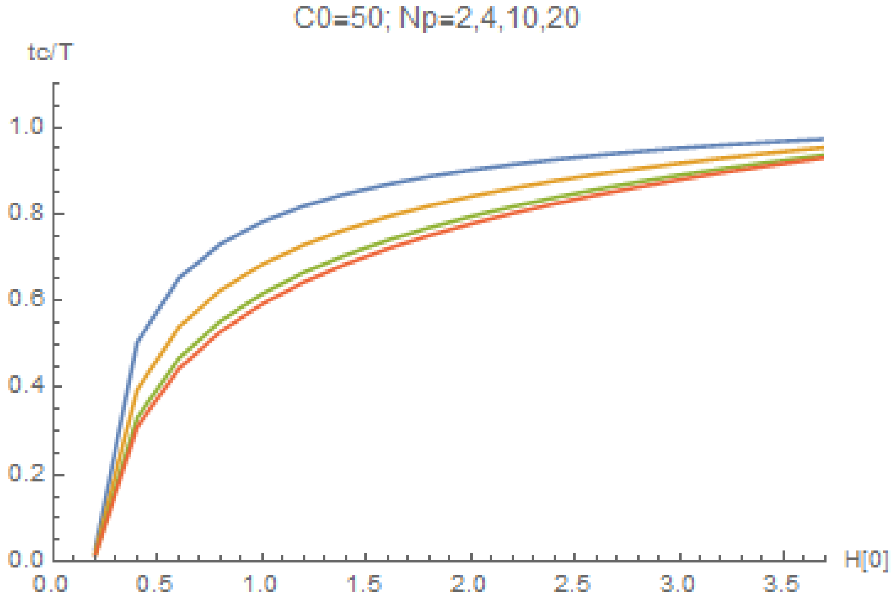
$$\varepsilon[t]=\frac{C0^{\frac{1}{2}-\frac{1}{Np}}\left(-1+C0^{\frac{1}{Np}}\right)(1+Np)\pi\left(1+\frac{\left(-1+C0^{-1/Np}\right)t}{T}\right)^{-1+\frac{Np}{2}}}{T\sqrt{AV0}}$$

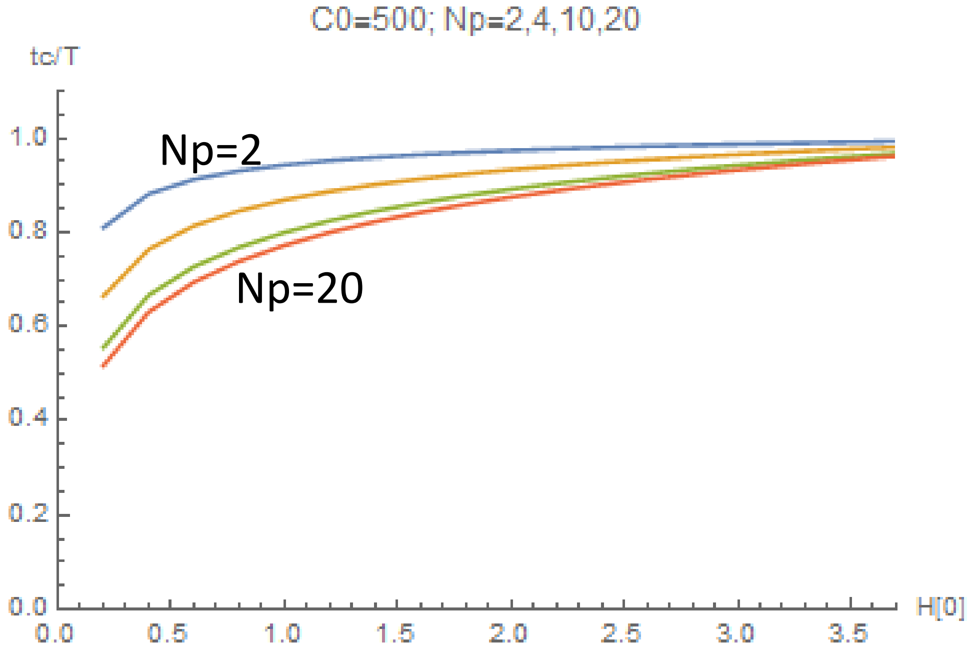


By inspection, ε = constant (iso-adiabatic) when Np = 2

Note: cases with Np>2 become MORE adiabatic at later times

Note: time of capture, tc, is increasingly skewed toward late times as C0 increases.





So we rank degree of adiabaticity according to ε(T) at end of capture

$$\epsilon[T] == \frac{\left(-1 + \mathrm{C0}^{\frac{1}{\mathrm{Np}}}\right)(1+\mathrm{Np})\pi}{T\sqrt{AV[T]}}$$

$$\frac{V'[T]}{V[T]} == \frac{\left(-1 + \mathrm{C0}^{\frac{1}{\mathrm{Np}}}\right)\mathrm{Np}}{T}$$

$$\epsilon[T] == \frac{(1+\mathrm{Np})\ \pi V'[T]}{\mathrm{Np}\sqrt{AV[T]}\,V[T]}$$

V'/V is a surrogate for ε[T] at fixed A.V[T]

$$\frac{V'/V[T]}{V'/V[0]} == \mathrm{C0}^{\frac{1}{\mathrm{Np}}}$$

V[t=0]=V[T]/C0
Ideally, we want C0 large as possible.
But, the smaller is V[t=0], the larger is ε[T]
But ε[T] is monotonic decreasing in Np.
The smallest value is

$$\epsilon[T, \mathrm{Np} == \infty] == \frac{\pi \mathrm{Log}[\mathrm{C0}]}{T\sqrt{AV[T]}}$$

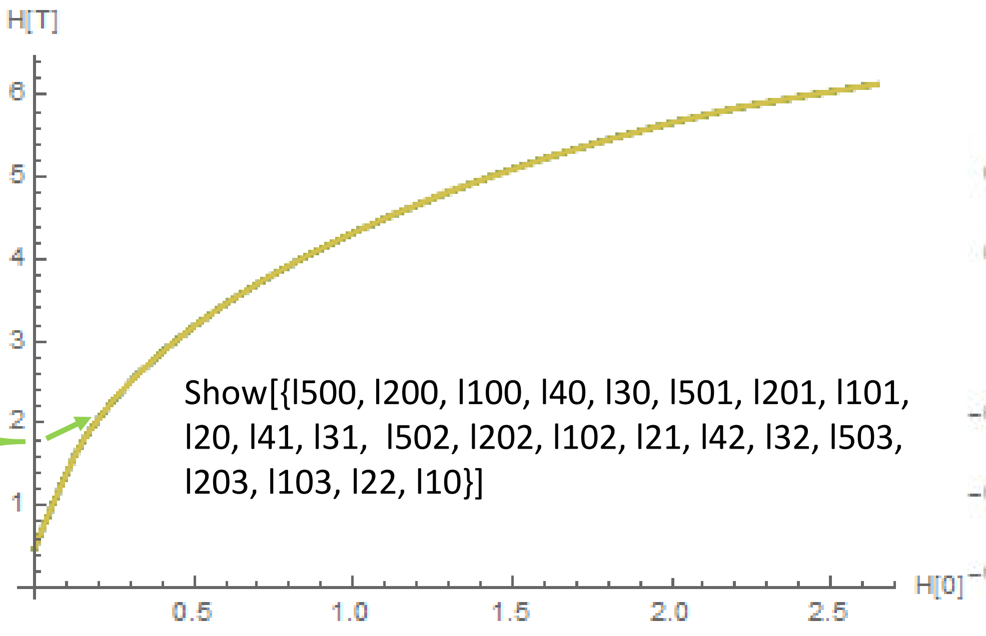
Taking the limit Np→∞, the voltage law becomes the exponential function

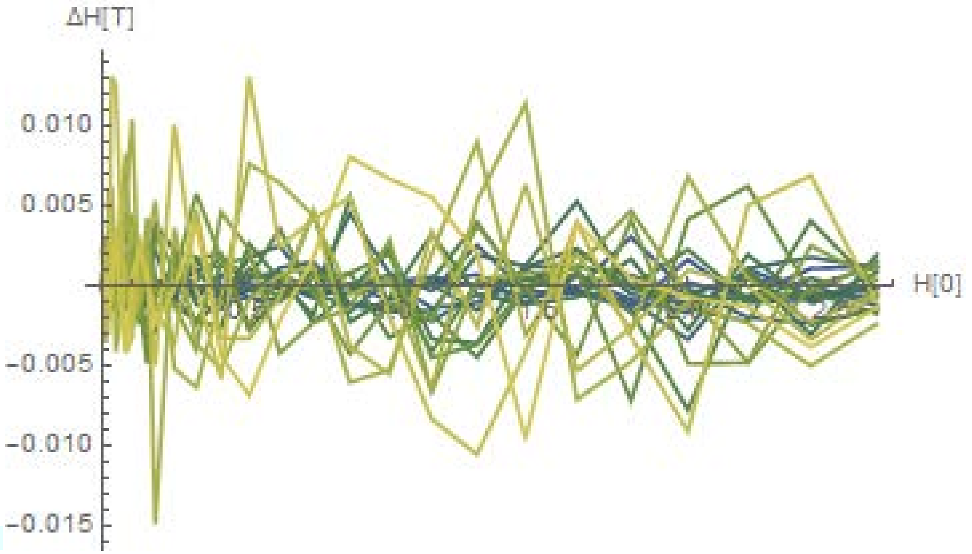
$$V[t] == \mathrm{C0}^{-1+\frac{t}{T}} V[T] \qquad \mathrm{V'[t]/V[t]} == \frac{\mathrm{Log[C0]}}{T}$$

<u>The exponential is the fastest possible adiabatic voltage law</u>

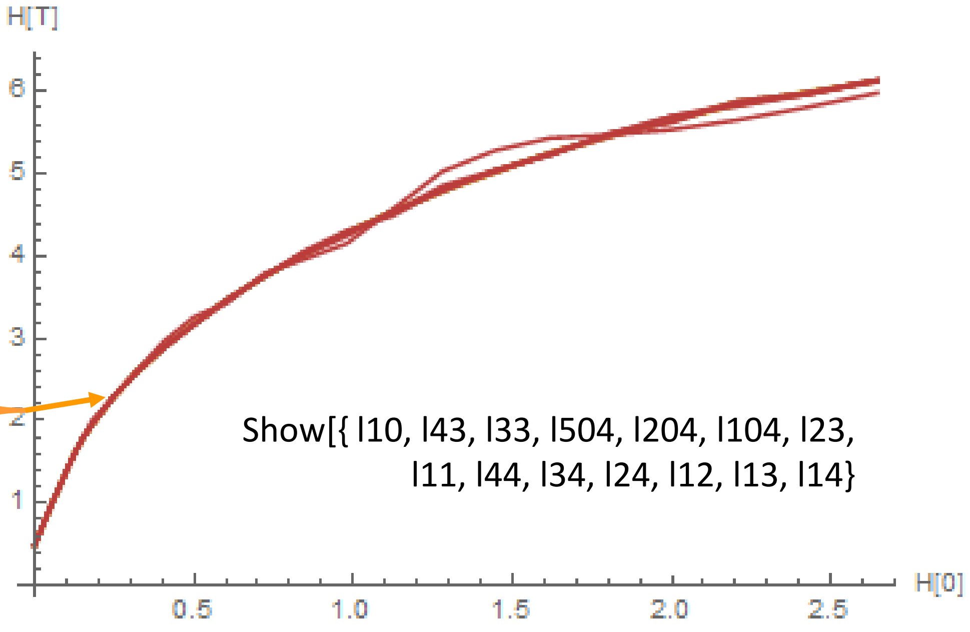
Allows C0>>1000 to be pushed to technological limit of LLRF control

C0	T	Np	V'[T]/V[T]	index
50	2000	50	0.002035	l500
50	2000	20	0.00216	l200
50	2000	10	0.002394	l100
50	2000	4	0.003318	l40
50	2000	3	0.004026	l30
50	1000	50	0.004069	l501
50	1000	20	0.004321	l201
50	1000	10	0.004788	l101
50	2000	2	0.006071	l20
50	1000	4	0.006637	l41
50	1000	3	0.008052	l31
50	500	50	0.008138	l502
50	500	20	0.008642	l202
50	500	10	0.009575	l102
50	1000	2	0.012142	l21
50	500	4	0.013273	l42
50	500	3	0.016104	l32
50	250	50	0.016277	l503
50	250	20	0.017283	l203
50	250	10	0.01915	l103
50	500	2	0.024284	l22
50	2000	1	0.0245	l10
50	250	4	0.026546	l43
50	250	3	0.032208	l33
50	125	50	0.032553	l504
50	125	20	0.034567	l204
50	125	10	0.038301	l104
50	250	2	0.048569	l23
50	1000	1	0.049	l11
50	125	4	0.053093	l44
50	125	3	0.064417	l34
50	125	2	0.097137	l24
50	500	1	0.098	l12
50	250	1	0.196	l13
50	125	1	0.392	l14

