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Basics of Transverse Beam Dynamics

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Abstract: This lecture is a brief introduction to the transverse dynamics of beams in particle accelerators. First, I present a few key concepts to understand the motion of a single particle around a reference orbit. Then, I introduce the idea of statistical beam parameters, and I show how to describe the evolution of an ensemble of particles as a whole. I also discuss the particular case of periodic structures which leads to the concept the Twiss/Courant-Snyder parameters. For simplicity I will mostly discuss the case of purely 1-dimensional transverse dynamics, although many definitions and properties presented here are readily generalized to the full 3-dimensional case. One of the main objectives of this lecture is to define some of the lingo used by accelerator physicists, so that you can understand what they talk about if you get caught in a meeting room between a two of them.

Contents

1 Preamble

Particle accelerators are, essentially, vacuum pipes with various components inside and/or around them to produce electromagnetic fields. Charged particles travel inside the vacuum pipe, and the electromagnetic fields are used to accelerate/decelerate the particles as well as to keep them confined within the pipe. The objective of this lecture is to study this transverse confinement. We will talk about acceleration and longitudinal confinement in the next lecture.

One can in principle calculate the trajectory of all charged particles in a global inertial frame of reference by integrating numerically:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}} = q\left(\boldsymbol{\mathcal{E}} + \mathbf{v} \times \mathbf{B}\right) , \tag{1}$$

where $\mathbf{p} = \gamma m \mathbf{v}$ is the particle's momentum, $\mathbf{v}$ its velocity, m and q are its mass and charge, γ its relativistic Lorentz factor[1], $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ are the electric and magnetic fields, and t is the time coordinate. Why not simply plug this equation into a numerical integrator to solve any beam dynamics problem? It seems so trivial. So what's the big deal?

2 Curvilinear Coordinate System

First of all, if you use a global frame of reference, you may set yourself up against extreme requirements on the precision of your numerical integrator. Think about the CERN Large Hadron Collider: a distance within the beam of 1 µm matters, in a machine nearly 10 km in radius. You would need to maintain the precision on at least 10 significant figures to carry out such a calculation.

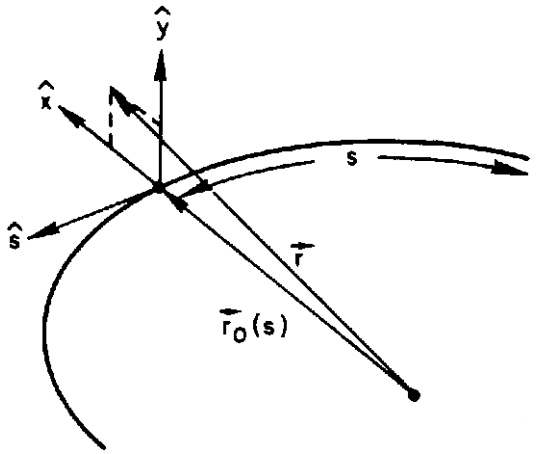

Figure 1: Frenet-Serret coordinate system (x, y, s) illustrated in the particular case of a planar reference trajectory. In this particular case the torsion of the curve is null, but the radius curvature ρ may vary along s.

Fortunately, this first hurdle is easily overcome by solving the problem, not in absolute space, but with respect to a reference trajectory. As this trajectory may be curved, this calls for the use of a curvilinear system of coordinates, see Fig. 1. Doing classical mechanics in

[1] $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ where c is the speed of light

such a non-inertial frame of reference requires some elementary precautions. If you choose to stick to Newtonian mechanics ($\mathbf{F} = m\mathbf{a}$), you have to add the appropriate 'centrifugal' and 'Coriolis' force terms to Eq. (1) (see for instance [1], Chapter 3).

Another approach is to switch to Hamiltonian mechanics, and to apply the appropriate canonical transformation to move from the Cartesian to the Frenet-Serret coordinate system, see Appendix B of Courant and Snyder's famous paper [2]. Although this would be a very interesting exercise, we will not re-derive the Courant and Snyder Hamiltonian in this lecture. I encourage you to give it a try, if you are interested, and to contact me directly if you experience any trouble with the derivation.

