Minimum Emittance Growth During RF Phase Sweep

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Previously, FFA2021, we answered the questions:

- **when does a particle bunch in longitudinal phase space follow movement of the RF phase?**
- **what is the emittance growth for linear RF phase sweep?**

Here, IPAC2022 and FFA2022, we answer two questions:

- **How to calculate emittance growth for arbitrary time law of phase sweep, f(t)?**
    - **We use value of the Hamiltonian as surrogate for emittance**
    - **We use a phenomenological Hamiltonian for the particle dynamics**

- **What time law(s) of rf phase variation generates absolute minimum emittance growth?**
    - **This is a brachistochrone-type problem: minimize an integral (subject to constraints) with respect to choice of path**

The *brachistochrone* is the curve that minimizes the time of flight of a bead sliding on a frictionless wire under the force gravity. It was the motivating problem for the calculus of variations, wherein an integral is minimized w.r.t. the form of some unknown function.
It is also the inspiration for thinking about the optimum phase slip f(t) in an abstract way.

RECAPITULATION

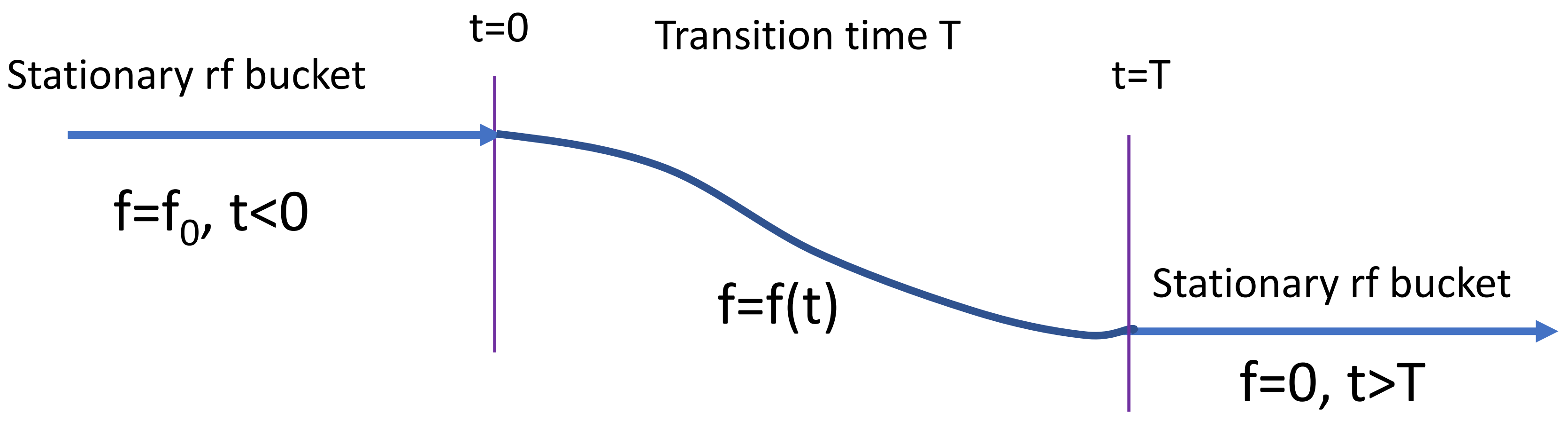
At FFA20

- David Kelliher’s presented an experimental study of emittance growth during the sweep of synchronous acceleration phase; and attempts to minimize the growth.
- I pointed out that there are two processes: (1) variation of the confining potential; and (2) variation of the centre of focusing.
- I noted that there is a large literature concerning the adiabaticity of (1); but no literature investigating the adiabaticity of (2).
- I posed the question “when does a particle beam follow a moving RF phase?”
- Stephen Brooks made the insightful comment this is equivalent to the question “when does a pendulum follow motion of its pivot and come to rest?”

At FFA21

- I presented start-to-end matching conditions for the bunch centroid
    - The RF phase sweep must be completed in an integer number of synchrotron oscillations
    - For some RF sweep laws, f(t), matching conditions cannot be found unless there is an initial (or terminal) fast RF phase jump
- I presented analytic calculation of emittance growth for linear RF phase sweep (in terms of Jacobi elliptic functions) and confirmed by numerical simulation
- I noted you cannot perform this calculation for arbitrary f(t) because there is no equivalent of the particular integral and complementary function for the biased pendulum oscillator.

MISSION STATEMENT



We shall find 3 **things**:

1) Matching conditions between t=0 and t=T that bring the centroid coincident with f(t) at those times.
2) Emittance growth of the ensemble with respect to the centroid
3) RF phase-sweep law f(t) that generates absolute minimum emittance growth for a given number of oscillations of the bunch centroid.

Original Equations $x'(t) = p(t)$ $p'(t) = -\omega^2 \sin(f(t) + x(t))$

$$\Delta H = \omega^2 \int_0^T f'(t) \sin(f(t) + x(t))\, dt$$

Transformations $x(t) = x2(t) - f(t)$ $p(t) = p2(t) - g(t)$

$g(t) = f'(t)$ momentum offset ***g(t)*** generates phase slip ***f(t)***

Transformed Equations $x2'(t) = p2(t)$ $p2'(t) = f''(t) - \omega^2 \sin(x2(t))$

$$\Delta H = \omega^2 \int_0^T g(t) \sin(x2(t))\, dt$$

Centroid Matching Conditions

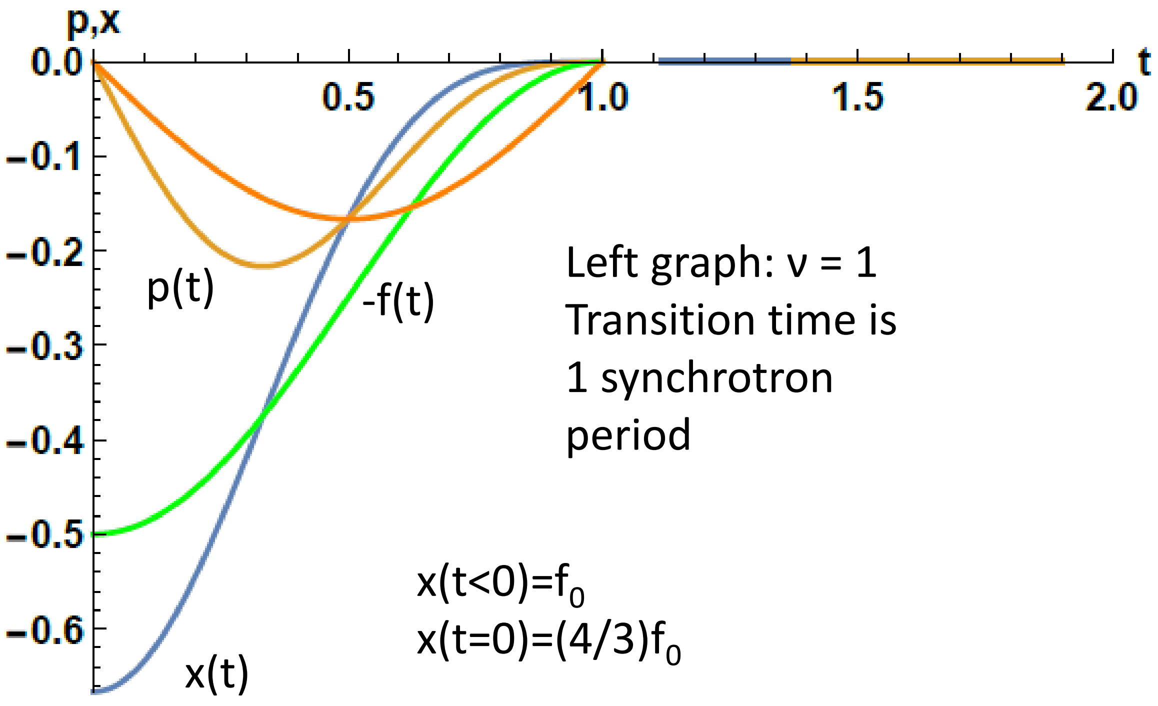
$$\Delta x2_c = \int_0^T p2_c\, dt = 0 \;\;\&\;\; \Delta p2_c = -\omega^2 \int_0^T x2_c\, dt = 0$$