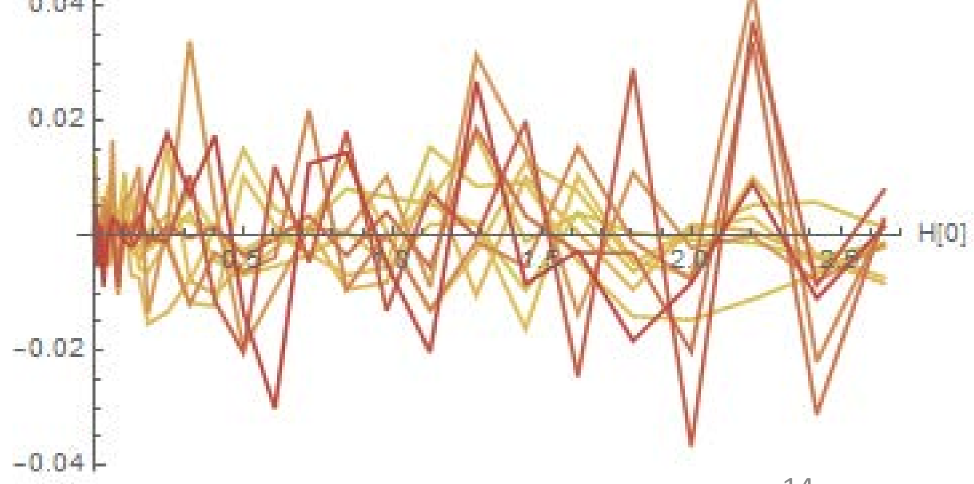




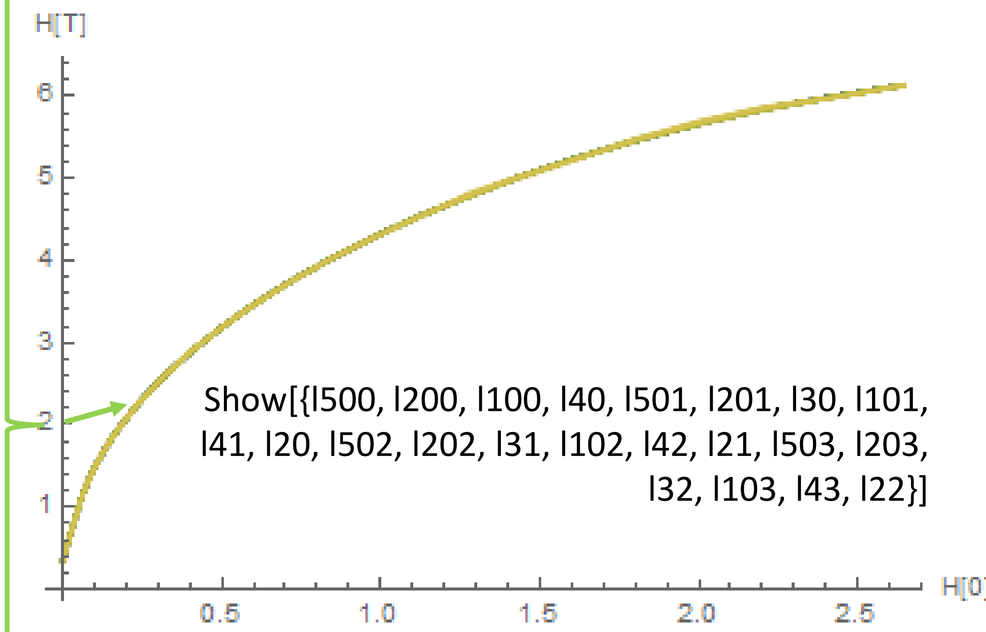
ΔH[T] is the deviation from the apparent smooth curve H[T]



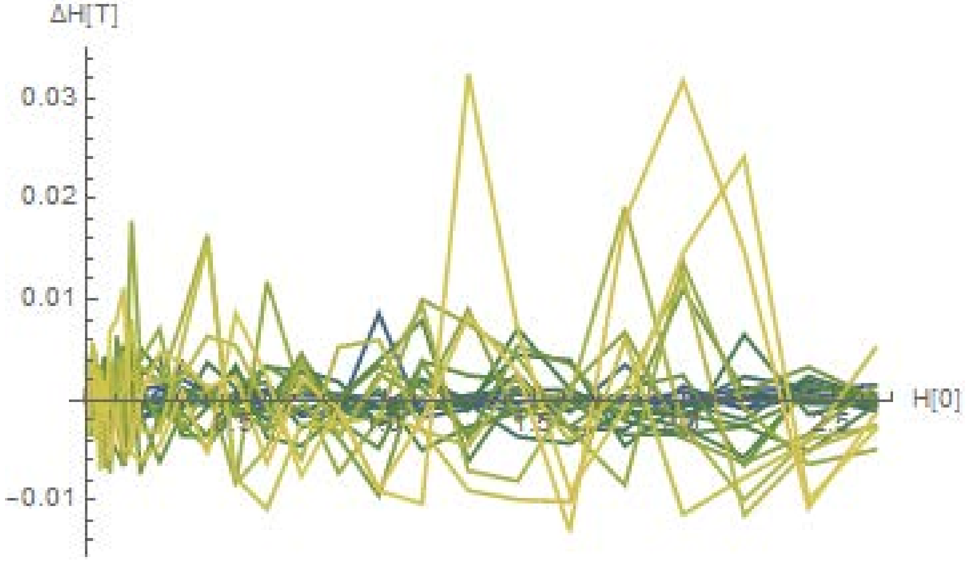
{l24,l12,13,14} omitted – because difference is large



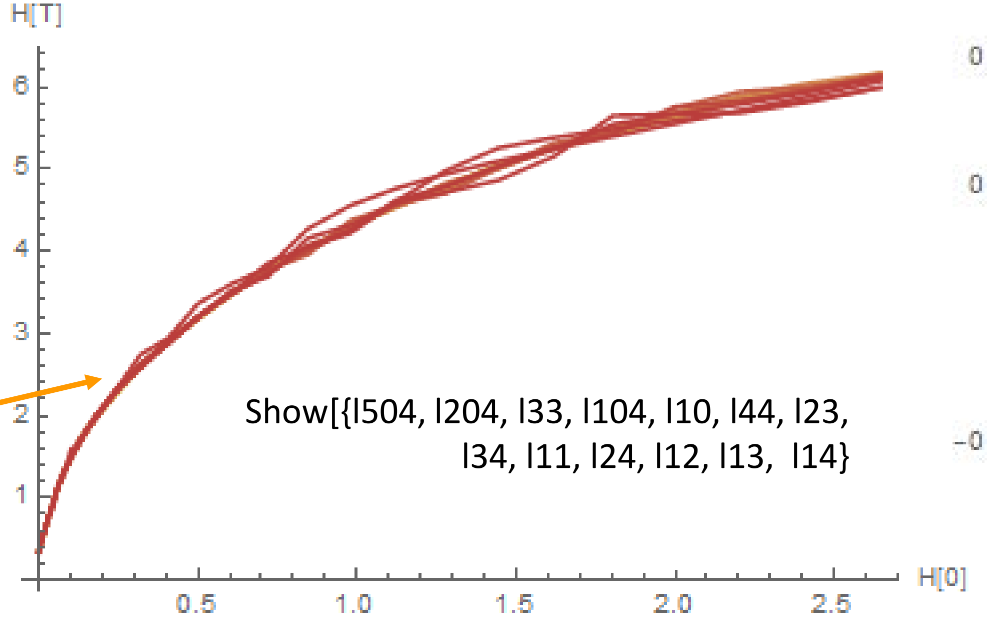
C0	T	Np	V'[T]/V[T]	index
100	2000	50	0.002412	l500
100	2000	20	0.0025893	l200
100	2000	10	0.0029245	l100
100	2000	4	0.0043246	l40
100	1000	50	0.0048239	l501
100	1000	20	0.0051785	l201
100	2000	3	0.0054624	l30
100	1000	10	0.0058489	l101
100	1000	4	0.0086491	l41
100	2000	2	0.009	l20
100	500	50	0.0096478	l502
100	500	20	0.010357	l202
100	1000	3	0.0109248	l31
100	500	10	0.0116979	l102
100	500	4	0.0172982	l42
100	1000	2	0.018	l21
100	250	50	0.0192956	l503
100	250	20	0.020714	l203
100	500	3	0.0218495	l32
100	250	10	0.0233957	l103
100	250	4	0.0345964	l43
100	500	2	0.036	l22
100	125	50	0.0385913	l504
100	125	20	0.0414281	l204
100	250	3	0.0436991	l33
100	125	10	0.0467915	l104
100	2000	1	0.0495	l10
100	125	4	0.0691929	l44
100	250	2	0.072	l23
100	125	3	0.0873981	l34
100	1000	1	0.099	l11
100	125	2	0.144	l24
100	500	1	0.198	l12
100	250	1	0.396	l13
100	125	1	0.792	l14



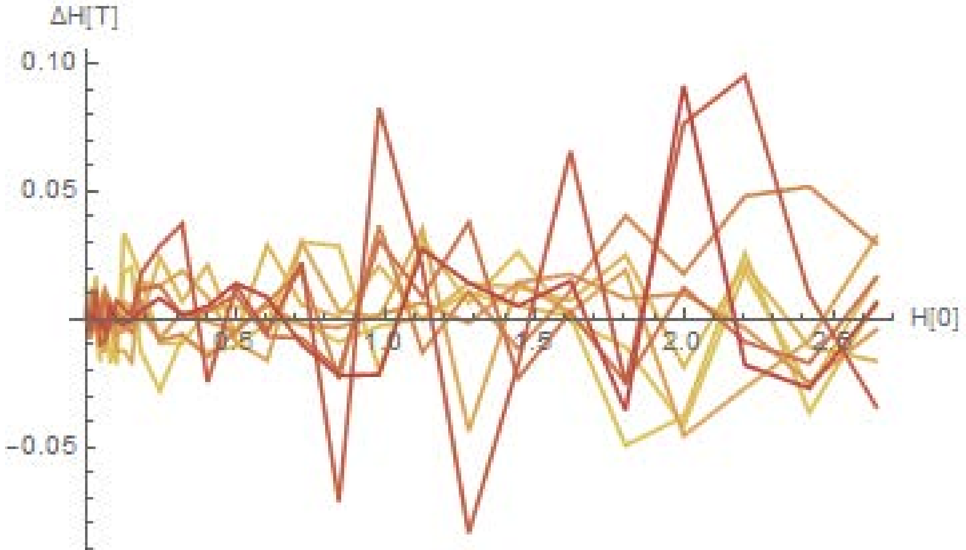
Show[{l500, l200, l100, l40, l501, l201, l30, l101, l41, l20, l502, l202, l31, l102, l42, l21, l503, l203, l32, l103, l43, l22}]



ΔH[T] is the deviation from the apparent smooth curve H[T]

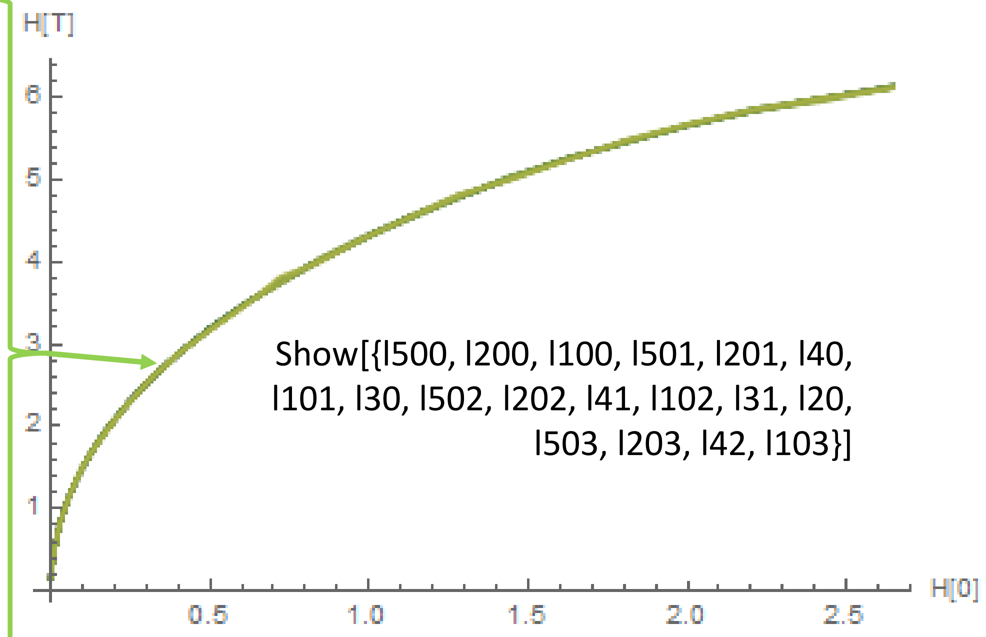


Show[{l504, l204, l33, l104, l10, l44, l23, l34, l11, l24, l12, l13, l14}]

