Transverse Beam Dynamics

FFA School 2022

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TRIUMF

Introduction

In a global **inertial** frame of reference:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = q(\boldsymbol{\mathcal{E}} + \mathbf{v} \times \mathbf{B})$$

Why don't we just through everything* at a decent numerical integrator. Runge-Kutta should do the trick, right?

*$\mathbf{B}(x, y, s)$, $\boldsymbol{\mathcal{E}}(x, y, s)$ and initial position and velocity of the particles

Introduction

In many cases it is the right thing to do. But what do you do if the result you get is not what you wanted? How do you change your design? And even if you do get what you wanted, how do you know that it is the correct answer and not the product of a human (you) or a numerical error?

- Break the problem into pieces you can chew, by using Taylor series for instance, and work your way out one order at a time.
- Identify conserved quantities to test your results against.

Contents

- Geometrical optics and phase space
- Closed orbits*
- Linearized motion and transfer matrices*
- Particle ensembles: beam matrix, RMS emittance
- Periodic focusing: Twiss parameters, betatron tunes

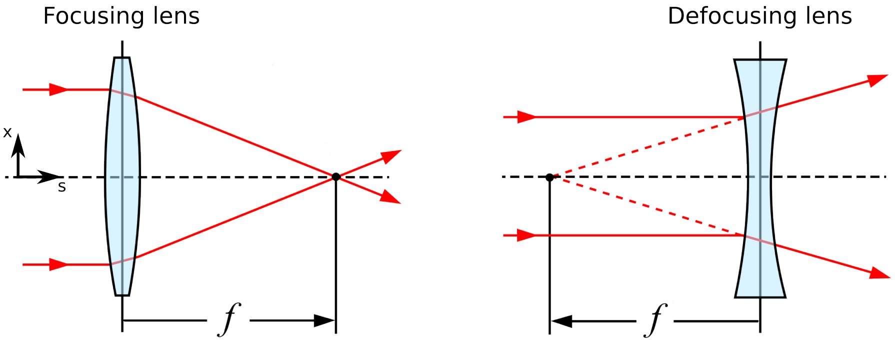
*and what makes FFAs special

Contents

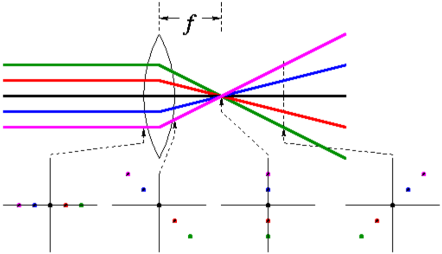
What we won't talk much about, unless you insist:

- Betatron resonances
- Non-linear optics
- Space charge

Geometrical Optics



Phase Space



Matrix Formalism

We can calculate the effect of linear optics with *transfer matrices*.

$$\begin{pmatrix} x_\mathrm{f} \\ {x'}_\mathrm{f} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_\mathrm{i} \\ {x'}_\mathrm{i} \end{pmatrix}$$

For a drift of length d, the matrix is:

$$M_\mathrm{drift} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},$$

for a focusing thin lens, the matrix is:

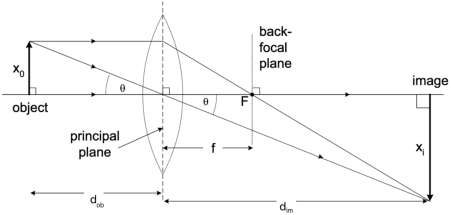
$$M_\mathrm{lens} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix},$$

etc.

Linear Optics

Let's use this to re-derive the optics rule

$$\frac{1}{d_{\text{ob}}} + \frac{1}{d_{\text{im}}} = \frac{1}{f}$$



$$\begin{pmatrix}1 & d_{\text{im}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & d_{\text{ob}}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1-\frac{d_{\text{im}}}{f} & d_{\text{ob}}+d_{\text{im}}-\frac{d_{\text{ob}}d_{\text{im}}}{f}\\ \frac{-1}{f} & 1-\frac{d_{\text{ob}}}{f}\end{pmatrix}.$$

For a total transfer matrix to take from an object, through a lens, to an image, which of the four matrix elements must be ZERO?