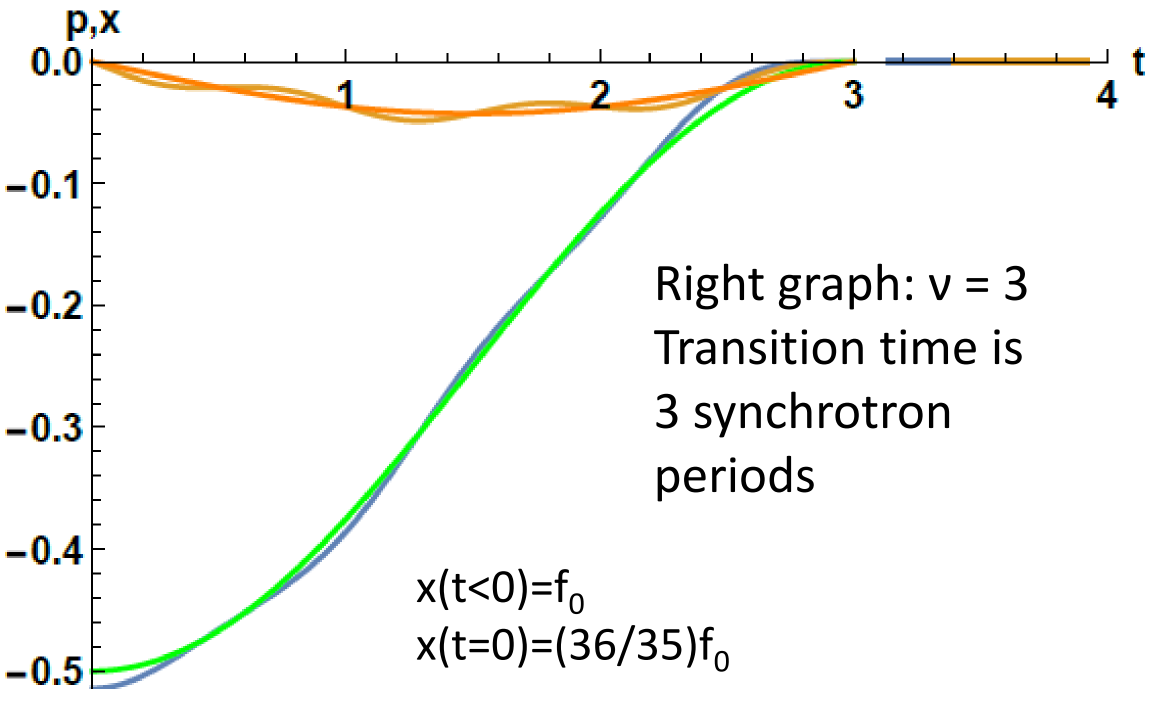
The primary constraint is that the sweep be completed in an integer number, *n*, of synchrotron oscillations of the centroid. Nevertheless, there is typically a small residual oscillation because the momentum offset caused by the RF sweep does not accrue enough phase slip of the bunch to catch up to the RF phase. The residual may either be accepted, or zeroed by making a “fast” RF-phase jump at start or end of the sweep.

SHO Matching conditions for centroid when $f(t) = (f_0/2)\,[1+\cos(\pi t/T)]$

$p(0)=p(T)=0$ when $T=2\pi\nu/\omega$. This implies $x(0) = -f_0(2\nu)^2/[(2\nu)^2-1]$

Which is contrary to assumption $x(0)=f_0$; and requires a phase jump rf-gymnastic





Centroid motion during and after phase slip.

Green curve: RF phase, -f. Blue curve: position. Gold curve: centroid momentum. Orange curve: momentum offset required to achieve phase slip, -g. Time in units of synchrotron oscillation period.

Provided $\nu \geq 2$ and phase jump at t=0 is accomplished, centroid oscillation about f(t) is smaller for ½-sinuosoid than for linear rf-ramp. For pendulum oscillator this results in smaller emittance growth.

Example Matching conditions for beam centroid

Hamiltonian = H= (1/2)A p^2+V[1-Cos[x]] $u == t\omega$ Jacobi $m == k^2$

Linear RF-phase ramp

Due to f(t)=(1-t/T)f0 centroid moves from (x,p)=(0,0) into lower left quadrant (-x,-p)

$$\mathrm{x2} = -2\,\mathrm{ArcSin}[\sqrt{m}\mathrm{JacobiSN}[u,m]]\qquad \mathrm{p2} = 2\sqrt{m}\omega\mathrm{JacobiCN}[u,m]$$

Period is $\tau == \frac{4\mathrm{EllipticK}[m]}{\omega}$

x[T] == 0 implies x2[T] ==0 because f[T]==0   p[T] == 0 implies p2[T] == -Poffset =f0/T

x[0] == -f0 implies x2[0] == 0 because f[0] == f0   p[0] == 0 implies p2[0] == - Poffset

For consistency, T=n 4K[m]/ω where n = # synchrotron oscillations and K= elliptic K

$$2\sqrt{m}\omega == \frac{\mathrm{f0}\omega}{4n\,\mathrm{EllipticK}[m]}\qquad \mathrm{K} == \frac{\pi}{2}+\frac{m\pi}{8}+\frac{9m^2\pi}{128}+\cdots$$

To first order $\sqrt{m} == \frac{\mathrm{f0}}{4n\pi}$

Bi-quadratic RF-phase ramp

$$\sqrt{m} == \mathrm{Sin}[\frac{3\mathrm{f0}}{8n^2\mathrm{EllipticK}[m]^2}]$$

½=cycle sine RF-phase ramp

$$\sqrt{m} == \mathrm{Sin}[\frac{\mathrm{f0}\pi^2}{4(\pi^2-16n^2\mathrm{EllipticK}[m]^2)}]$$

The emittance is proportional to the Hamiltonian of the bounding trajectory in phase space. Emittance growth is reduced by minimizing the change in Hamiltonian between the bounding trajectory and the bunch centroid (ΔH - ΔH_c)

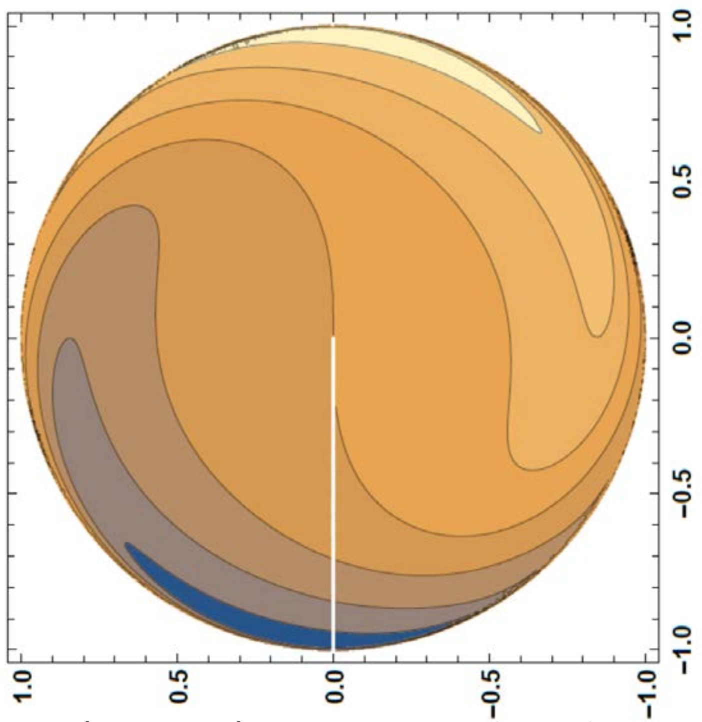
Linear RF phase sweep

Important, special case. Use it to bench mark numerical calculations and/or other analysis
Simple because f”=0, g=constant; and no phase jump, so no jump in Hamiltonian

Let $\sin(x2(u)/2) = k\,\mathrm{sn}(u+u0;m)$ Hamiltonian value is $H=2\omega^2 m$ Jacobi amplitude parameter $m=k^2$

Define the change $\Delta cn = cn(U+u0;m) - cn(u0;m)$.
$U=\omega T$ is the accrued phase. Δcn describes a dipole oscillation.
The normalized fractional change of Hamiltonian for the general trajectory is: $(\Delta H/H)\ (2\pi/\Delta f) = -\Delta cn/(n\ k)$.

ΔH of the bunch is minimized when we set duration $U=2\pi\ n$.
u0(m) is chosen to find the largest ΔH on the bounding trajectory. Note: for the linear ramp and short bunches, ΔH does not fall as $1/n$ because the de-phasing (between different oscillation amplitudes) is approximately proportional to n



Δcn(u0,U,m) versus initial angle u0 and radius k^2 = m

More details at FFA2021

FFA2021: technical difficulties

For time-varying rf phase sweep, we found particle coordinates evolve according to

$$\left\{\mathrm{p2}'[t] == \mathrm{VSin}[\mathrm{x2}[t]] - \frac{f''[t]}{A},\ \mathrm{x2}'[t] == -A\mathrm{p2}[t]\right\}$$

We take a few moments to explore why "calculating the emittance increase analytically via changes in the Hamiltonian" is so difficult when f"≠ 0.

- There are no "closed form" solutions (like *sn* and *cd*) for a biased pendulum.
- *cd* and *sn* are not a suitable starting point for perturbative or iterative expansion – because they have the wrong symmetries.
- There is no analog for constructing the particular integrals (PI); and if there were, we would find a different PI for every value of elliptic parameter *m*.