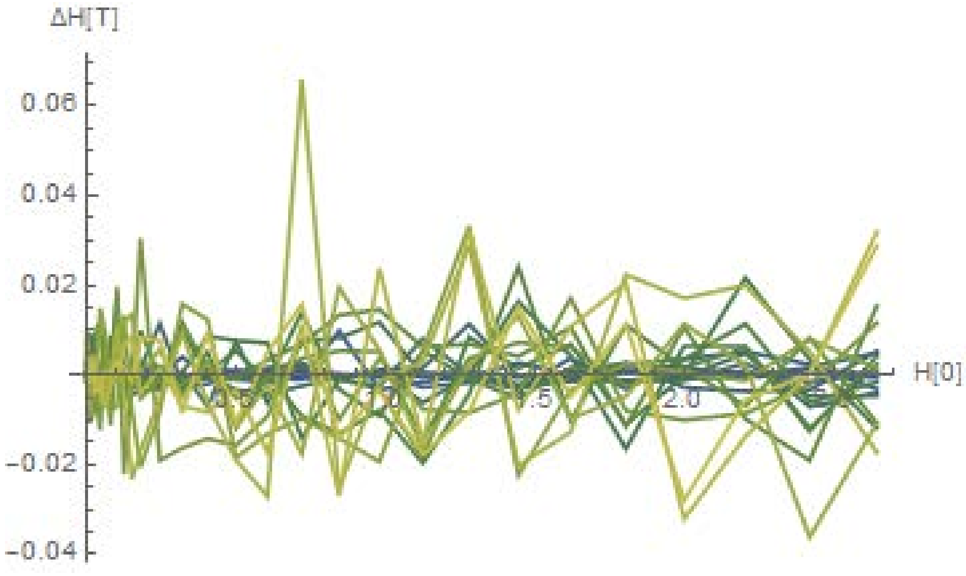


{l24,l12,13,14} omitted – because difference is too large

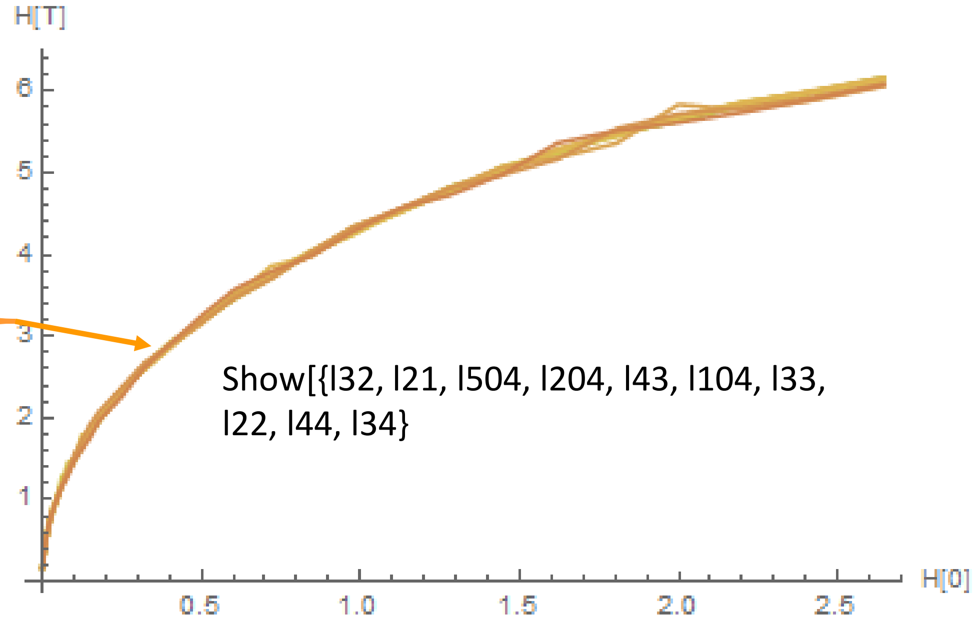
C0	T	Np	V'[T]/V[T]	index
500	2000	50	0.0033087	l500
500	2000	20	0.0036442	l200
500	2000	10	0.0043082	l100
500	1000	50	0.0066173	l501
500	1000	20	0.0072884	l201
500	2000	4	0.0074574	l40
500	1000	10	0.0086165	l101
500	2000	3	0.0104055	l30
500	500	50	0.0132347	l502
500	500	20	0.0145769	l202
500	1000	4	0.0149148	l41
500	500	10	0.0172329	l102
500	1000	3	0.020811	l31
500	2000	2	0.0213607	l20
500	250	50	0.0264693	l503
500	250	20	0.0291537	l203
500	500	4	0.0298297	l42
500	250	10	0.0344658	l103
500	500	3	0.041622	l32
500	1000	2	0.0427214	l21
500	125	50	0.0529387	l504
500	125	20	0.0583074	l204
500	250	4	0.0596593	l43
500	125	10	0.0689316	l104
500	250	3	0.0832441	l33
500	500	2	0.0854427	l22
500	125	4	0.1193187	l44
500	125	3	0.1664881	l34
500	250	2	0.1708854	l23
500	2000	1	0.2495	l10
500	125	2	0.3417709	l24
500	1000	1	0.499	l11
500	500	1	0.998	l12
500	250	1	1.996	l13
500	125	1	3.992	l14



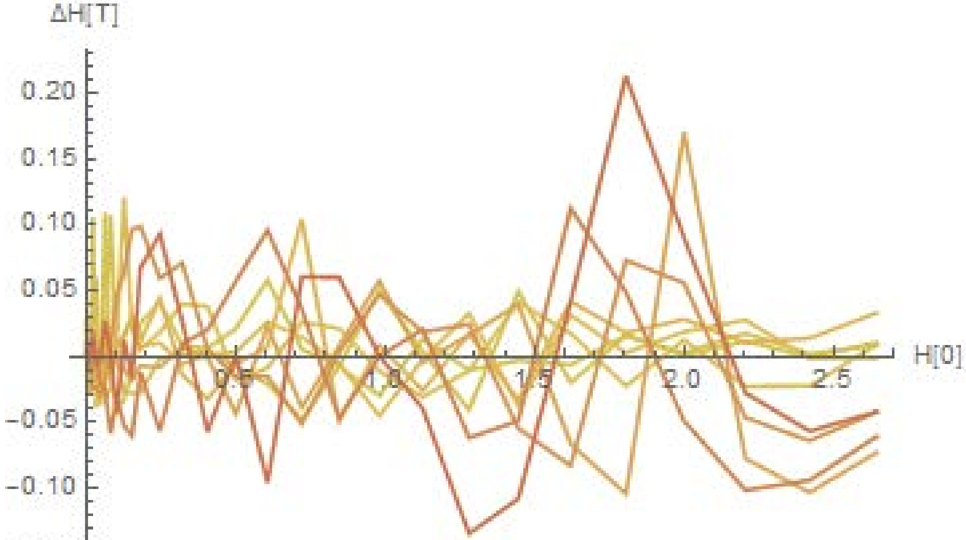
Show[{l500, l200, l100, l501, l201, l40, l101, l30, l502, l202, l41, l102, l31, l20, l503, l203, l42, l103}]



ΔH[T] is the deviation from the apparent smooth curve H[T]

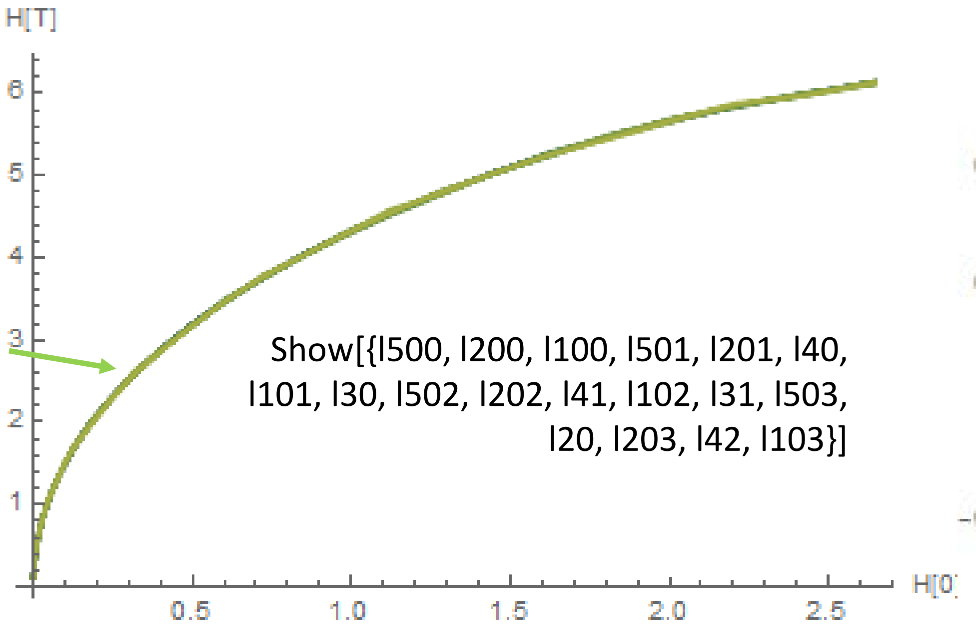


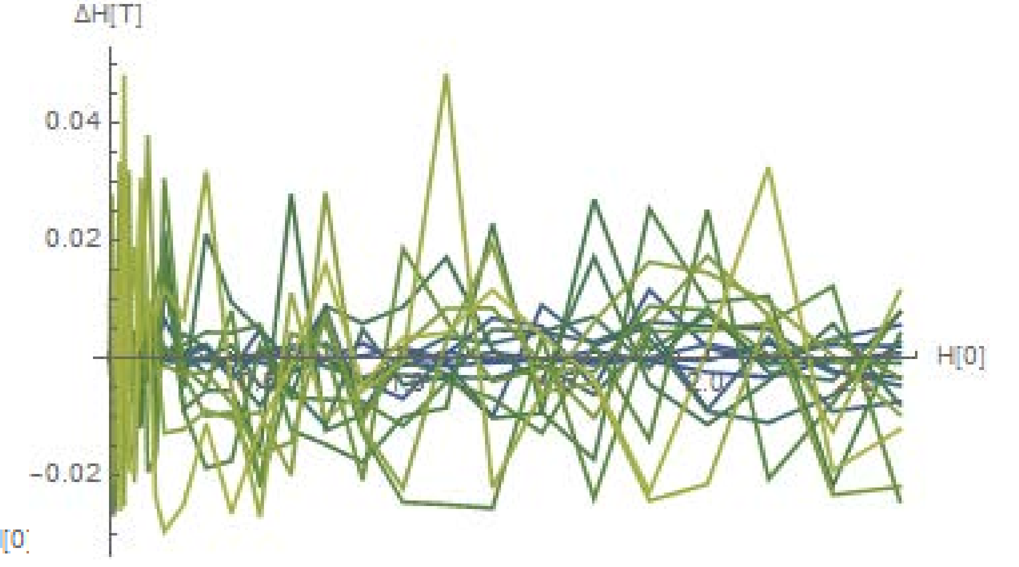
Show[{l32, l21, l504, l204, l43, l104, l33, l22, l44, l34}



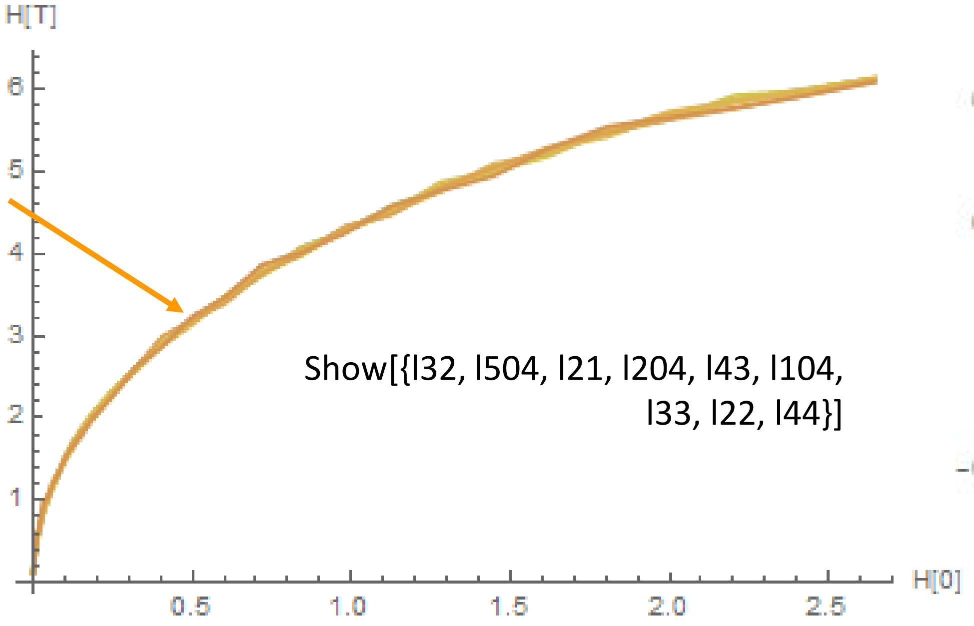
{l23, l10, l24, l11, l12, l13, l14} omitted because difference is too large