Choice of the Independent Variable

We tend to use s instead of t as independent variable:

- **Optics**: 'light' optics uses position along the optical axis as independent variable.
- **Diagnostics**: measurements are taken at a given s.

6D Phase Space

With distance s along the reference trajectory as independent variable, the full state vector of a particle is:

$$\mathbf{X} = \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ \Delta t \\ -\Delta E \end{pmatrix}$$

6D Phase Space

With distance s along the reference trajectory as independent variable, the full state vector of a particle is:

$$\mathbf{X} = \begin{pmatrix} x \\ \frac{P_x}{P_0} \\ y \\ \frac{P_y}{P_0} \\ \Delta t \\ \frac{-\Delta E}{P_0} \end{pmatrix}$$

6D Phase Space

With distance s along the reference trajectory as independent variable, the full state vector of a particle is:

$$\mathbf{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \Delta t * \beta_0 c \\ \frac{-\Delta E}{P_0} * \frac{1}{\beta_0 c} \end{pmatrix}$$

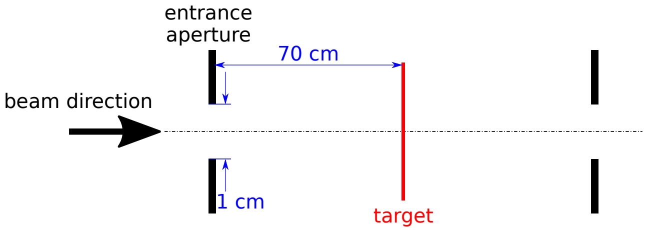
6D Phase Space

With distance s along the reference trajectory as independent variable, the full state vector of a particle is:

$$\mathbf{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ z' \end{pmatrix}$$

Test your understanding

- An experimenter has an apparatus that is 1.4 m long and an inlet aperture diameter of 1 cm, with a target that the beam must hit at the centre of the apparatus. No electric or magnetic fields are allowed inside the 1.4 m drift. If all of the beam is inside an elliptical phase space area of $10\pi\,\text{mm}\times\text{mrad}$ in both the transverse directions, what is the smallest diameter beam spot that can be created at the target location?
- Draw a phase space diagram and indicate the elliptical phase space area corresponding to the two locations: the injection aperture, and the target.



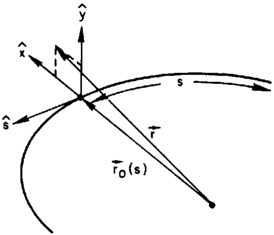
What optical axis?

Synchrotrons are built around a desired obit. The magnets are ramped up during acceleration to keep the beam circulating around it.

Figure: Aerial view of the CERN LHC

Non-inertial frame of reference

Frenet-Serret coordinate system (x, y, s):



Non-inertial frame: the Newtonian approach is to add the appropriate 'centrifugal' and 'Coriolis' forces: [Goldstein et al., 2002], Chapter 3.

Personal opinion: mechanics in non-inertial is way easier with Hamiltonian mechanics: [Courant and Snyder, 1958], Appendix B.

Closed orbits in FFAs

In FFAs, the reference orbit is not known a priori. And it changes with the energy of the beam. Before you can do any transverse optics, you must find the reference orbit, the one that has the same periodicity than the lattice. This is done numerically.

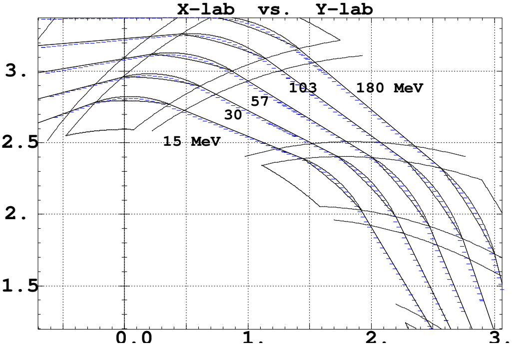


Figure: Closed orbits in the RACCAM FFAG ring [Planche et al., 2009]

Lineraized Motion

It describes well “small” deviations w.r.t. the reference orbit. How small should small be will depend on the details of your FFA. . .