FFFA2021: Good Ideas Are Needed and Welcome…

IPAC2022: phenomenological Hamiltonian

$H[a,t] == \frac{p[t]^2}{2} + \frac{1}{2}(f[t] + x[t])^2 \omega[a]^2$ a harmonic oscillator with an artificial frequency spread

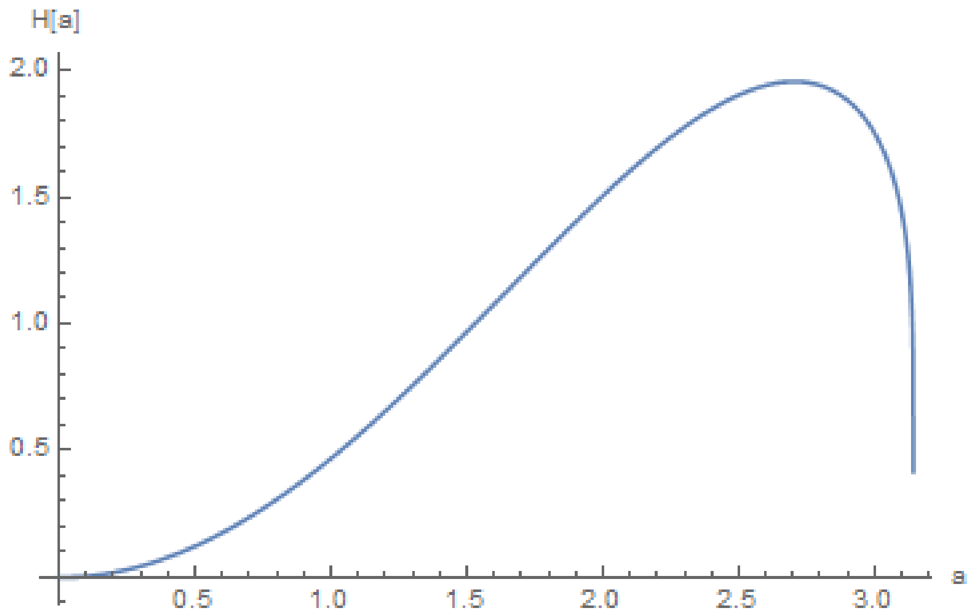
$x'[t] == p[t]$ $p'[t] == -(f[t] + x[t])\omega[a]^2$ $\omega[a] == \frac{\pi\omega[0]}{2\,\text{EllipticK}[\text{Sin}[\frac{a}{2}]^2]}$ $a == [0, \text{Pi}]$

a = initial amplitude

$\omega(a)$ = pendulum oscillator dispersion $\omega[\pi] == 0$

Suppose f[t]=0.

$$H[a] == \frac{1}{2} a^2 \omega[a]^2$$



Definitely this Hamiltonian has "issues". When we come to form ΔH/H, H[a] will have to be "regulated" to avoid division by zero squared.
The regulated form for the <u>initial</u> Hamiltonian is

$$H[a] \to \frac{1}{2} a^2 \omega[0] \omega[a]$$

Employ same transformations as before

$$x(t) = \mathrm{x2}(t) - f(t) \qquad p(t) = \mathrm{p2}(t) - g(t)$$

$g(t) = f'(t)$ momentum offset ***g(t)*** generates phase slip ***f(t)***

Transformed Equations $\quad \mathrm{x2}'(t) = \mathrm{p2}(t) \qquad \mathrm{p2}'(t) = f''(t) - \omega^2 \sin(\mathrm{x2}(t))$

$$\Delta\mathrm{H} = \omega(a)^2 \int_0^T g(t)\mathrm{x2}(t)\,\mathrm{d}t$$

Equations can be solved for x2 as particular integral (PI) & complementary function (CF);
and ΔH calculated analytically for arbitrary ***f(t).***
The PI is proportional to 1/ω[a], because a weaker restoring force results in a larger amplitude x2.
Hence ΔH is proportional to ω[a]. Therefore, taking initial $H[a] \to \frac{1}{2} a^2 \omega[0]\omega[a]$ will regulate ΔH/H.

Centroid Matching Conditions

$$\Delta x2_c = \int_0^T p2_c\,dt = 0 \;\; \& \;\; \Delta p2_c = -\omega[0]^2 \int_0^T x2_c\,dt = 0$$

Matching leads to integer number n of synchrotron oscillations; and may require a fast phase jump.

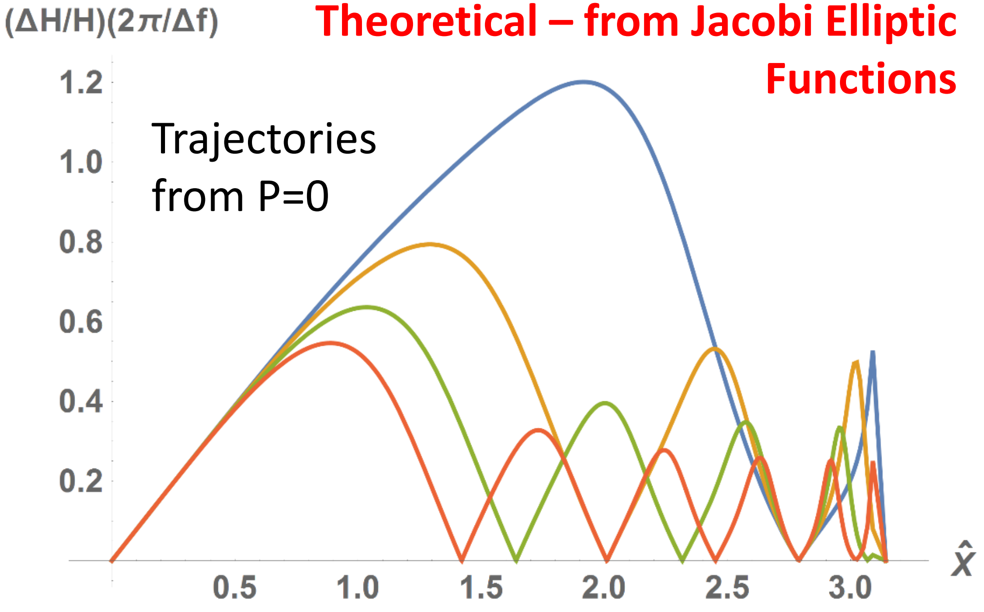
Absolute Minimum Emittance Growth

- Ideally, the difference (ΔH - ΔHc) would be minimized with respect to the choice of the phase sweep, f(t), according to the calculus of variations, in the manner of the brachistochrone problem.
- However, we have not found a suitable variational principle; and so have resorted to trial and error in the choice of f(t).
- The trial functions are: (1) linear ramp, (2) ½-cycle sinusoid, (3) bi-quadratic, (4) dual-sinusoid; (5) cubic; and (6) linear plus sinusoid – all ramped between t=0 and t=T.
- The procedure is: 1st compute centroid matching conditions; 2nd compute (ΔH - ΔHc) for the trajectories as perturbed by the sweep f(t). Repeat for variety of phase ramps.
    - Here "compute" does not mean "numerical computation", rather it means compute analytic formulae that can be evaluated later
    - Note, a fast phase jump is the cause of a jump in H, but jump values are cancelled out when the difference (ΔH - ΔHc) is formed.

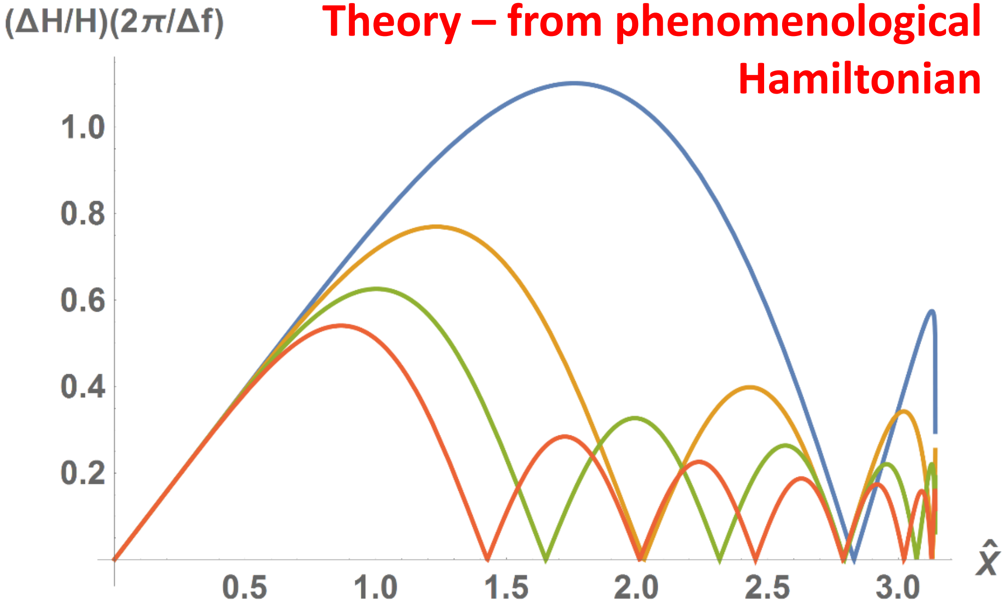
We start by benchmarking numerical particle tracking and the phenomenological Hamiltonian against the known exact case Jacobian elliptic functions driven by a linear RF phase ramp