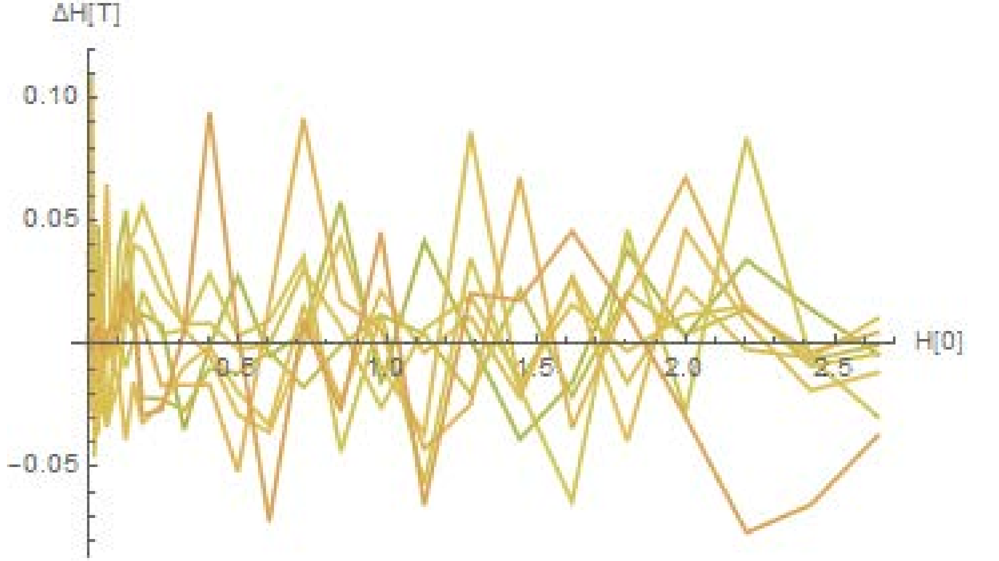
C0	T	Np	V'[T]/V[T]	index
1000	2000	50	0.003704	l500
1000	2000	20	0.004125	l200
1000	2000	10	0.004976	l100
1000	1000	50	0.007408	l501
1000	1000	20	0.008251	l201
1000	2000	4	0.009247	l40
1000	1000	10	0.009953	l101
1000	2000	3	0.0135	l30
1000	500	50	0.014815	l502
1000	500	20	0.016502	l202
1000	1000	4	0.018494	l41
1000	500	10	0.019905	l102
1000	1000	3	0.027	l31
1000	250	50	0.029631	l503
1000	2000	2	0.030623	l20
1000	250	20	0.033003	l203
1000	500	4	0.036987	l42
1000	250	10	0.03981	l103
1000	500	3	0.054	l32
1000	125	50	0.059261	l504
1000	1000	2	0.061246	l21
1000	125	20	0.066006	l204
1000	250	4	0.073975	l43
1000	125	10	0.079621	l104
1000	250	3	0.108	l33
1000	500	2	0.122491	l22
1000	125	4	0.147949	l44
1000	125	3	0.216	l34
1000	250	2	0.244982	l23
1000	125	2	0.489964	l24
1000	2000	1	0.4995	l10
1000	1000	1	0.999	l11
1000	500	1	1.998	l12
1000	250	1	3.996	l13
1000	125	1	7.992	l14





ΔH[T] is the deviation from the apparent smooth curve H[T]



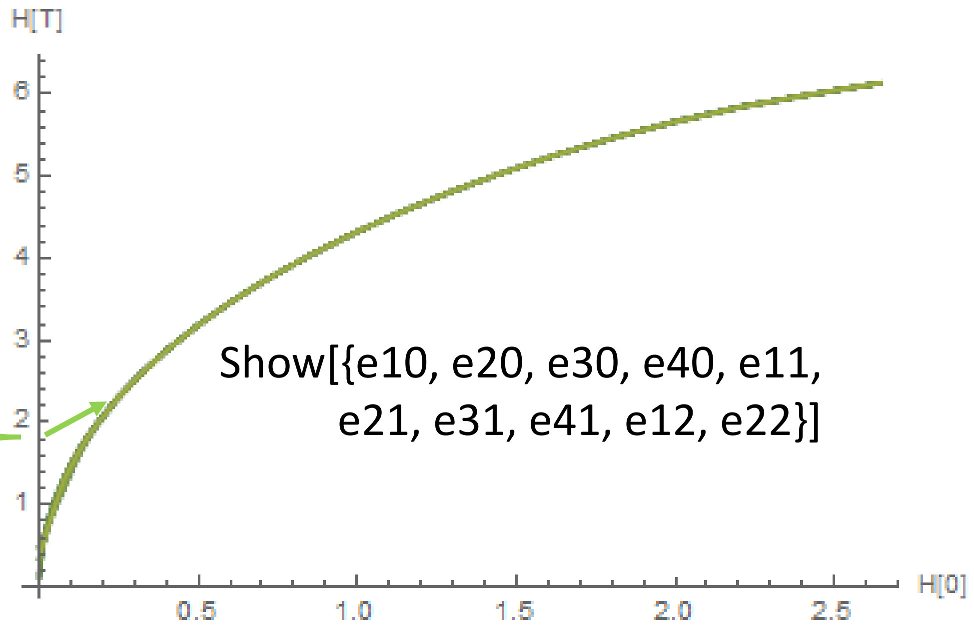


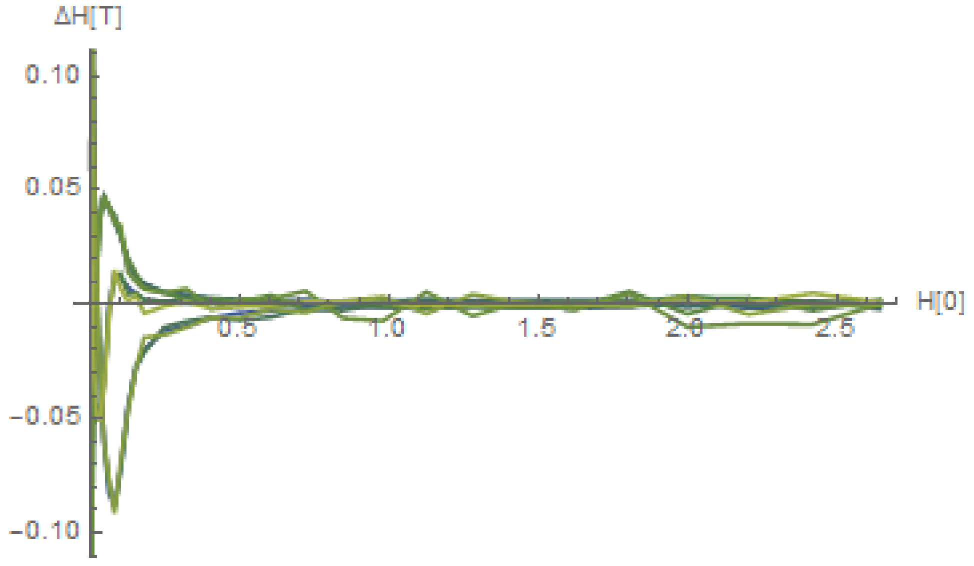
{l34, l23, l24, l10, l11, l12, l13, l14} omitted because large difference

Exponential Voltage Law V'/V=Log[C0]/T

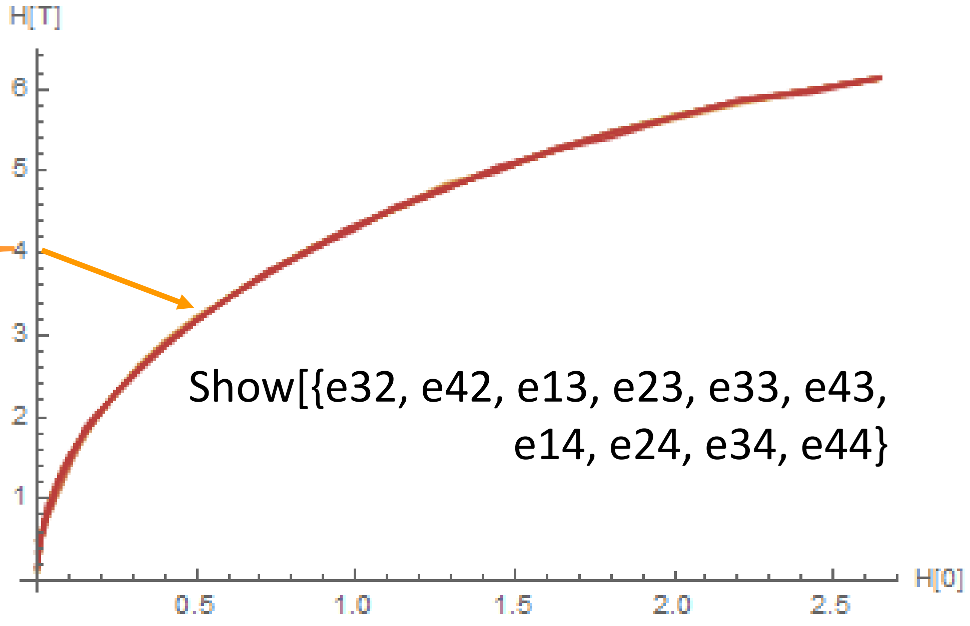
C0	T	V'[T]/V[T]	index
50	2000	0.001956	**E10**
100	2000	0.002303	**E20**
500	2000	0.003107	**E30**
1000	2000	0.003454	**E40**
50	1000	0.003912	**E11**
100	1000	0.004605	**E21**
500	1000	0.006215	**E31**
1000	1000	0.006908	**E41**
50	500	0.007824	**E12**
100	500	0.00921	**E22**
500	500	0.012429	**E32**
1000	500	0.013816	**E42**
50	250	0.015648	**E13**
100	250	0.018421	**E23**
500	250	0.024858	**E33**
1000	250	0.027631	**E43**
50	125	0.031296	**E14**
100	125	0.036841	**E24**
500	125	0.049717	**E34**
1000	125	0.055262	**E44**

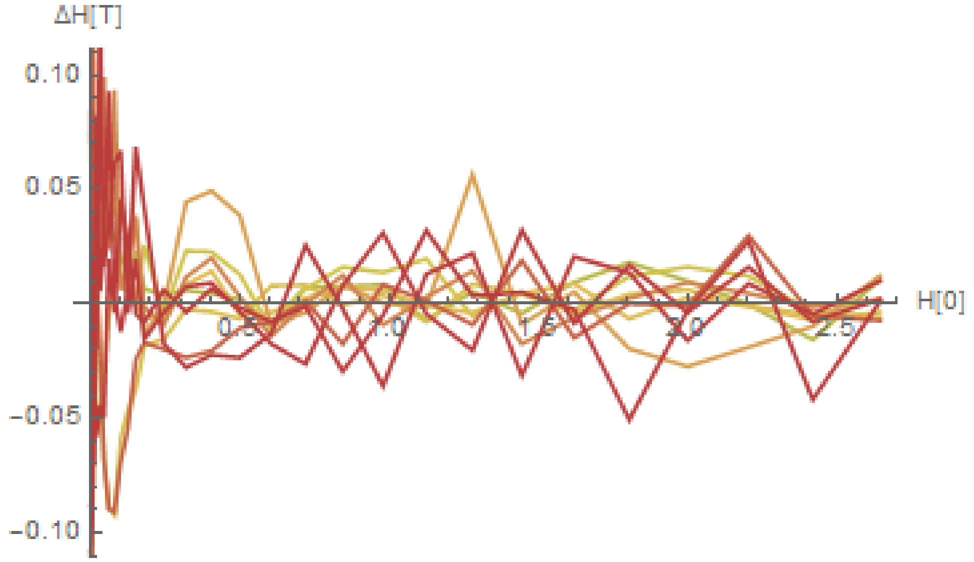
Final capture voltage V[T]=3





ΔH[T] is the deviation from the apparent smooth curve H[T]





We have shown that H[T] is almost a universal function of H[0], almost independent of C0, Np and T, provided that adiabaticity parameter ε(T)< few %

Question: But "how do the voltage laws differ?"
Answer: The r.m.s. values of δH[T] about H[T]

Ideally one value H[0] maps uniquely to a single H[T].
However, there is a point spread function such that each H[0] maps to a narrow band H[T]±δH[T]
δH represents phase mixing due to imperfect adiabaticity.

There are two separate non-adiabatic processes which generate spreads δH

1) Sudden voltage turn on at t=0
2) Crossing the separatrix (m=1) at time tc, for H[0] > 2V[T]/C0

Sudden voltage turn-on lifts ALL values of Hamiltonian at t=0

Sudden voltage turn-on adds spread δH to ALL values of Hamiltonian at t=0

$H[0] \rightarrow H[0] + (1 - \mathrm{Cos}[x])(V[T]/\mathrm{C0})$ V[0]=V[T]/C0

Average lift increment ΔH = V[0]Integrate[(1-Cos[x]),{x,-π,+π}]/(2π) = V[T]/C0

Common spread= $\delta\mathrm{H} == -\frac{\mathrm{VCos}[x]}{\mathrm{C0}}$ x={-π,+π}

We integrate δH^2 over x={-π,+π} to find the variance. The common r.m.s spread at t=0 is V/C0/Sqrt[2].

How this initial spread evolves depends on whether H[0] is inside or outside the initial RF bucket at t=0.