F

Take any Hamiltonian $H(x, P_x; s)$, the equation of motion for x is:

$$\frac{\mathrm{dx}}{\mathrm{ds}} = \frac{\partial H}{\partial P_x}$$

F

Expand this equation in Taylor series around a reference orbits:

$$\frac{\mathrm{dx}}{\mathrm{ds}} = \left.\frac{\partial H}{\partial P_x}\right|_{x=P_x=0} + P_x \left.\frac{\partial^2 H}{\partial P_x^2}\right|_{x=P_x=0} + x \left.\frac{\partial^2 H}{\partial x \partial P_x}\right|_{x=P_x=0} + ...$$

If the reference trajectory is an actual particle trajectory:

$$\left.\frac{\partial H}{\partial P_x}\right|_{x=P_x=0} = 0$$

F

Expand this equation in Taylor series around a reference orbits:

$$x' = \left.\frac{\partial H}{\partial P_x}\right|_{x=P_x=0} + P_x \left.\frac{\partial^2 H}{\partial P_x^2}\right|_{x=P_x=0} + x \left.\frac{\partial^2 H}{\partial x \partial P_x}\right|_{x=P_x=0} + ...$$

If the reference trajectory is an actual particle trajectory:

$$\left.\frac{\partial H}{\partial P_x}\right|_{x=P_x=0} = 0$$

Since you can do the same with P_x (mind the sign difference) you get:

$$\mathbf{X}' = \mathbf{F}\mathbf{X} + \text{ higher order terms}$$

where

$$\mathbf{F} = \begin{pmatrix} \frac{\partial^2 H}{\partial P_x \partial x} & \frac{\partial^2 H}{\partial P_x^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial P_x} \end{pmatrix}$$

with partial derivatives evaluated on the reference orbit.

$\mathbf{F} \to \mathbf{M}$

$$\mathbf{X}' = \mathbf{F}\mathbf{X}$$

$$\mathbf{X_f} = \mathbf{M}\mathbf{X_i}$$

How to get $\mathbf{M}$ from $\mathbf{F}$?

$\mathbf{F} \rightarrow \mathbf{M}$ with F independent of s

$$\mathbf{X_f} = e^{FL}\,\mathbf{X_i}$$

where $\mathbf{X_f} = \mathbf{X}(s = L)$ and $\mathbf{X_i} = \mathbf{X}(s = 0)$.

$$\mathbf{M} = e^{FL}$$

for a hard-edge element of length L. $\mathbf{F}$ is a matrix, so this is a matrix exponential:

$$e^{\mathbf{F}} \equiv \sum_{i=0}^{\infty} \frac{1}{i!}\mathbf{F}^i$$

$\mathbf{F} \to \mathbf{M}$: example of a hard-edge quadrupole

In Mathematica:

```
MatrixExp[( 0    1  0   0
            -k^2 0  0   0
            0    0  0   1
            0    0  k^2 0 ) L] // ExpToTrig // MatrixForm
```

returns

$$\mathbf{M} = \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) & 0 & 0 \\ -k\sin(kL) & \cos(kL) & 0 & 0 \\ 0 & 0 & \cosh(kL) & \frac{1}{k}\sinh(kL) \\ 0 & 0 & k\sinh(kL) & \cosh(kL) \end{pmatrix}$$

$\mathbf{F} \rightarrow \mathbf{M}$ with $F(s)$

If $\mathbf{F}(s)$ you can still calculate $\mathbf{M}$ from it, but you will have to use a numerical integrator.

Hamitonian of linear motion in horizontal FFAs

$$h = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{x'^2}{2} + \frac{y'^2}{2} - \frac{z'x}{\rho} + \frac{z'^2}{2\gamma^2},$$

where ρ is the curvature of the **closed** orbit, and n is the local field index evaluated on the closed orbit:

$$n = -\frac{\rho}{B_0}\left.\frac{\partial B}{\partial x}\right|_{x=y=0}.$$

Since the close orbit is an actual particle trajectory we also have:

$$P = qB_0\rho$$

And that's it. And what about VFFA?