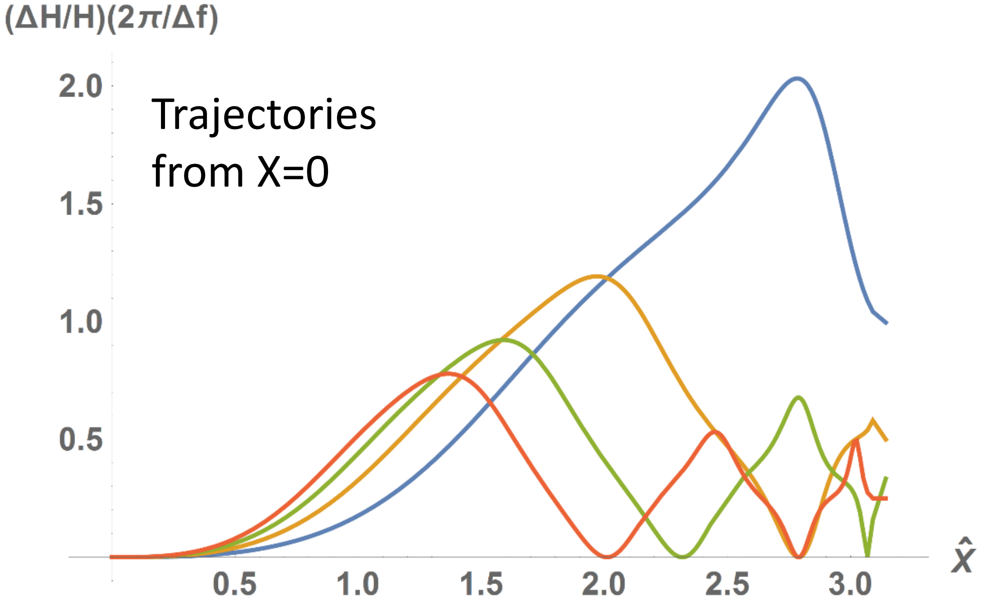
Benchmarking – Linear RF phase ramp

Theoretical – from Jacobi Elliptic Functions

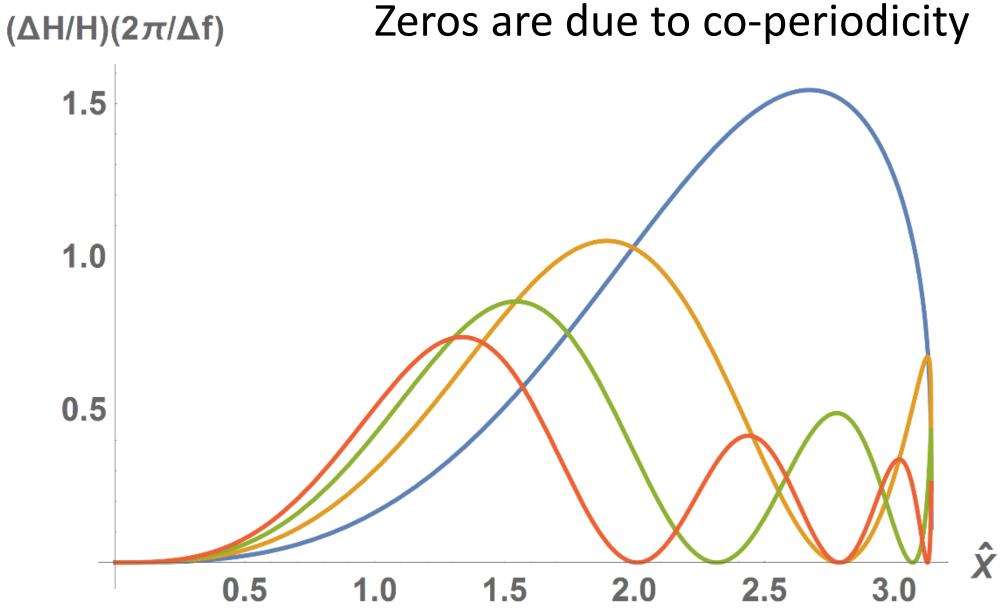


Theory – from phenomenological Hamiltonian





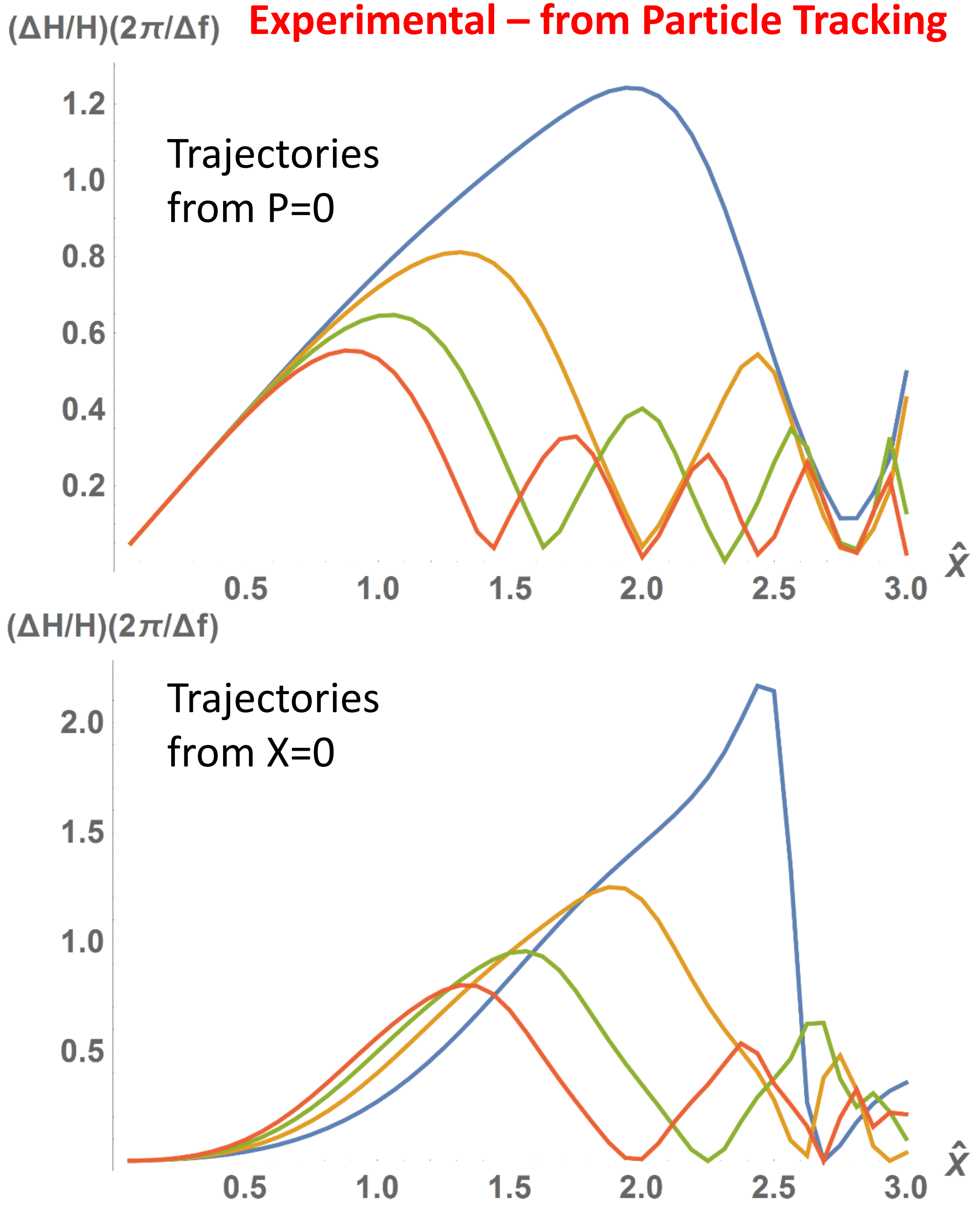
Zeros are due to co-periodicity



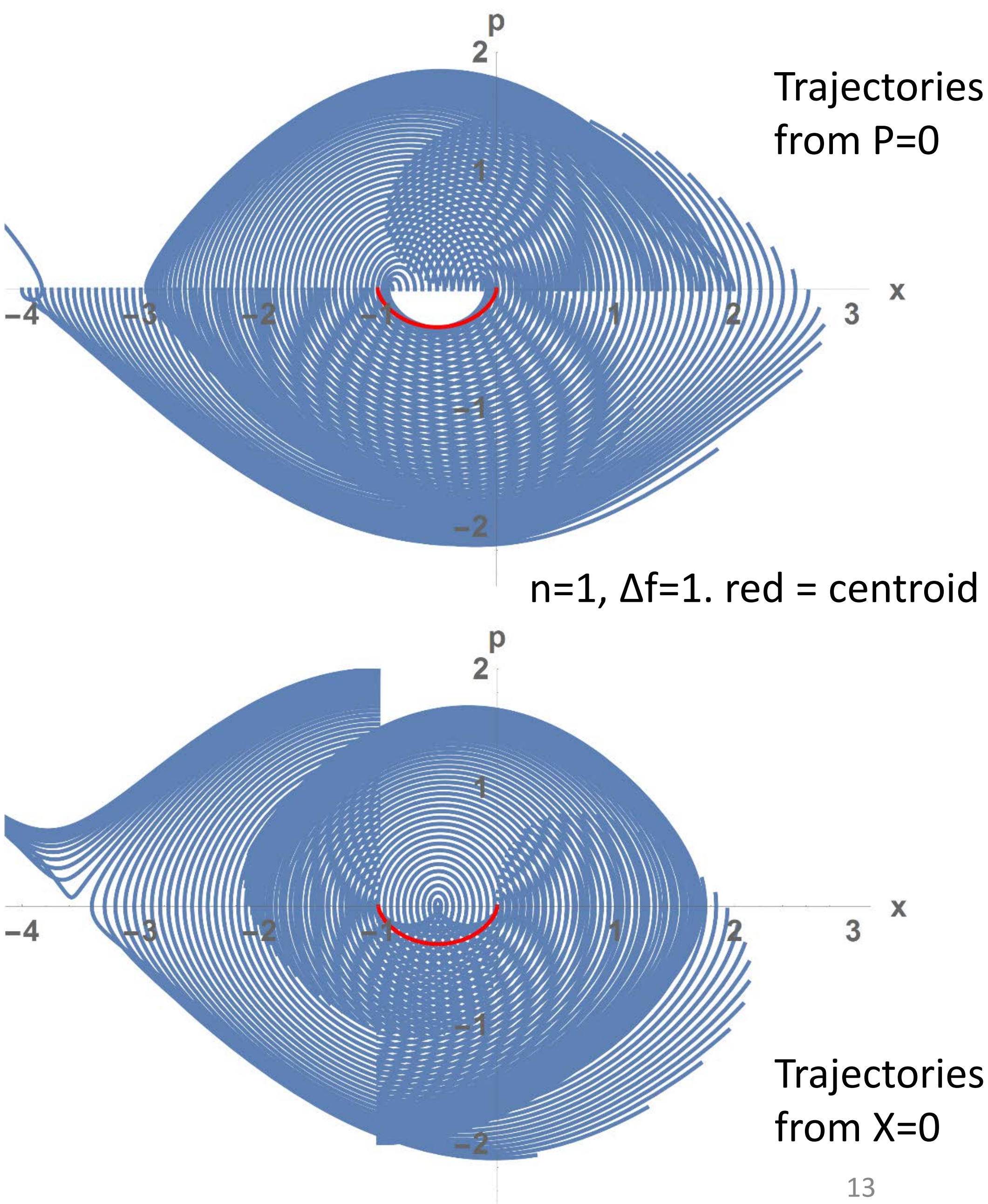
n=1, blue; n=2, gold; n=3, olive; n=4 synchrotron oscillations coral