For those trajectories captured at t=0, i.e. for H[0] < 2V[T]/C0, the Hamiltonian will be inflated to H[T]=H[0] Sqrt[V[T]/V[0]] = H[0] Sqrt[C0].

But $H[0] \rightarrow H[0] + (V[T]/\mathrm{C0})$

So the lift at t=T becomes ΔH[T]= (V[T]/C0)Sqrt[C0] = V[T]/Sqrt[C0]

The r.m.s spread at t=0 is V/C0/Sqrt[2]. This will be inflated to t=T by Sqrt[C0].

For trajectories having Hamiltonian value H[0]< 2V[T]/C0 prior to voltage turn-on, each H[0] value will acquire a lift ΔH = V[T]/Sqrt[C0] and an r.m.s spread V/Sqrt[2C0] at t=T

For those trajectories NOT captured at t=0, i.e. for H[0] > 2V[T]/C0, the Hamiltonian will be inflated to H[T]= a_0Sqrt[H[0]]

But $H[0] \to H[0] + (V[T]/\mathrm{C0})$ $\quad H[T] + \Delta\mathrm{H}[T] == a_0\sqrt{H[0] + \frac{V[T]}{\mathrm{C0}}} == \sqrt{H[0]}a_0\sqrt{1 + \frac{V[T]}{\mathrm{C0}H[0]}}$

So the lift ΔH at t=T becomes

$$\Delta\mathrm{H}[T] \to \frac{a_0V[T]}{2\mathrm{C0}\sqrt{H[0]}} - \frac{a_0V[T]^2}{8\mathrm{C0}^2H[0]^{3/2}}$$

To 2[nd] order

The r.m.s spread at t=0 is δH[0]= V[T]/C0/Sqrt[2]. This will be evolved to δH[T] in the same way as above.

$$\sqrt{2}\delta\mathrm{H}[T] \to \frac{a_0V[T]}{2\mathrm{C0}\sqrt{H[0]}} - \frac{a_0V[T]^2}{8\mathrm{C0}^2H[0]^{3/2}}$$

To 2[nd] order

So, for H[0] > 2V[T]/C0, the lift and spread both fall monotonically, roughly as $\frac{1}{\sqrt{H[0]}}$

If there are other processes generating spreads, then the initial spread due to sudden turn-on will set the baseline

There is in fact a third process which is potentially non-adiabatic. However, it is often masked/obscured by propagation of the initial sudden turn-on, unless C0 is large and/or T is too short

The instantaneous rate of change of the Hamiltonian is

$$H'[t] == (1 - \mathrm{Cos}[x[t]])V'[t]$$

Previously we assumed that we could form the average rate over one up/down or down/up

$$\overline{H'[t]} == V'[t]\ \mathrm{Integrate}[(1 - \mathrm{Cos}[x[t]]), \left\{t, -\frac{\tau}{4}, \frac{\tau}{4}\right\}]/(2\tau)$$

This assumption breaks down if either: (a) V'/V, or (b) Jacobi m[t] change too quickly.

We consider case (a) that V'[t] cannot be considered constant during the half period $\left\{t - \frac{\tau}{4}, t + \frac{\tau}{4}\right\}$

We presume this condition occurs during separatrix crossing at t=tc; because m→1 and the period is longest at this time.

(For the linear voltage law V'[t]=constant, both assumptions are broken.)

Non-adiabatic growth of Hamiltonian during separatrix crossing

We assume the crossing to be a one-time event* during which H[tc] acquires a spread due to the dependence on the phase of the oscillation immediately prior to crossing. This spread is later inflated by Sqrt[V[T]/V[tc]] inside the bucket.

We approximate V'[t] during the crossing by V'[tc]+V''[tc](t-tc) $\delta H'[t] == (1 - \mathrm{Cos}[x[t]])V''[tc]t$

We integrate over $\left\{t, -\frac{\tau}{4}, \frac{\tau}{4}\right\}$ to find δH[tc]

***The assumption that H-spread is acquired only in a one-time event (and is not a continuous process) is a CONJECTURE, yet to be proven.**

Outside

$$\delta H[tc] == \int_{-\frac{\tau}{4}}^{\frac{\tau}{4}} 2t\,\text{JacobiCD}\left[\sqrt{m}(q+tw),\frac{1}{m}\right]^2 V''[tc]\,dt$$

"q" is the initial phase of the oscillation prior to separatrix crossing

Inside

$$\delta H[tc] == \int_{-\frac{\tau}{4}}^{\frac{\tau}{4}} 2mt\,\text{JacobiSN}[q+tw,m]^2 V''[tc]\,dt$$

w = small-amplitude synchrotron frequency

The integrals can be performed as series expansions if Jacobi^2 is substituted by its Fourier series (Whittaker and Watson)

To lowest order in Jacobi "m" parameter

Outside $$\delta H[tc] == \frac{\pi \text{Sin}[2\pi Q]V''[tc]}{2mw^2} = \frac{\pi \text{Sin}[2\pi Q]V''[tc]}{2AmV[tc]}$$

Inside $$\delta H[tc] == \frac{m\pi \text{Sin}[2\pi Q]V''[tc]}{2w^2} = \frac{m\pi \text{Sin}[2\pi Q]V''[tc]}{2AV[tc]}$$

"A" has dimensions of 1/t^2

Q is a rescaling of initial phase q into the range [0,1]

Outside vs inside results are identical in limit m→1

The average of δH over Q is zero, but the r.m.s. is non zero.

$$\text{Sqrt}[<\delta H[tc]^2>] == \frac{\pi\, V''[tc]}{4w^2}$$

Valid for trajectories that do NOT experience the sudden capture at t=0, namely initial Hamiltonian values that satisfy H[0]>2V[T]/C0

Now propagate δH inside the RF bucket as if it were any other H[t]

$$H[T] + \delta H[T] == \sqrt{\frac{V[T]}{V[tc]}}(H[tc] + \delta H[tc])$$

$$\delta H[T] == \sqrt{\frac{V[T]}{V[tc]}}\,\delta H[tc]$$

But we know that $V[tc] \to \frac{H[T]^2}{4V[T]}$ and $H[T] \to \frac{\pi\sqrt{H[0]}\sqrt{V[T]}}{\sqrt{2}}$

$$\delta H[T] == \frac{\sqrt{2}\mathrm{Sin}[2\pi Q]\sqrt{\frac{V[T]}{H[0]}}V''[tc]}{A\,V[tc]}$$

The scaling law for the point-spread function

$$\mathrm{RMS}\{\delta H[T]\} == \frac{\sqrt{\frac{V[T]}{H[0]}}V''[tc]}{\sqrt{2}\,A\,V[tc]}$$

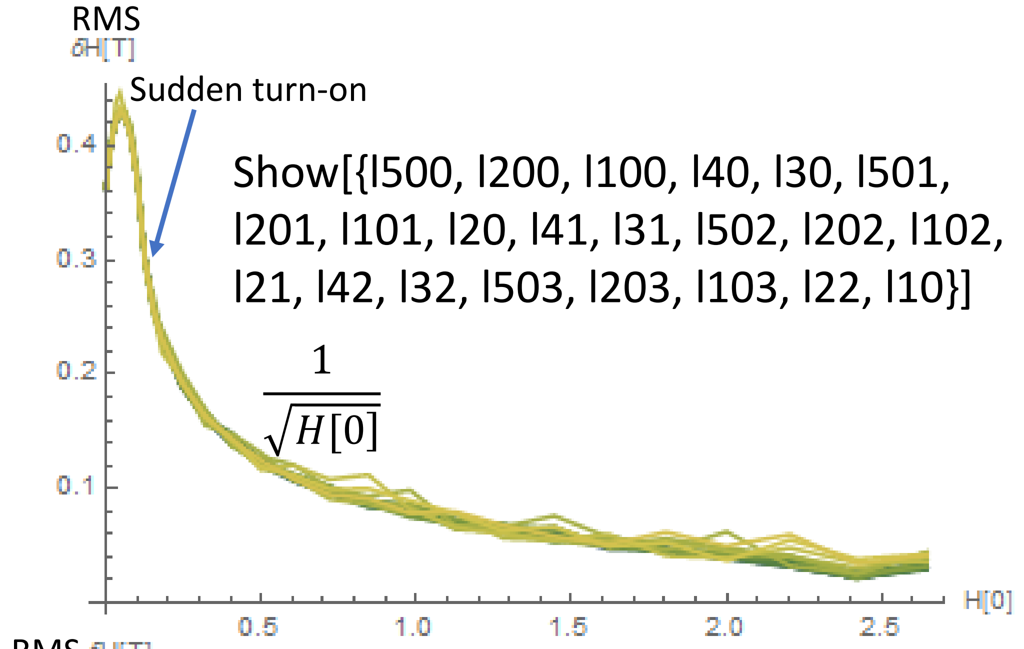
Recapitulation: Ideally one value H[0] maps uniquely to a single H[T].
However, there is a point-spread function such that each H[0] maps to a narrow band H[T]±δH[T].
δH represents phase mixing due to imperfect adiabaticity.

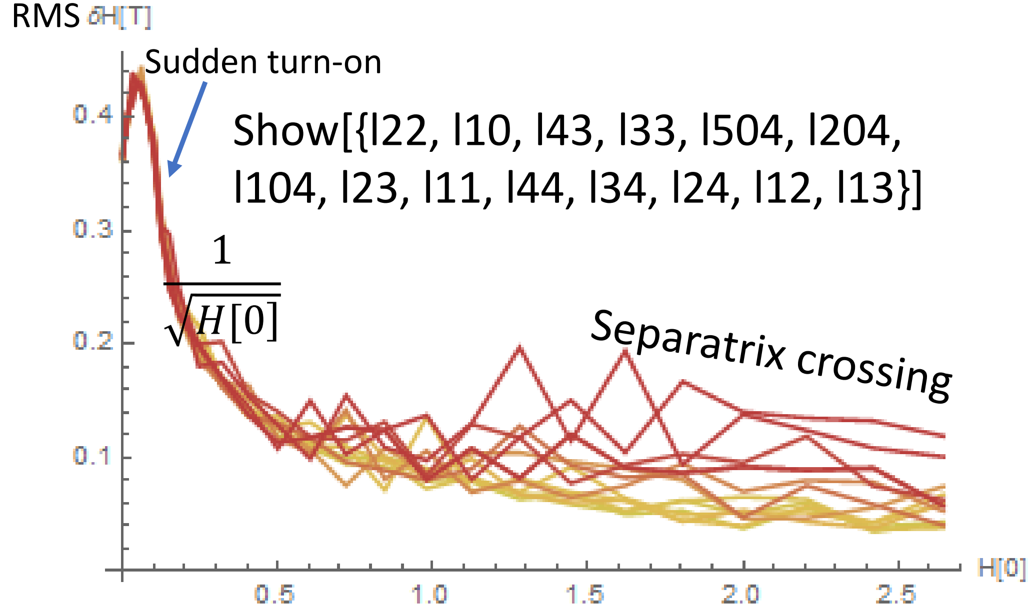
If RMS[δH] is large, there will be noticeable r.m.s. (and 100% envelope) emittance growth.
Note: V"/V is largest at at t=T. V"/V increases as C0 is increased or T reduced or Np reduced.

Tailoring V''/V allows to adjust the emittance growth of the core versus the tails.
However, the dominant growth of the core will be due to the sudden turn-on, unless C0 can be made very large.
Final note: we know tc as a function of H[0]. Therefore, RMS{δH[T]} can be written as a function solely of H[0].