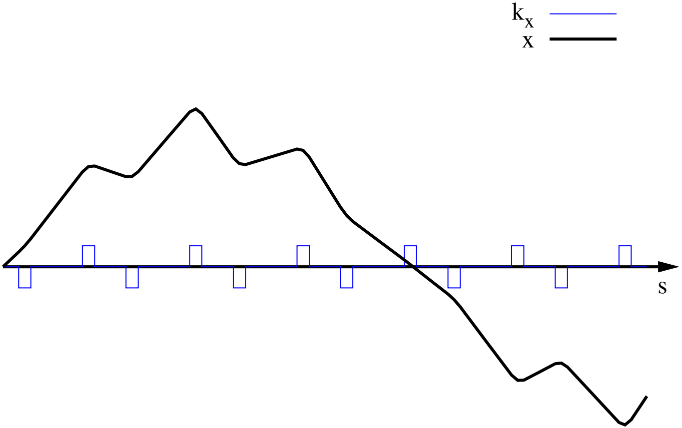
Concatenation of Transfer Matrices

Matrices for long sections are obtained by concatenating pieces:

$$M_{A \to D} = M_{C \to D} \cdot M_{B \to C} \cdot M_{A \to B}$$

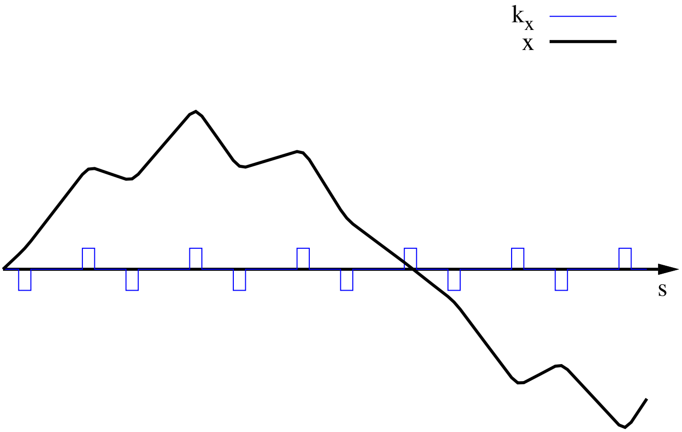
Strong Focusing

I used $\mathbf{X}_\mathsf{B} = \mathbf{M} \cdot \mathbf{X}_\mathsf{A}$ multiple times to produce this plot:



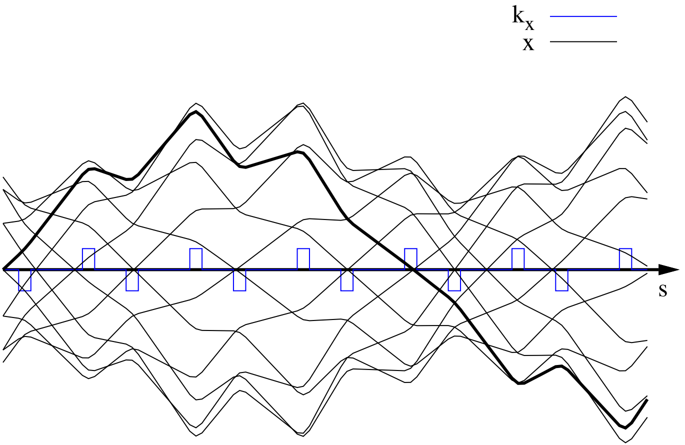
Single Particle Trajectory

I used $\mathbf{X}_\mathsf{B} = \mathbf{M} \cdot \mathbf{X}_\mathsf{A}$ multiple times to produce this plot:



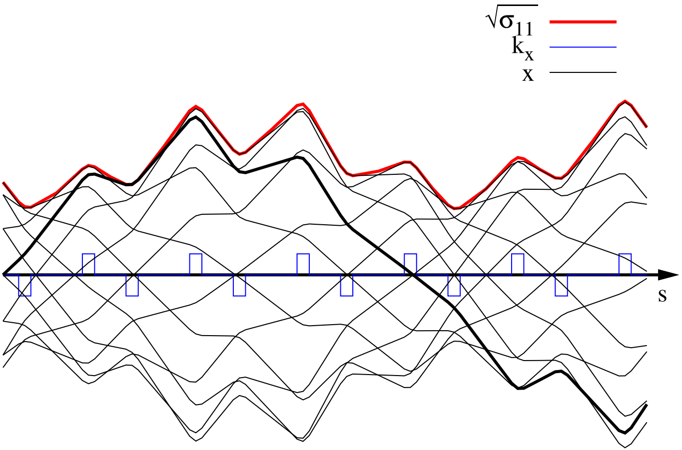
Multiple Trajectories

But beams are often made of millions of particles. With only 12:



Beam Envelope

How to calculate directly the red curved for any number of particles?