Benchmarking – Linear RF phase ramp

Experimental – from Particle Tracking

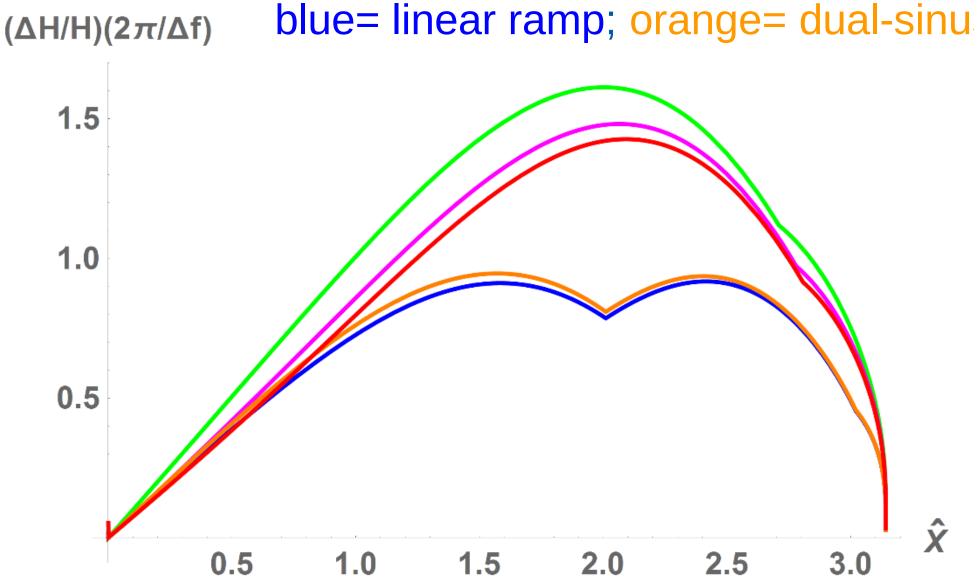


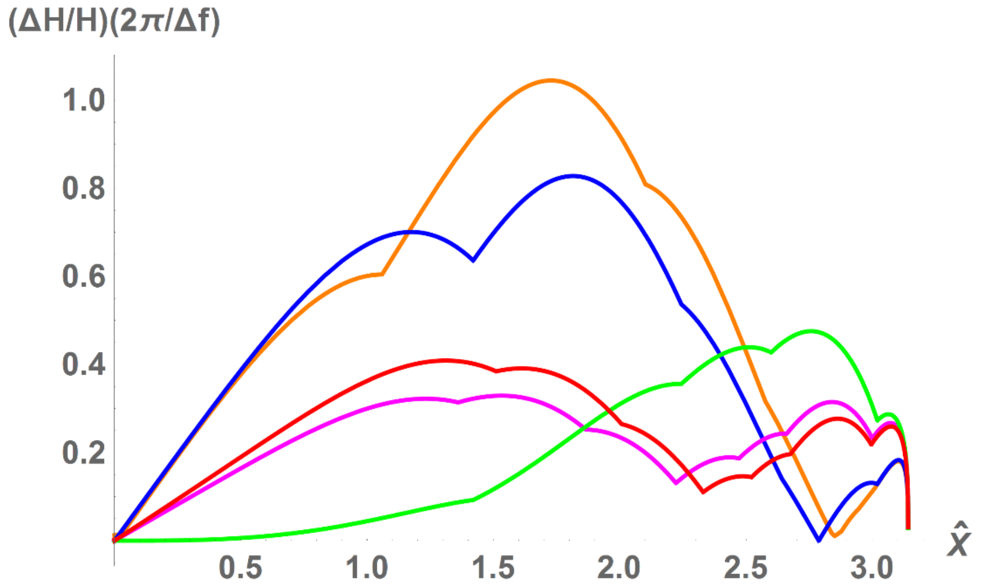
Example Particle Tracking



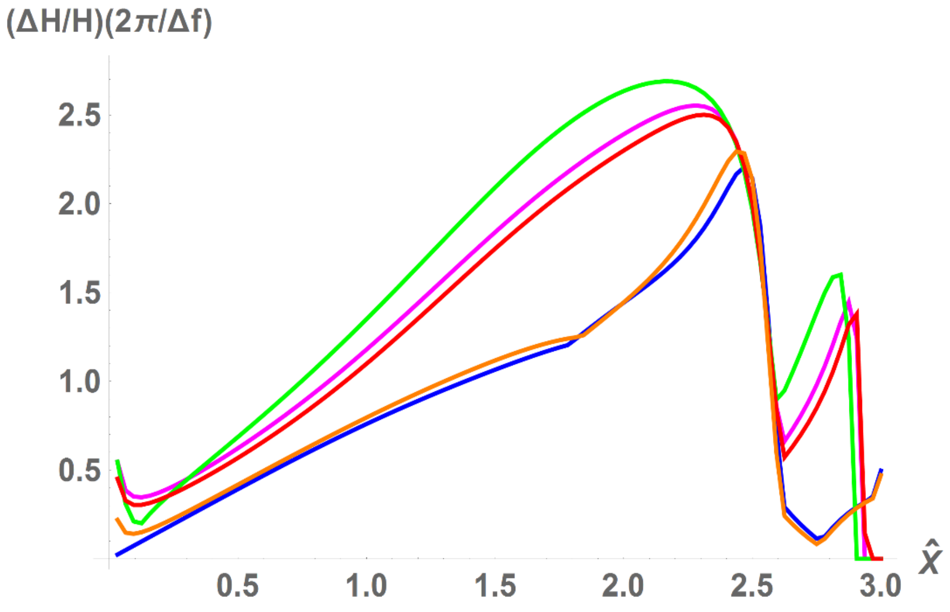
blue= linear ramp; orange= dual-sinusoid; magenta= ½-cycle sinusoid; green= bi–quadratic; red= cubic ramp