Question: But "how do the voltage laws differ?"
Answer: The r.m.s. values of δH[T] about H[T]

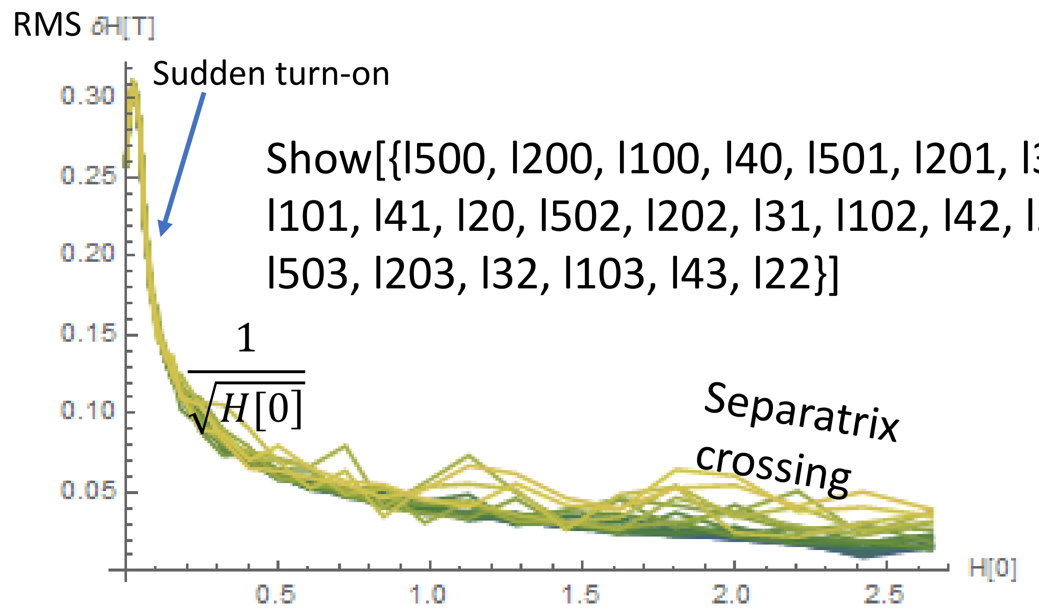
C0=50; V[0]=2%V[T]

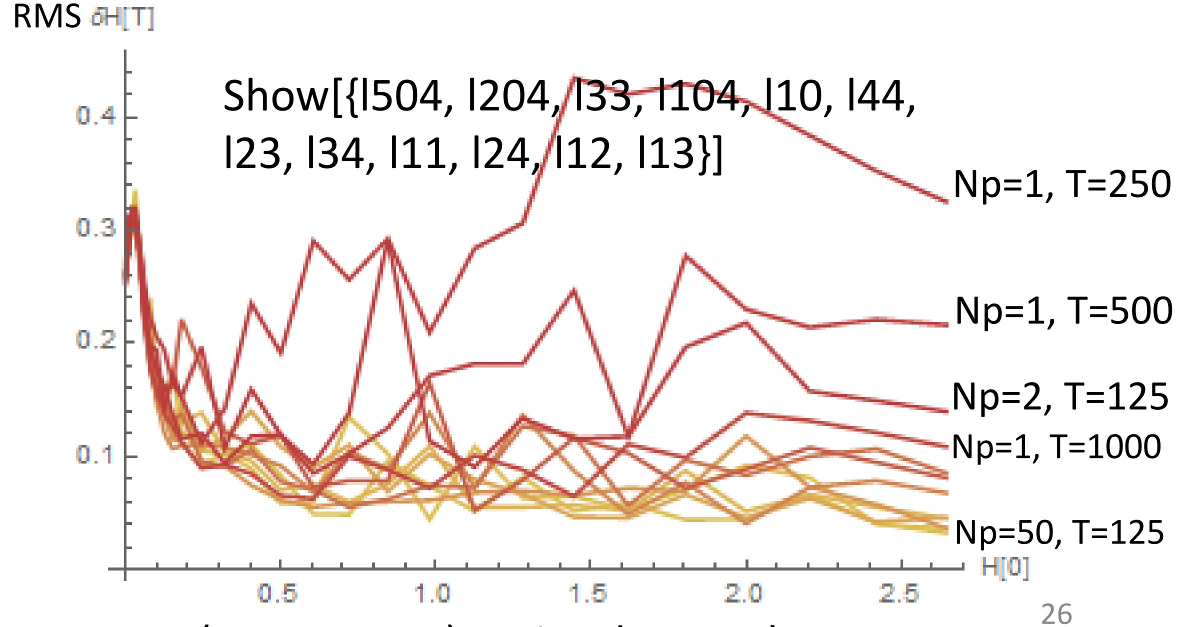




L14 (Np=1,T=125) omitted as too large

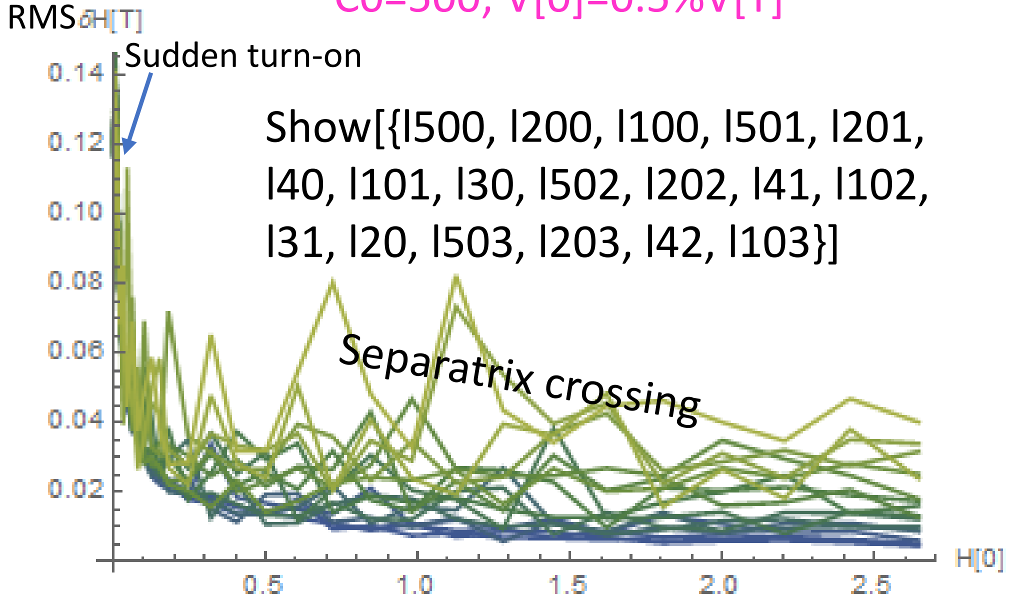
C0=100; V[0]=1%V[T]



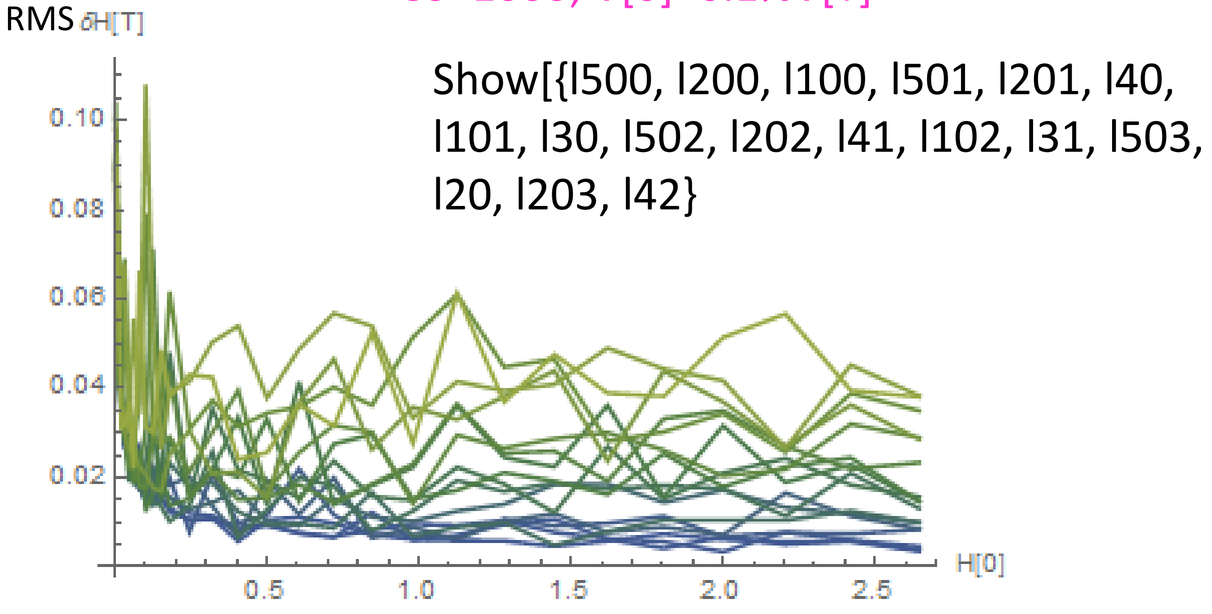


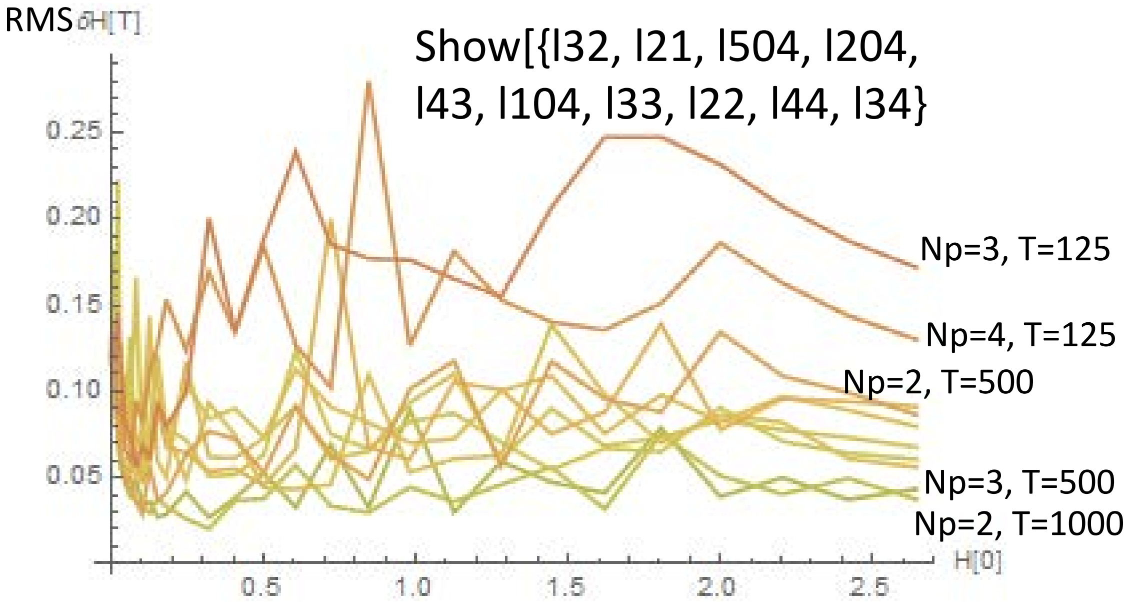
L14 (Np=1,T=125) omitted as too large

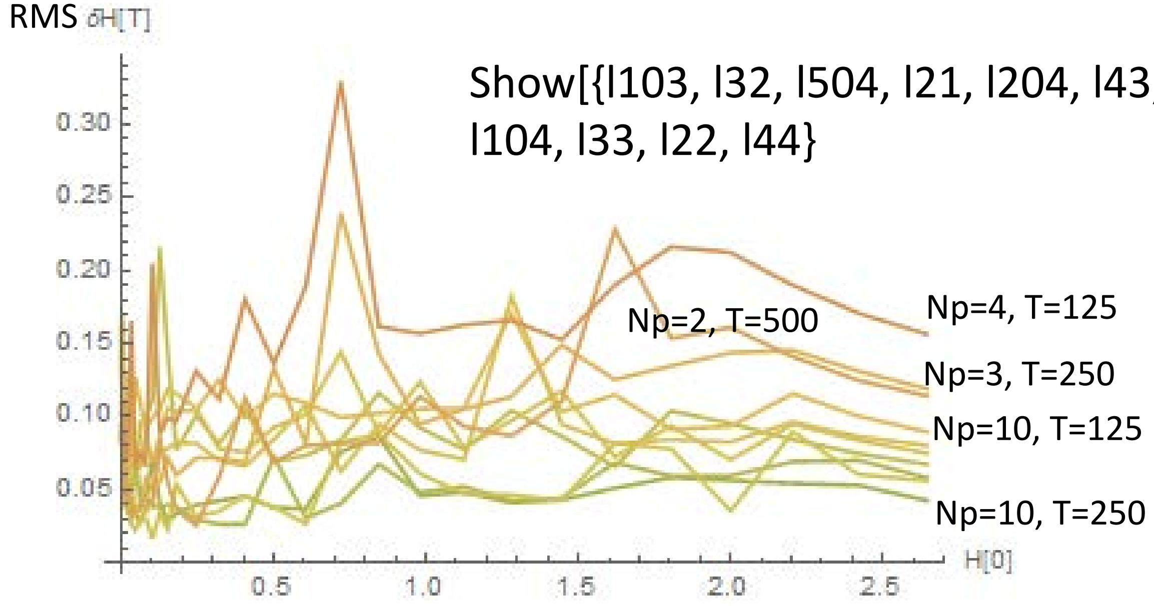
C0=500; V[0]=0.5%V[T]



C0=1000; V[0]=0.1%V[T]







{l23, l24}=(Np=2, T=250,125) omitted as too large
{l10, l11, l12, l13, l14}=(Np=2, T=2000, … 125) omitted

{l34, l23, l24, l10, l11, l12, l13, l14} omitted as too large

Conclusion from data plots and trending

Increasing C0 reduces core damage – reduces effect of sudden voltage turn-on

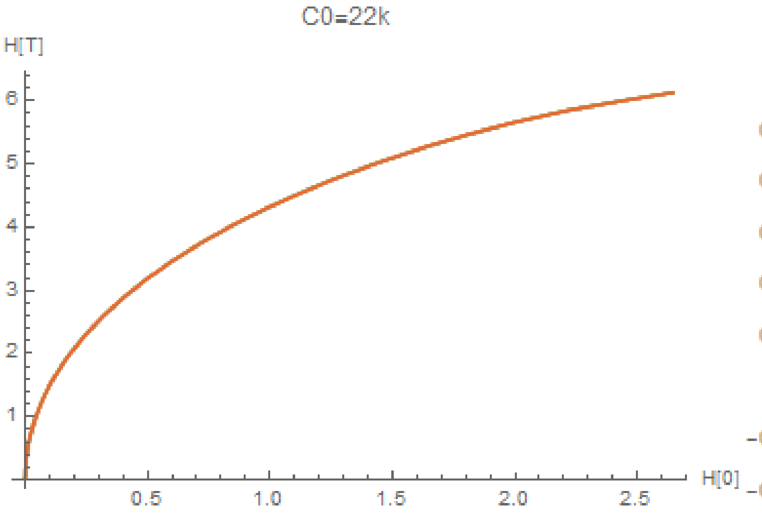
But to reduce body and tail damage, T must be increased and/or Np increased >2