Beam Matrix

How to measure the 'size' of a particle distribution?

$$\mathbf{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{X} \cdot \mathbf{X}^{\mathsf{T}}$$

called the covariance matrix, or the beam matrix, or the 'sigma' matrix.

Works with $\mathbf{X}$ a 2-d, 4-d or 6-vector.

Beam Matrix: 1-D

In 1-dimension:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$\sqrt{\sigma_{11}}$ is the RMS beam size,

$\sqrt{\sigma_{22}}$ is the RMS beam divergence, and

$\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ is the statistical correlation between x and x'.

Beam Matrix

Combining:

$$\mathbf{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{X} \cdot \mathbf{X}^{\mathsf{T}}$$

with:

$$\mathbf{X}_{\mathsf{B}} = \mathbf{M} \cdot \mathbf{X}_{\mathsf{A}}$$

leads to:

$$\mathbf{\Sigma}_{\mathsf{B}} = \mathbf{M} \cdot \mathbf{\Sigma}_{\mathsf{A}} \cdot \mathbf{M}^{\mathsf{T}}$$

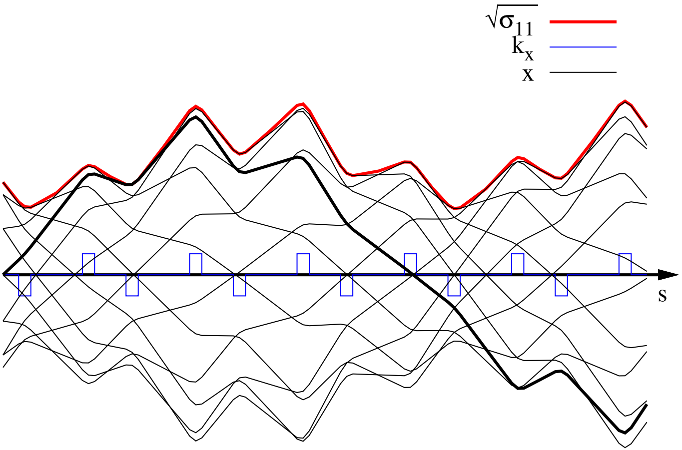
Test your understanding

You can also integrate directly the evolution of $\mathbf{\Sigma}$ from $\mathbf{F}$ without using at transfer matrix as a go-between.

Can you figure out the equation of motion $\mathbf{\Sigma}' = ???$

Beam Envelope

I used $\mathbf{\Sigma}_\mathsf{B} = \mathbf{M} \cdot \mathbf{\Sigma}_\mathsf{A} \cdot \mathbf{M}^\mathsf{T}$ multiple times to calculate the red envelope:



Digression: Symplecticity of the Transfer Matrix

Transfer matrix for quadrupoles:

$$\mathbf{M} = \begin{cases} \begin{pmatrix} \cos(\sqrt{k}L) & \dfrac{\sin(\sqrt{k}L)}{\sqrt{k}} \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix} & \text{, if } k > 0 \text{ (focusing)} \\ \\ \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} & \text{, if } k = 0 \text{ (drift space)} \\ \\ \begin{pmatrix} \cosh(\sqrt{|k|}L) & \dfrac{\sinh(\sqrt{|k|}L)}{\sqrt{|k|}} \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix} & \text{, if } k < 0 \text{ (defocusing)} \end{cases}$$

Calculate the determinants of these matrices…

Digression: Symplecticity of the Transfer Matrix

They are all equal to 1! This is no accident. It is a consequence of the fact that (x, x') are canonically conjugated variables.

Looking a little closer, $x' = \frac{P_x}{P_0}$ is actually canonically conjugated to x only when P_0 constant. P_x is the canonical momentum.

Acceleration: determinant is not 1 but $\frac{P_A}{P_B}$, the ratio between the initial and the final reference momentum.

Symplecticity of 6×6 Matrices

With s as independent variable the full state vector of a particle is:

$$\mathbf{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ z' \end{pmatrix}$$

15 symplectic conditions on a 6×6 matrix (15 = n(2n-1), n = 3).