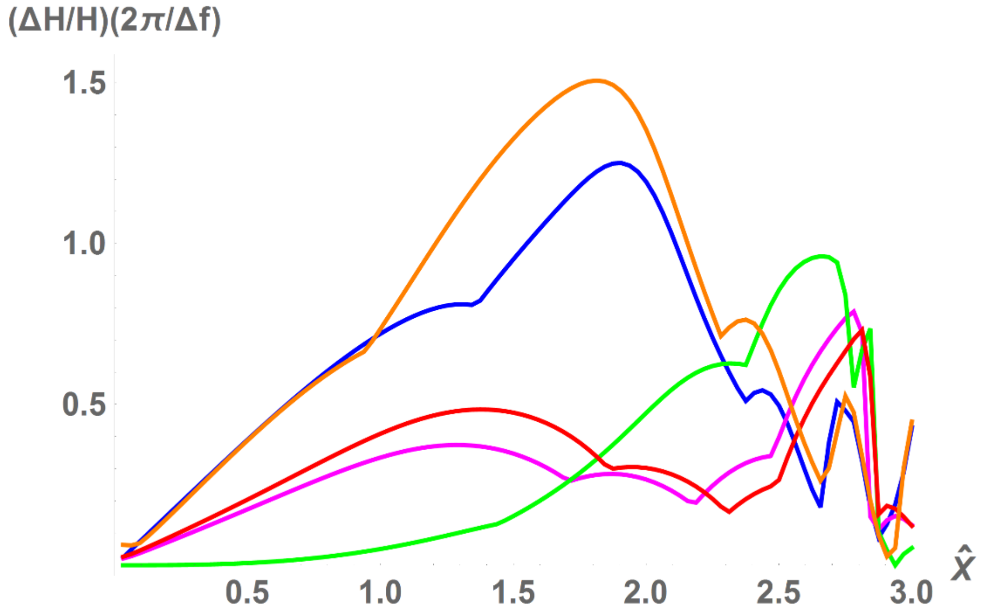
Predict – phenomeno-logical Hamiltonian





Confirm - Tracking

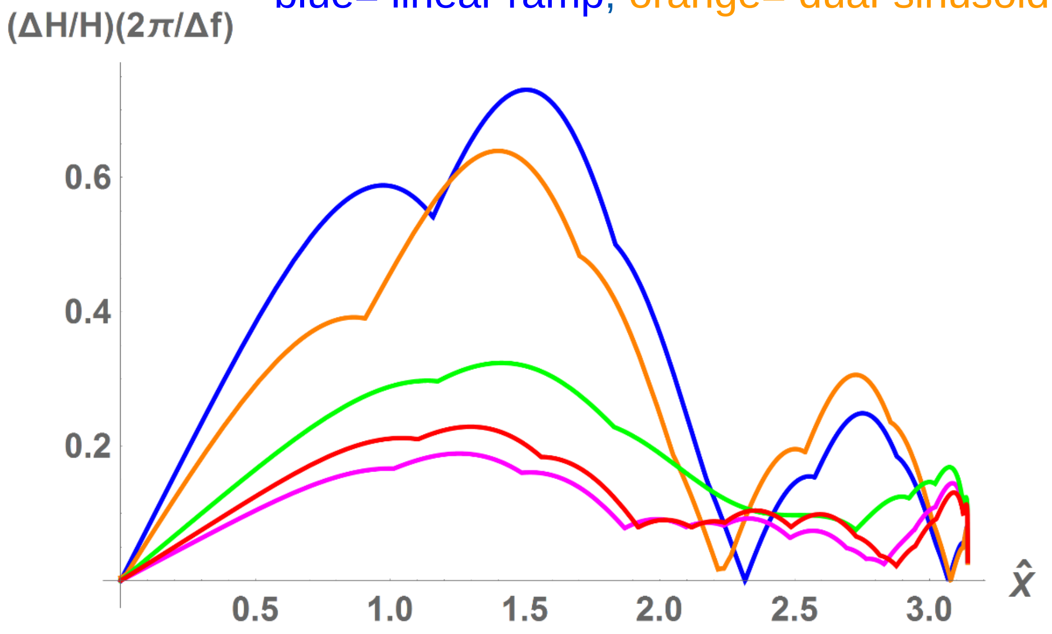


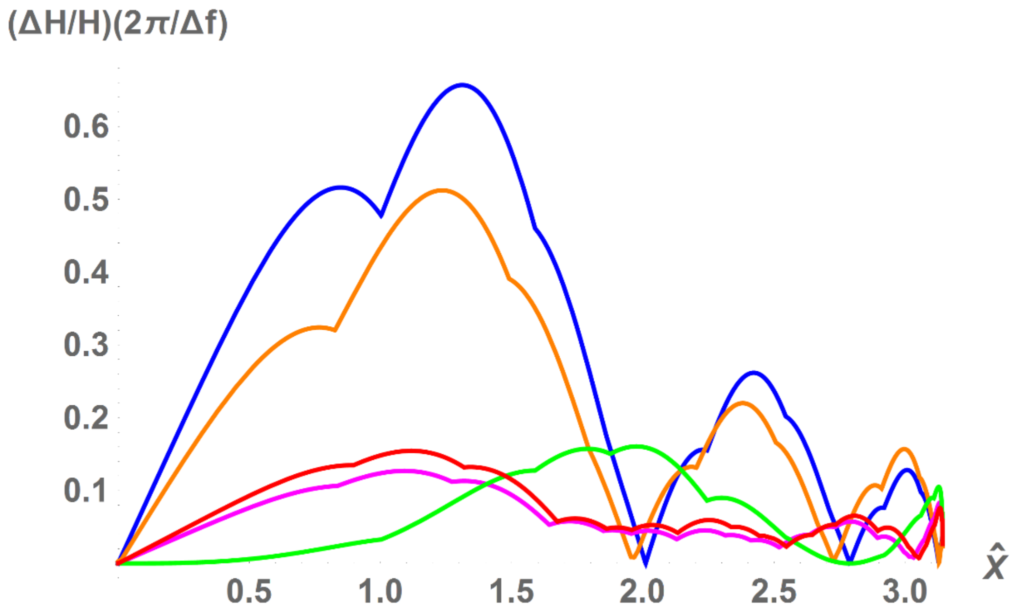


Conclude: "linear" is best for n = 1.
Also: if insist linear ramp is used and amplitudes <0.5 radian, then no point using n>1

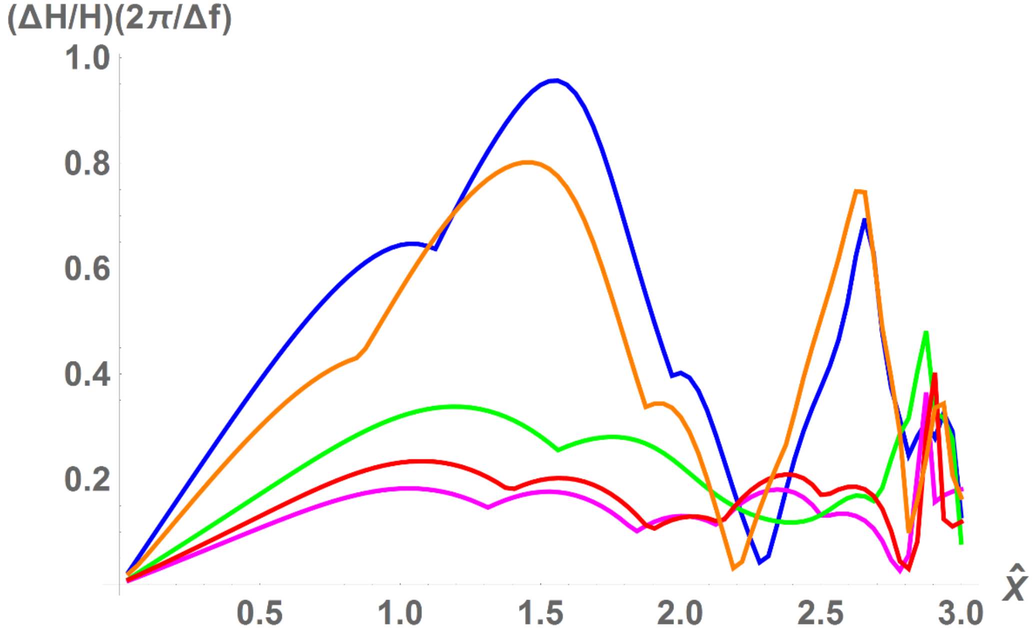
Conclude: "bi-quadratic" is best for n = 2 and amplitudes < 2 radian blue= linear ramp; orange= dual-sinusoid; magenta= ½-cycle sinusoid; green= bi–quadratic; red= cubic ramp