Strategy for Voltage Law Selection

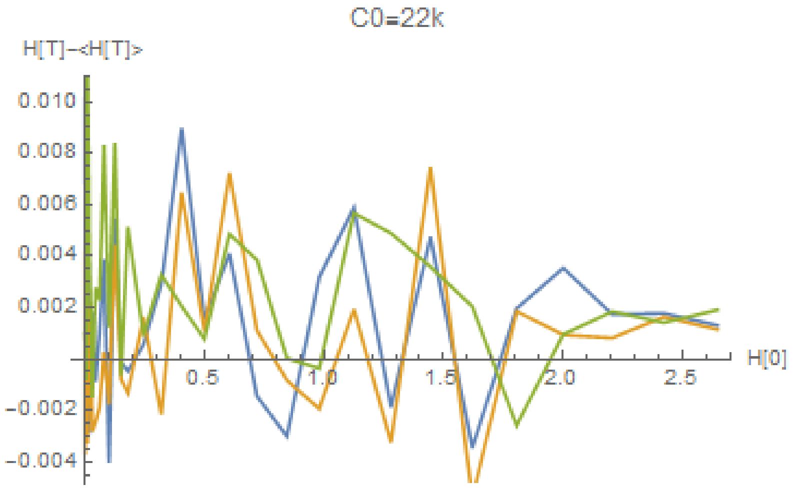
Push C0 to technological limit (control of very small RF cavity voltage)

Take exponential law and shortest duration T consistent with adiabaticity parameter, say, ε(T)<0.02. Then progressively increase T as consistent with other machine operation constraints, and take Np consistent with ε(T). For example

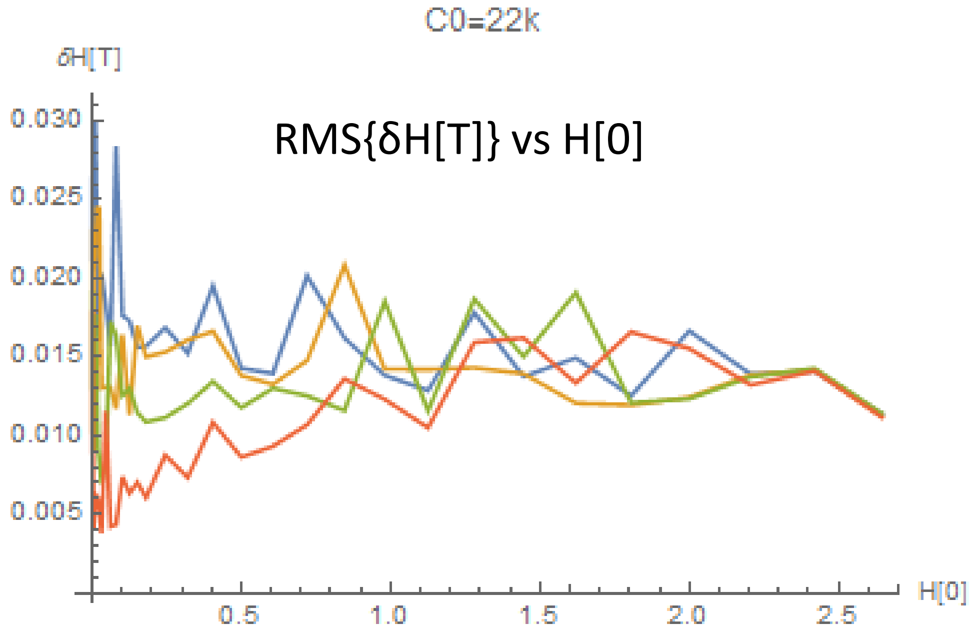
C0	T	Np	V'[T]/V[T]	index
22K	1000	Exp	0.01	E1
22K	1200	28	0.01	N28
22K	2000	8	0.01	N8
22K	4400	4	0.01	N4



Show[{e1, n28, n8, n4}]



Blue=e1; gold=n28; green = n8

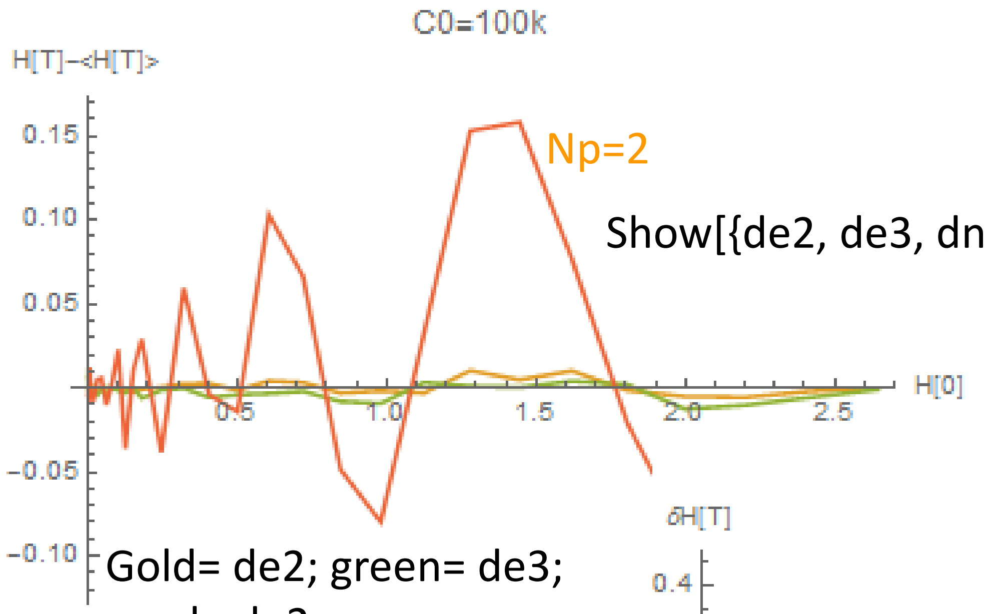


Blue=e1; gold=n28; green = n8; coral=n4

Four times longer duration leads to four times smaller core damage

Back-to-back Comparison of Exponential law versus Iso-adiabatic (Np=2) for an extreme value of C0

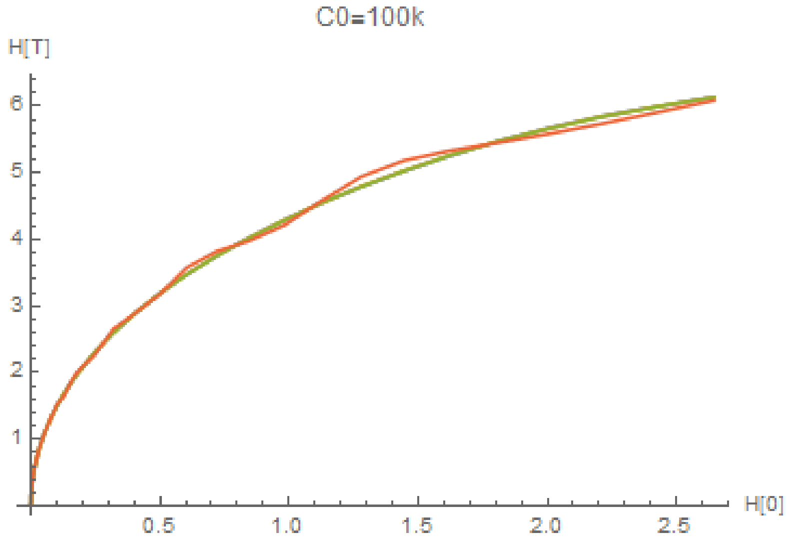
C0	T	Np	index
100K	2000	Exp	E1
100K	1000	Exp	E2
100K	500	Exp	E3
100K	2000	2	N2

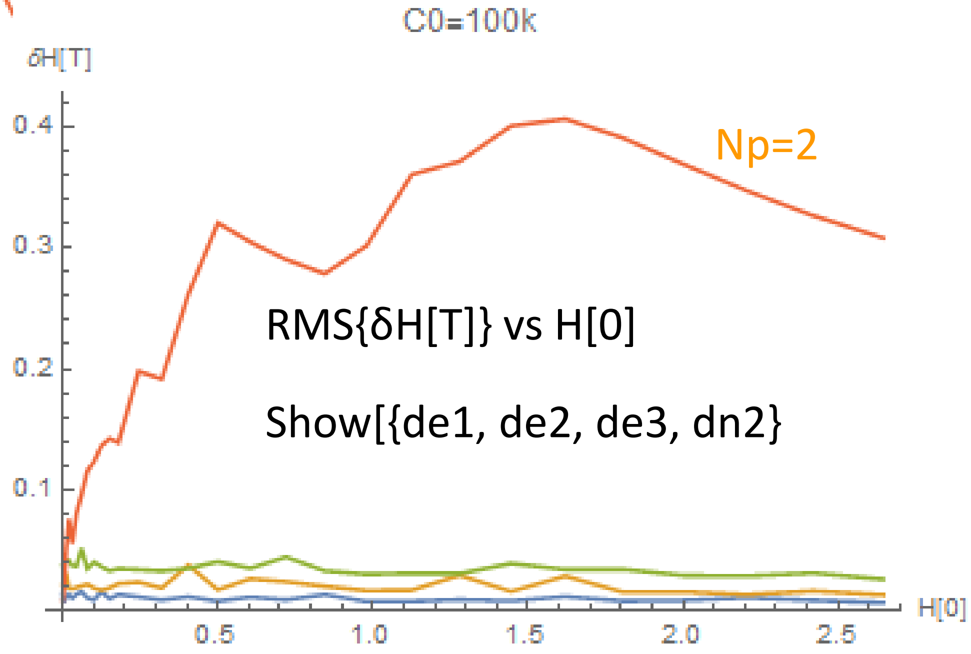


Show[{de2, de3, dn2}]

Gold= de2; green= de3; coral= dn2

Show[{e1, e2, e3, n2}]





RMS{δH[T]} vs H[0]

Show[{de1, de2, de3, dn2}

Blue= de1; gold= de2; green= de3; coral= dn2

BUNCH LONGITUDINAL PROFILE

Knowing the relationship between H[T] and H[0] facilitates the construction of the bunch longitudinal profile and bunch momentum spectrum from the momentum spectrum of the initial un-bunched coasting beam.

Momentum spectrum of un-bunched beam

$$\rho[p] == \frac{2\left(1-\frac{p^2}{\text{pmax}^2}\right)^n \text{Gamma}[\frac{3}{2}+n]}{\sqrt{\pi}\,\text{pmax Gamma}[1+n]}$$

transformation

$$\left\{p \to \frac{\sqrt{2}\sqrt{\text{H0}}}{\sqrt{A}}\right\}$$

Hamiltonian spectrum of un-bunched beam

$$\rho[\text{H0}] == \frac{\left(1-\frac{\text{H0}}{\text{H0max}}\right)^n \text{Gamma}[2+n]}{\text{H0max Gamma}[1+n]}$$

transformation $\left\{\text{H0} \to \frac{\text{HT}^2}{\text{a0}^2}\right\}$

Hamiltonian spectrum of bunched beam

$$\rho[\text{HT}] == \frac{2\left(1-\frac{\text{HT}^2}{\text{HTmax}^2}\right)^n \text{Gamma}[\frac{3}{2}+n]}{\text{HTmax}\sqrt{\pi}\text{Gamma}[1+n]}$$

Bunch Shape is obtained by integrating over the product of Hamiltonian spectrum and Dwell function

$$H[x,p,t] == 1-\text{Cos}[x[t]]+\frac{p[t]^2}{2}$$

$$\text{dwell} == \frac{1}{xdot} == \frac{1}{p[t]} == \frac{1}{\sqrt{2}\sqrt{-1+\text{HT}+\text{Cos}[x]}}$$

$$\text{norm} = 2\text{Integrate}[\text{dwell},\{x,0,\text{ArcCos}[1-\text{HT}]\}]$$

$$\text{dwell} \to \frac{\text{dwell}}{\text{norm}} == \frac{1}{2\sqrt{2}\sqrt{-1+\text{HT}+\text{Cos}[x]}\,\text{EllipticK}[\frac{\text{HT}}{2}]}$$

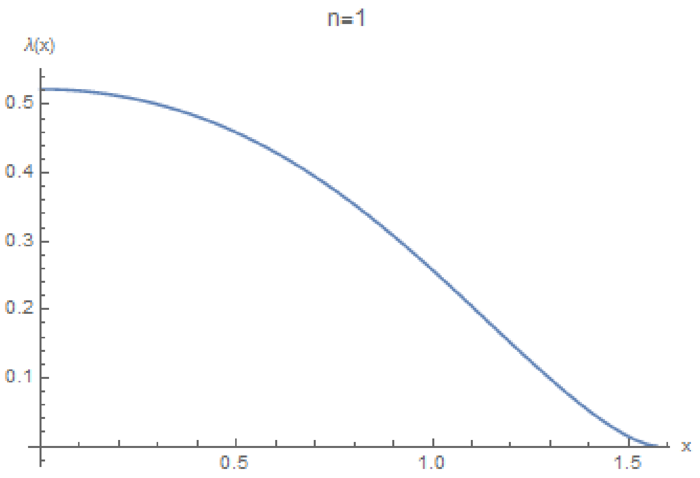
Note : we can also define dwell function outside of the bucket

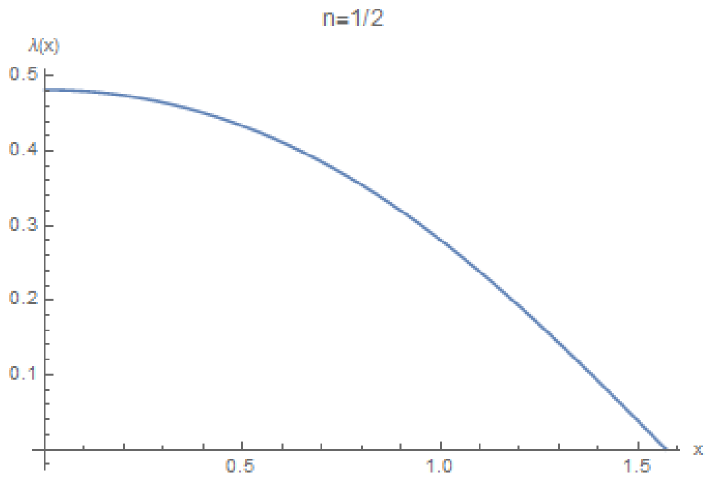
Note the implication: if ρ[p] is quadratic, then ρ[HT] is also quadratic.