Geometrical Beam Emittance

The RMS beam emittance is the square root of the determinant of the beam matrix:

$$\epsilon = \sqrt{|\mathbf{\Sigma}|}$$

Acceleration Damping

Since:

$$\mathbf{\Sigma}_{\mathsf{B}} = \mathbf{M} \cdot \mathbf{\Sigma}_{\mathsf{A}} \cdot \mathbf{M}^{\mathsf{T}}$$

$$\epsilon_B = |M| \epsilon_A$$

leading:

$$\epsilon_B = \frac{P_A}{P_B} \epsilon_A$$

The geometrical emittance shrinks when the beam is accelerated.

Normalized Beam Emittance

$$\epsilon_n = \frac{P_0}{mc}\epsilon\,,$$

is constant throughout the acceleration process. Typical values of the normalized RMS beam emittance is of the order of 1 μm.

Normalized Beam Emittance

$$\epsilon_n = \beta\gamma\epsilon\,,$$

is constant throughout the acceleration process. Typical values of the normalized RMS beam emittance is of the order of 1 μm.

Take a pen and a piece of paper. . .

The geometrical RMS x (and y) emittance of the electron beam out of our 300 kV gun is about 2 μm.

What is the normalized transverse RMS emittance of the e^- beam?

Take a pen and a piece of paper…

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Answer: $\approx$2.5 μm

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Answer: $\approx$2.5 μm

What value of the transverse geometrical RMS emittance should we expect at 30 MeV?

Take a pen and a piece of paper...

The geometrical RMS x (and y) emittance of the electron beam out of our 300 kV gun is about 2 μm.

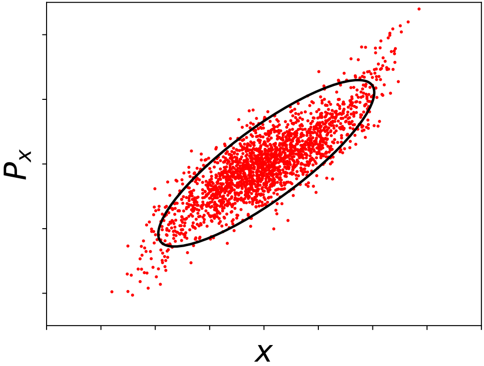
What is the normalized transverse RMS emittance of the e$^-$ beam?
Answer: $\approx$2.5 μm

What value of the transverse geometrical RMS emittance should we expect at 30 MeV?
Answer: $\approx$0.04 μm

Beam Ellipse

The beam emittance is related to the area occupied by the beam in phase space. Try to draw around any 2-D particle distribution:

$$\sigma_{22}x^2 - 2\sigma_{12}xP_x + \sigma_{11}P_x = (4\,\epsilon_{\mathrm{rms}})^2$$



4×RMS Emittance

$$\sigma_{22}x^2 - 2\sigma_{12}xP_x + \sigma_{11}P_x = (4\,\epsilon_{\mathsf{rms}})^2$$

The factor 4 is somewhat arbitrary: see homogenous elliptical distribution. For other types of distributions the fractions of particles inside the '4 RMS' ellipse varies, but it is typically of the order of 90%:

Phase-space area	Gaussian	Uniform	Maxwellian
$\pi\epsilon_{\rm rms}$	39%	25%	35%
$4\pi\epsilon_{\rm rms}$	86%	100%	93%
$\zeta\pi\epsilon_{\rm rms}$	$1-\exp(-\zeta/2)$	$\zeta/4$	$\mathrm{erf}(\pi\zeta/\sqrt{96})$

The factor π

You will sometimes see emittance numbers given like this:

$$\epsilon = 10\,\pi\,\text{mm}\,\text{mrad}$$

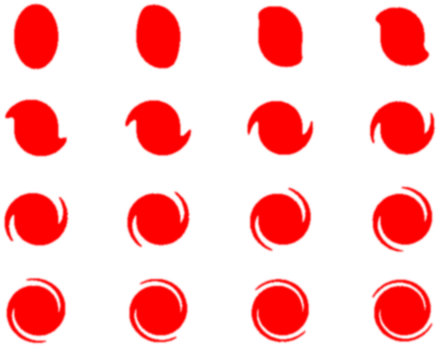
with the extra π factor. This notation is archaic and confusing, see [Becker and Herrmannsfeldt, 2006],[Baartman, 2015].
Don't use it.