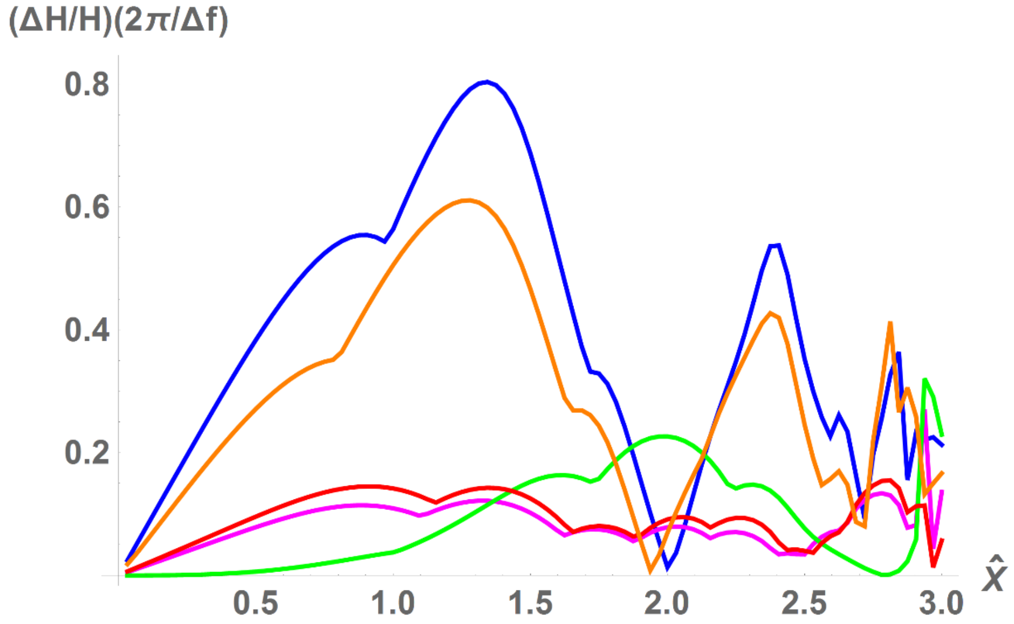
Predict – phenomeno-logical Hamiltonian





Confirm - Tracking





Conclude: "1/2-cycle sine" is best for n = 3

Conclude: "bi-quadratic" is best for n = 4 and amplitudes < 1.5 radian

We could stop here – But the expressions for x2 and p2 developed in terms of PI and CF for the phenomenological Hamiltonian allow to compute additional quantities, and also power series expansions that improve our insight as to why one phase ramp law is better than another.

$$\Delta\mathrm{x2} - \Delta\mathrm{x2}_c == \int_0^T (\mathrm{p2}[t] - \mathrm{p2}_c[t])\,\mathrm{d}t$$

$$\Delta\mathrm{p2} - \Delta\mathrm{p2}_c == \int_0^T (\mathrm{x2}[t]\omega[a]^2 - \omega[0]^2\mathrm{x2}_c[t])\,\mathrm{d}t$$

$$\Delta\mathrm{H2} - \Delta\mathrm{H2}_c == \int_0^T (\mathrm{x2}[t]\omega[a]^2 - \omega[0]^2\mathrm{x2}_c[t])f'[t]\,\mathrm{d}t$$

- All 3 quantities are measures of the effect of dephasing due to synchrotron frequency spread.
- All 3 quantities are identically zero if there is no oscillation frequency spread.

An example: the linear RF phase ramp f(t)=Δf/T

$\Delta\mathrm{x2}_c$ & $\Delta\mathrm{p2}_c$ & $\Delta\mathrm{H2}_c$ are all zero.

Let $T\omega[0] \to 2n\pi$ $\quad T\omega[a] \to n(2\pi + \psi)$ $\quad$ Ψ is the increment in the phase-advance due to frequency spread

$\psi < 0$ if $\omega[a] < \omega[0]$

$$\Delta\mathrm{x2} == -a\mathrm{Cos}[\phi] + a\mathrm{Cos}[\phi + n\psi] + \frac{\Delta\mathrm{f}\,\mathrm{Sin}[n\psi]}{n(2\pi + \psi)}$$

$$\Delta\mathrm{p2} == \{(\Delta\mathrm{f} - \Delta\mathrm{f}\mathrm{Cos}[n\psi] - an(2\pi + \psi)\mathrm{Sin}[\phi] + an(2\pi + \psi)\mathrm{Sin}[\phi + n\psi])\omega[0]\}/(2n\pi)$$

$$\Delta\mathrm{H2} == \{\Delta\mathrm{f}(\Delta\mathrm{f} - \Delta\mathrm{f}\mathrm{Cos}[n\psi] - an(2\pi + \psi)\mathrm{Sin}[\phi] + an(2\pi + \psi)\mathrm{Sin}[\phi + n\psi])\omega[0]^2\}/(4n^2\pi^2)$$

ϕ is the initial oscillation phase $\qquad$ If $\psi \to 0$, then $\Delta\mathrm{x2}$ & $\Delta\mathrm{p2}$ & $\Delta\mathrm{H2}$ are all zero.

$$\psi == \frac{2\pi(-\omega[0] + \omega[a])}{\omega[0]}$$

Suppose we know ω[a] and its derivatives at a=0, then ψ becomes a Taylor series in amplitude "a"

$$\psi \rightarrow \frac{a^2\pi\omega''[0]}{\omega} + \frac{a^4\pi\omega^{(4)}[0]}{12\omega} + \ \ldots$$

Because ω is a function of a^2, the odd derivatives are zero

Hence the power series expansions

$$\Delta\text{x2} == \frac{a^2\Delta\text{f}\,\omega''[0]}{2\omega} - \frac{a^3 n\pi\text{Sin}[\phi]\omega''[0]}{\omega} + \frac{a^4\Delta\text{f}\left(-6\omega''[0]^2 + \omega\omega^{(4)}[0]\right)}{24\omega^2}$$

$$\Delta\text{p2} == a^3 n\pi\text{Cos}[\phi]\omega''[0] + \frac{a^4 n\pi\Delta\text{f}\omega''[0]^2}{4\omega}$$

$$\Delta\text{H2} == \frac{1}{2}a^3\Delta\text{f}\omega\text{Cos}[\phi]\omega''[0] + \frac{1}{8}a^4\Delta\text{f}^2\omega''[0]^2$$

At this order, ΔH appears independent of duration. The 1/T in (Δf/T) cancels against the phase advance which is proportional to T. This is correct for small amplitudes, and indicates that for small amplitudes there is no reason to use n>1 if you insist to use a linear ramp.
Numerical simulations confirm this to be correct.
For the linear phase ramp, the Δf/T reduction of ΔH is seen in the LARGE amplitudes.

Power series expansions for Δx2, Δp2 & ΔH2 can be obtained for all six RF phase sweep laws (and many others). It is particularly instructive to do so for the bi-quadratic ramp

Bi-quadratic ramp is defined piecewise before and after t=T/2;
and is continuous in value and derivative across t=T/2

$$0 < t < \frac{T}{2} \to f[t] == \frac{2t^2\Delta f}{T^2} + f[0] \qquad T > t > \frac{T}{2} \to f[t] == -\frac{2t^2\Delta f}{T^2} + \frac{4t\Delta f}{T} + 2f[0] - f[T]$$

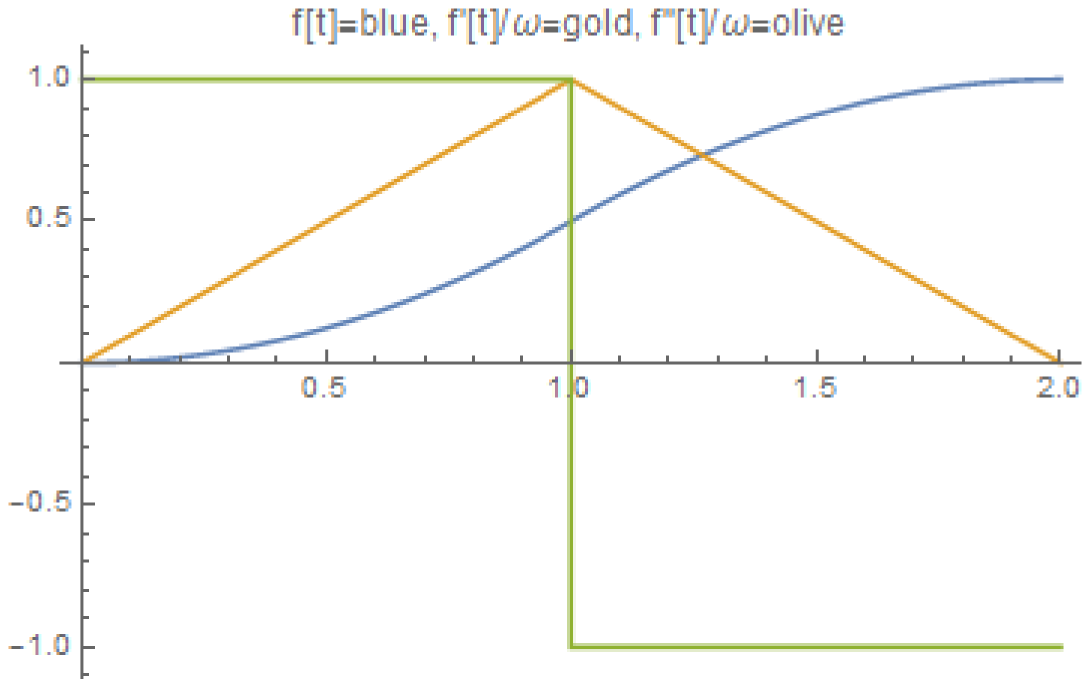
No fast-phase-jump is required if n is even.