Also note: if the point-spread function is significant, then one must form the convolution of δH[T] with ρ[HT]

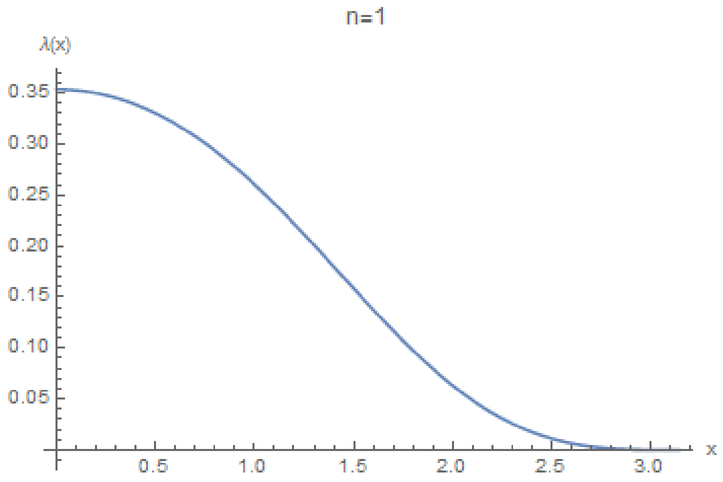
$$\text{BunchProfile}[x] == \int_{1-\text{Cos}[x]}^{\text{HTmax}} \frac{\left(1-\frac{\text{HT}^2}{\text{HTmax}^2}\right)^n \text{Gamma}[\frac{3}{2}+n]}{\text{HTmax}\sqrt{2\pi}\sqrt{-1+\text{HT}+\text{Cos}[x]}\,\text{EllipticK}[\frac{\text{HT}}{2}]\,\text{Gamma}[1+n]} d\text{HT}$$

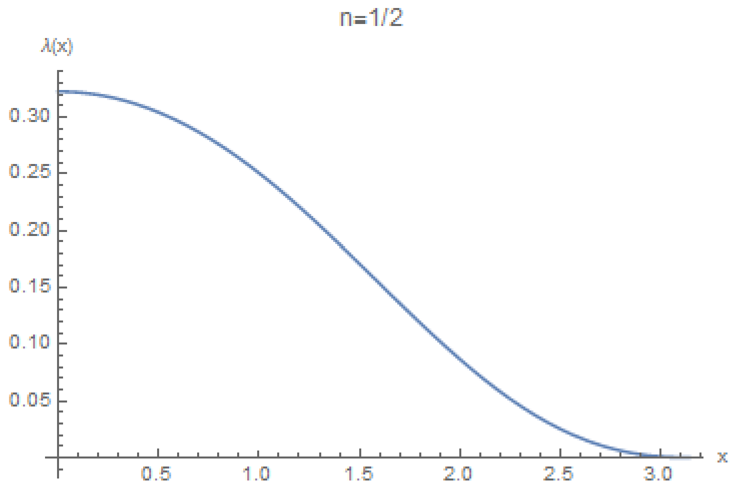
1/4 Full RF bucket
Htmax →1





Full RF bucket
Htmax →2





BUNCH MOMENTUM SPECTRUM

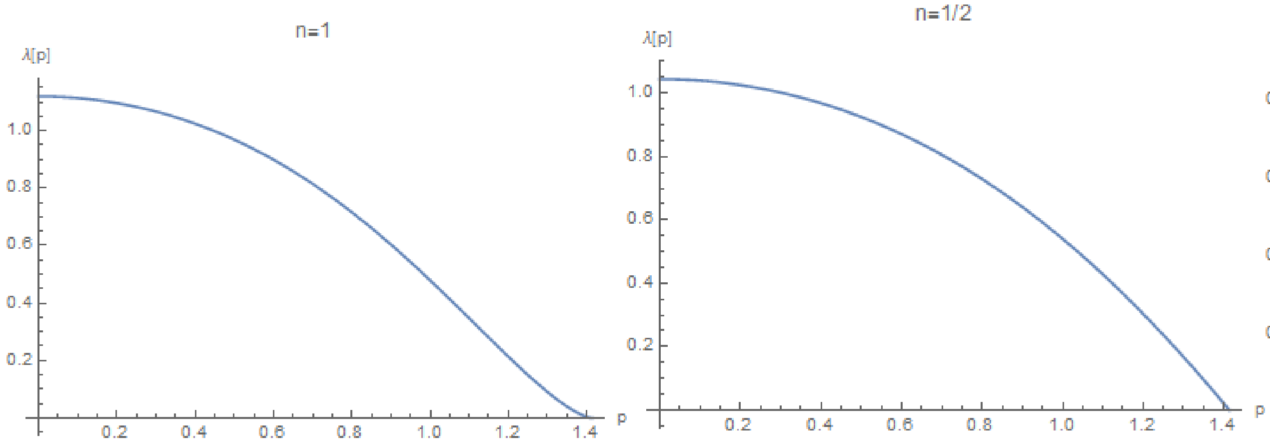
Bunch spectrum is obtained by integrating over the product of Hamiltonian spectrum and Dwell function

$$\text{dwell} == \frac{1}{\text{pdot}} == -\frac{1}{\text{Sin}[x[t]]} == \frac{1}{\sqrt{1-\frac{1}{4}(2-2\text{HT}+p^2)^2}} == \frac{1}{\sqrt{2\text{HT}-p^2}\sqrt{1-\frac{\text{HT}}{2}+\frac{p^2}{4}}}$$

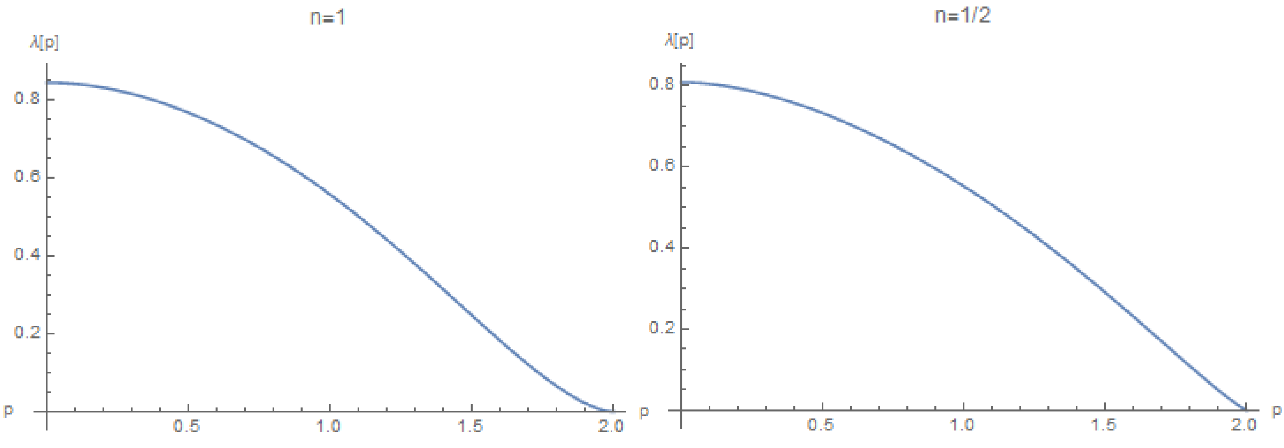
$$\text{norm} == 2\text{integrate}[\text{dwell},\{p,0,\sqrt{2}\sqrt{\text{HT}}\}] == \frac{\text{EllipticK}[\frac{\text{HT}}{-2+\text{HT}}]}{\sqrt{1-\frac{\text{HT}}{2}}} == \text{EllipticK}[\frac{\text{HT}}{2}]$$

$$\text{dwell} \rightarrow \frac{\text{dwell}}{\text{norm}} == \frac{1}{\sqrt{2\text{HT}-p^2}\sqrt{1-\frac{\text{HT}}{2}+\frac{p^2}{4}}\ \text{EllipticK}[\frac{\text{HT}}{2}]}$$

$$\text{Spectrum}[\text{p}] = \text{Integrate}[\text{RhoHT.dwell},\{\text{HT},p\hat{}2/2\,,\text{HTmax}\}]$$



1/4 RF bucket: HTmax →1



Full RF bucket: HTmax →2

Incidentally, an extremely good approximation

$$\text{EllipticK}[m] \rightarrow \frac{\pi}{2(1-m)^{1/4}}$$

CONCLUSIONS

- Thus ends 63 years without a detailed, useful, predictive theory of longitudinal adiabatic capture.
- The relationship between H[T] and H[0] is predictable: almost a universal function independent of V[t], H[T] ~Sqrt[H[0]], provided adiabaticity parameter ε=V"[T]/V'[T]/Sqrt[A V[T]] is few % or less.
- Of course, there are refinements to this theory (e.g. better estimation of the ^(1/2) power law and constants of proportionality) that lead to more accurate predictions for H[t] – but space/time prevents their presentation here.
- The point-spread function δH[T] quantifies <u>and predicts</u> the effect of non-adiabatic processes
  - sudden voltage turn-on which affect core, and separatrix crossing which affects tails.
- Lilliequist & Symon (1959) replaced linear voltage law by the iso-adiabatic voltage law (Np=2); an improvement because linear voltage law is potentially very damaging to the beam core if T too small.
  - But, ironically, sudden turn-on can be major source of δH[T] unless C0 very large
- However, the power law family with index Np>2 can out-perform Np=2; choose Np & C0 to optimize adiabaticity for given T.
- The fastest adiabatic capture is given by the exponential voltage law.
- The universal relationship between H[T] and H[0] facilitates prediction of bunch profile and spectrum.
- Synchrotrons (& storage rings) around the world that do adiabatic capture (or debunching) should revisit this topic – there are improvements to be made in beam quality and/or faster processes.

CAVEATS

- Conjecture (see slide #23) needs a proof (or refutation)
  - See IPAC 2023
- We have compared members of the adiabatic family (as a function of Np & C0) against one another
- But we have not made a direct comparison with other voltage laws such that V[t=0] =0, such as linear V[t]=V(t/T) or quadratic $V[t]=V(t/T)^2$. These do not suffer the sudden turn on, but do suffer from ε>>1 or ε>1 non-adiabaticity around t=0 -- which is core damaging
  - The theory presented here does not predict their behavior of H[T,H[0]] or δH[T,H[0]] as H[0]->0
- Direct comparison of the adiabatic family against linear & quadratic laws requires either to reduce the initial voltage step V[T]/C0 from the former or introduce it into the latter.
  - This is scope for further study
  - See IPAC 2023

Acknowledgement

Present author thanks Massimo Giovannozzi (CERN) for drawing my attention to the large body of work about adiabaticity in relation to a time-varying potential well.
Few if any of the results presented here seem to appear in the previous literature.

Analysis of adiabatic trapping for quasi-integrable area-preserving maps

Armando Bazzani (Bologna U.), Christopher Frye (Central Florida U. and CERN), Massimo Giovannozzi (CERN), Cédric Hernalsteens (CERN and Ecole Polytechnique, Lausanne, LPAP)
2014

22 pages
Published in: *Phys.Rev.E* 89 (2014) 4, 042915
DOI: 10.1103/PhysRevE.89.042915
Report number: CERN-ACC-2014-0042
View in: CERN Document Server

Provides a long list of references