Write instead:

$$\epsilon = 10\,\mu\text{m}\,.$$

RMS Emittance Increase

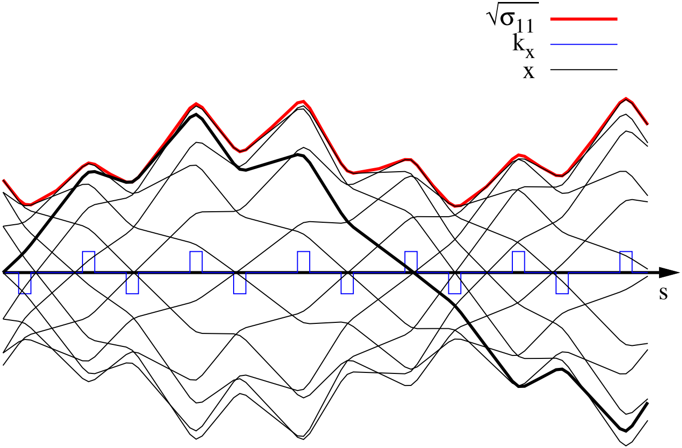
So in phase space, a non-linear lens leads to an effective emittance increase. Below are successive snapshots in phase space of a beam acted upon by lenses with small nonlinearity. (Proceed left to right and top to bottom.)



Note that the initial elliptical emittance diagram filaments into a shape to such an extent that the effective emittance has grown by 50%.

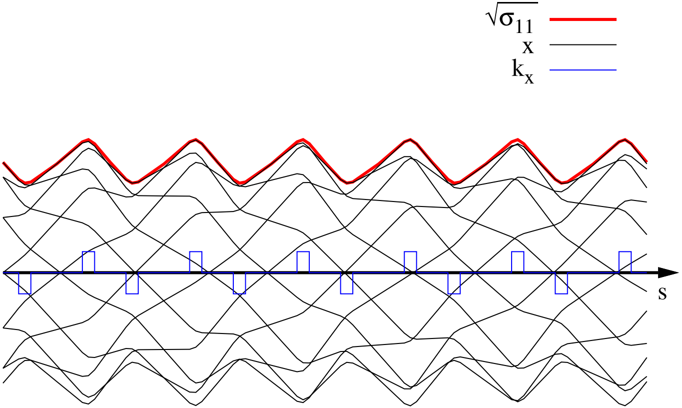
Periodic Structure

There is an infinity of possible envelopes (depending on the initial Σ):



Periodic Structure: Matched Beam Envelope

Periodic $k_x(s)$: a 'matched' envelope has the same periodicity as $k_x(s)$:



The Twiss/Courant-Snyder Parameters

$$\epsilon_{\mathrm{rms}} \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}_{\mathrm{matched}}$$

Be careful to not confuse the Twiss/Courant-Snyder parameters β_x, and γ_x with the relativistic Lorentz factors: they have nothing to do with each other!

Twiss or Courant-Snyder? To forge your own opinion, read: [Twiss and Frank, 1949], and [Courant and Snyder, 1958].

The Twiss/Courant-Snyder Parameters

Similarly:

$$\epsilon_{\mathrm{rms}} \begin{pmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \gamma_y \end{pmatrix} = \begin{pmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{34} & \sigma_{44} \end{pmatrix}_{\mathrm{matched}}$$

In the rest of this lecture I will use $_x$ interchangeably with $_y$.

Relations between α_x, β_x, and γ_x

Because of this scaling the determinant of the $\begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$ matrix is 1, leading to the first relation between the 3 Twiss parameters:

$$\gamma_x = \frac{1+\alpha_x^2}{\beta_x} \,.$$

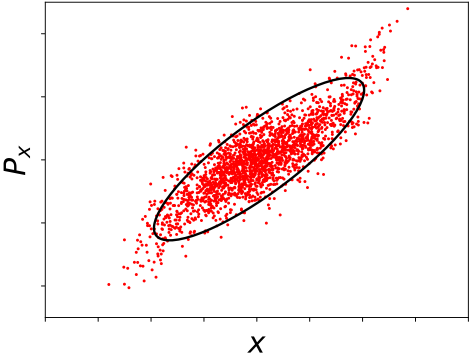
There is another relation between the slope $\frac{\mathrm{d}\beta_x}{\mathrm{d}s} = \beta'_x$ and α_x given by:

$$\beta'_x = -2\alpha_x \,.$$

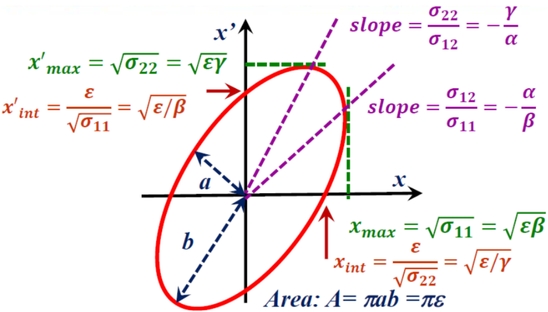
For a proof, see Ref. [Courant and Snyder, 1958].

Note that $\beta_x \geq 0$ and $\gamma_x \geq 0$, while α_x can change sign.

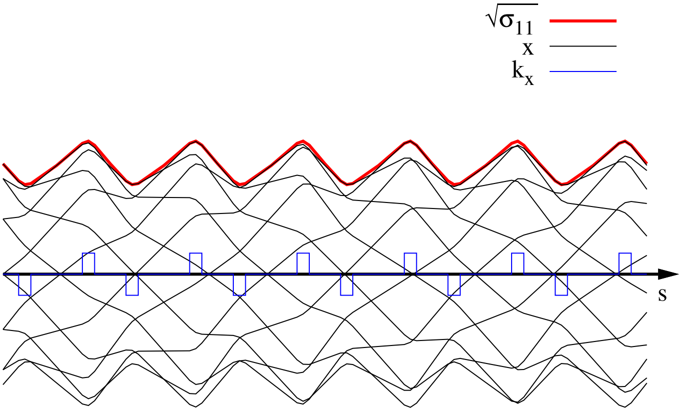
Relations



Relations


$slope = \frac{\sigma_{22}}{\sigma_{12}} = -\frac{\gamma}{\alpha}$
$x'_{max} = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$
$x'_{int} = \frac{\varepsilon}{\sqrt{\sigma_{11}}} = \sqrt{\varepsilon/\beta}$
$slope = \frac{\sigma_{12}}{\sigma_{11}} = -\frac{\alpha}{\beta}$
x'
x
a
b
$x_{max} = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta}$
$x_{int} = \frac{\varepsilon}{\sqrt{\sigma_{22}}} = \sqrt{\varepsilon/\gamma}$
Area: A= πab =πε

Matching Condition



Matching Condition

$$\mathbf{\Sigma}_{\mathsf{B}} = \mathbf{M} \cdot \mathbf{\Sigma}_{\mathsf{A}} \cdot \mathbf{M}^{\mathsf{T}}$$

If $\mathbf{M}$ is the matrix of 1 period, the 'matching condition' is:

$$\mathbf{\Sigma} = \mathbf{M} \cdot \mathbf{\Sigma} \cdot \mathbf{M}^{\mathsf{T}}$$

Stability Condition

Or more explicitly in 1-D:

$$\begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \cdot \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{\mathsf{T}}$$

This quadratic system has a solution if and only if:

$$\frac{|m_{11} + m_{22}|}{2} \leq 1\,.$$

This is the **condition of stability.**

Phase Advance

The solution is unique and can be written:

$$\mathbf{M} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu\,,$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{J} = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}$, and:

$$\cos\mu = \frac{m_{11} + m_{22}}{2}\,.$$

The angle μ is called the phase advance.

Transverse Tunes

In a circular accelerator the phase advance per turn, divided by 2π, is called the tune:

$$\nu = \frac{\mu_{\text{one turn}}}{2\pi} .$$

The tune is the number of betatron oscillations that a particle makes around the reference orbit each turn. The values of the horizontal tune ν_x and vertical tune ν_y play a central role in the study of betatron resonances in circular accelerators, but this is beyond the scope of this lecture.

Recommended Text Books

Book in open access: [Wiedemann, 2015] (see Chapter 5).

Other books I have used to prepare this lecture:[Wille, 2000], [Conte and MacKay, 2008], [Bryant and Johnsen, 2005].

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