The bi-quadratic ramp achieves the smallest peak value of f"[t]
Remember, it is f" which drives the particular integral

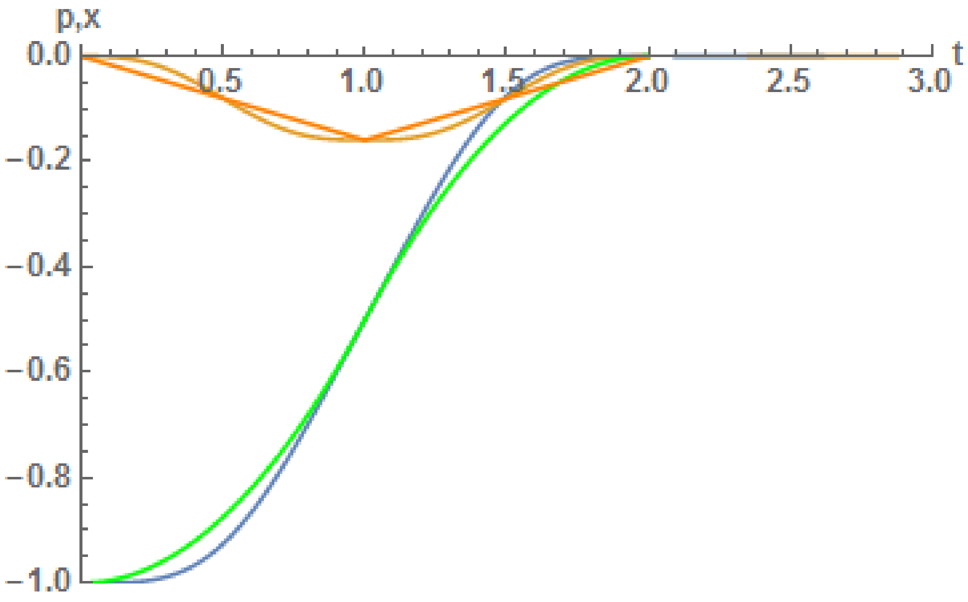
$$\boldsymbol{p2'(t) = f''(t) - \omega^2 \sin(x2(t))}$$

$$\boldsymbol{x2'(t) = p2(t)}$$

So we might expect the bi-quadratic ramp to launch the smallest synchrotron oscillations

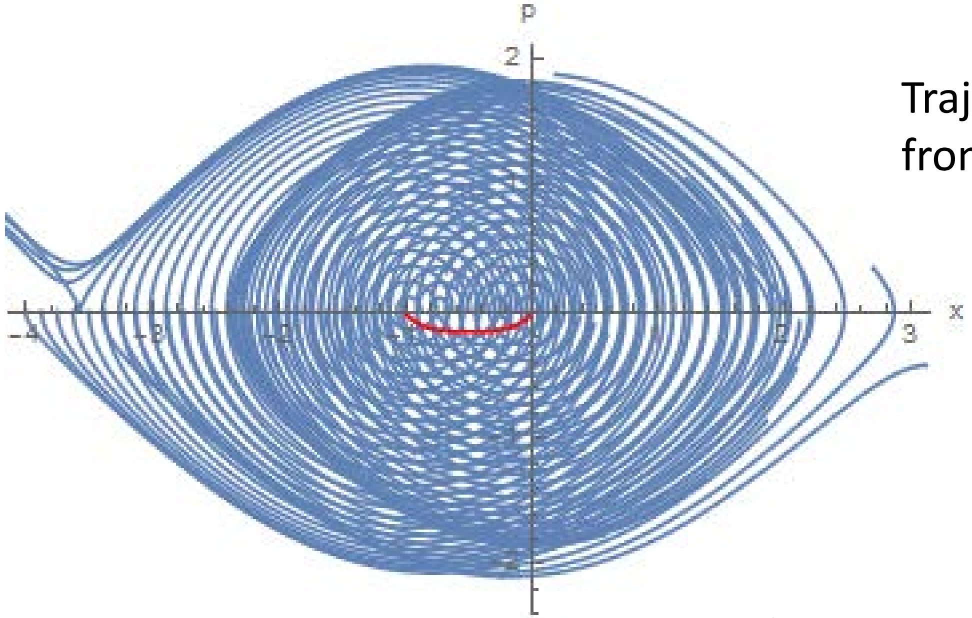


Bi-quadratic ramp

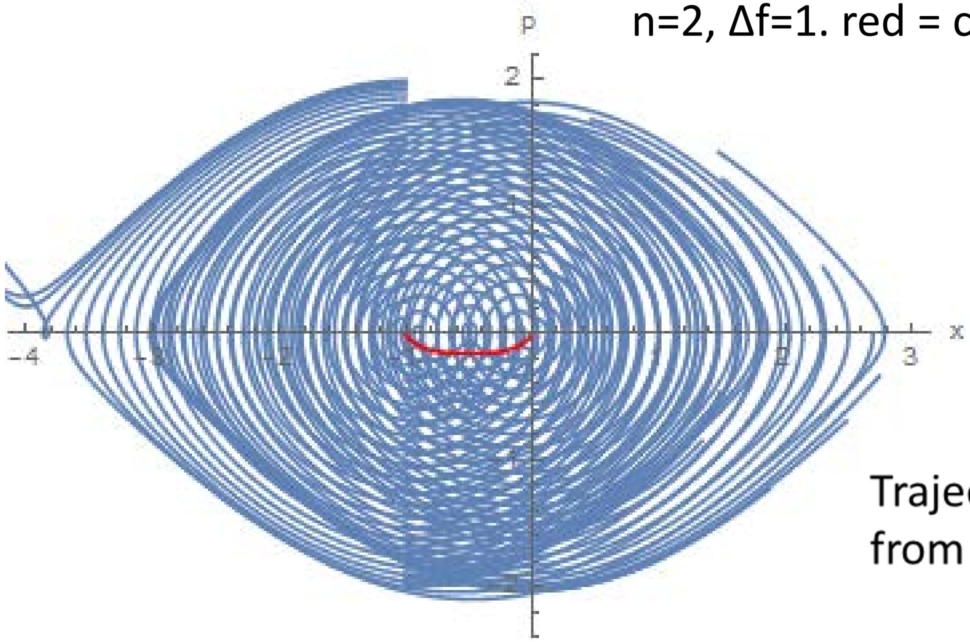


Blue = x, Gold = p, Green = -phase ramp = -f(t),
Orange = momentum offset = g(t)

Trajectories
from P=0



n=2, Δf=1. red = centroid



Trajectories
from X=0

Power series expansions for Δx2, Δp2 & ΔH2 for bi-quadratic ramp and integer n even

$$\Delta\text{x2} == -\frac{a^3 n\pi \text{Sin}[\phi]\omega''[0]}{\omega} + \frac{a^4 \Delta\text{f}\omega''[0]^2}{4\omega^2} - \frac{a^5 n\pi\left(6n\pi\text{Cos}[\phi]\omega''[0]^2 + \omega\text{Sin}[\phi]\omega^{(4)}[0]\right)}{12\omega^2}$$

$$\Delta\text{p2} == a^3 n\pi\text{Cos}[\phi]\omega''[0] + \frac{a^5 n\pi\left(6(\text{Cos}[\phi] - n\pi\text{Sin}[\phi])\omega''[0]^2 + \omega\text{Cos}[\phi]\omega^{(4)}[0]\right)}{12\omega}$$

$$\Delta\text{H2} == \frac{1}{4}a^5\Delta\text{f}\text{Cos}[\phi]\omega''[0]^2 + \frac{1}{24}a^7\Delta\text{f}\omega''[0]\left(-\frac{3n\pi\text{Sin}[\phi]\omega''[0]^2}{\omega} + \text{Cos}[\phi]\omega^{(4)}[0]\right)$$

The high power law a^5 in ΔH implies the change in Hamiltonian, and hence emittance increase, grows very slowly for amplitude a <1

CONCLUSIONS

Here we answered two questions:

- **How to calculate longitudinal emittance growth for arbitrary time law of RF phase ramp?**
  - **Use the phenomenological Hamiltonian**
  - **Predictions are in good agreement with particle tracking, particularly the ordering of the "best" ramp versus number of synchrotron oscillations**
- **What time law(s) of RF phase variation generates minimum growth of the bounding emittance?**
  - **The choice of "best ramp" depends on the maximum oscillation amplitudes in the bunch**
  - **It is probable that for n=1 the linear ramp generates the absolute minimum**
  - **It is probable that for n=2 & amplitude <2 radian, the bi-quadratic ramp generates the absolute minimum**

If the figure of merit is the r.m.s. emittance growth, then one must form the weighted average over the distribution F[H] within the bunch. This will skew choice of "best" toward small amplitudes

$$\int_0^{\mathrm{H0max}} \frac{\Delta \mathrm{H} F[H]}{H}\, \mathrm{d}H$$

The function $x=\sin(\omega t)$ is that which $\omega^2 x$ is the same as taking the 2nd derivative d^2x/dt^2
The function $x=2\mathrm{Arcsin}[k.\mathrm{JacobiElliptic}(\omega t,k^2)]$ is that which $\omega^2\sin(x)$ is the same as taking the 2nd derivative

Note it is a property of all periodic functions that if $x(t)$ is a trajectory solution, then the loop integrals over one cycle of motion of ALL derivatives (x', x'', etc) is identically zero.