An important aspect of the Courant-Snyder approach, is that it uses as independent variable the coordinate s instead of the time t as in Eq. (1). This is important for two reasons:

- **Diagnostics**: most beam diagnostics equipment measures beam properties, such as position, profile, or phase-space distribution, right at the location where it is installed. They provide a measurement at a given s, and not a snapshot at a given t. Comparison between simulations and measurements is far easier when you use s as the independent variable.

- **Optics**: to apply all the tools of conventional optics to charged particles optics, such as transfer matrices (see Section 3), one must define the 'optical axis' – accelerator physicists call it the reference trajectory – and transport the 'rays' from a point A to point B along this axis. Once again, this calls for using s instead of t as the independent variable.

Although it is straightforward to numerically integrate Eq. (1), it requires an accurate description of the electromagnetic fields in 3-dimensional space to calculate the trajectory of individual particles, which can be hard to put together. Various codes allows you to do just that (Geant4, G4beamline, zgoubi, etc), but it is often much easier to describe a beamline, or an accelerator, as a succession of optical elements placed along a optical axis. In the rest of this lecture I will show you how you can do beam optics using tools and concept borrowed from conventional linear optics.

3 Simple Harmonic Motion

As mentioned in the preamble, all particle accelerators are designed to keep the beam confined withing the beampipe. To understand how this is done, let's consider the motion of a single particle within the beam. Let's choose for reference trajectory an actual particle trajectory: a particle placed right along it will remain on it. Most accelerator electromagnets are designed to provide a **linear force** towards (or away from) the reference trajectory. Let's now consider a section of the accelerator where this linear transverse force is constant, and where no acceleration happens (the reference momentum P_0 is constant, fields are static). In this case, the motion of a single particle is that of a simple harmonic oscillator:

$$x'' + k_x\, x = 0\,, \tag{2}$$

where x and s are Frenet-Serret coordinates (see Fig. 1), and $x'' = \frac{\mathrm{d}^2 x}{\mathrm{d}s^2}$. The motion of this particle in the vertical y direction follows from a similar equation, although generally $k_y \neq k_x$. The solution of this equation is given by the linear matrix relation:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \tag{3}$$

and **M**, called the 'transfer' matrix, is given by:

$$\mathbf{M} = \begin{cases} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{\sin(\sqrt{k}s)}{\sqrt{k}} \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} & \text{, if } k > 0 \text{ (focusing)} \\ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} & \text{, if } k = 0 \text{ (drift space)} \\ \begin{pmatrix} \cosh(\sqrt{|k|}s) & \frac{\sinh(\sqrt{|k|}s)}{\sqrt{|k|}} \\ \sqrt{|k|}\sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s) \end{pmatrix} & \text{, if } k < 0 \text{ (defocusing)} \end{cases} \tag{4}$$

Note that with our sign convention, if $k > 0$ the motion is bounded, while if $k < 0$ the distance x to the reference orbit diverges exponentially.

In practice, k_x and k_y are functions of s: in that case Eq. (2) is no longer a simple harmonic oscillator equation, but is called **Hill's equation**, and this can be analytically solved, see Ref. [2].

Since $k_x(s)$ can often be approximated as being piece-wise constant, which is a way to idealize magnets that is often referred to as the **hard-edge** approximation, the solution given in Eq. (4) still applies piece-wise. The overall transfer matrix can be obtained by concatenating the matrix for each piece:

$$M_{A\to D} = M_{C\to D} \cdot M_{B\to C} \cdot M_{A\to B}\,, \tag{5}$$

it is just like playing with lego blocks. Note that even in the cases that the 'hard-edge' is no longer a good approximation[2], one can always use a numerical integrator to calculate the effective transfer matrix for a **soft-edge** electromagnet. The lego block approach still applies.

Let's now consider the particular case of a quadrupole lense. As we will see in the following lecture on magnet design, quadrupoles are such that $k_x = -k_y$. In other words, a quadrupole that is focusing in the horizontal direction is defocusing in the vertical direction, and vice versa. Overall focusing is achieved by combining quadrupoles of opposite polarities into doublets (called FD doublet, or sometime FODO cells – for focusing-drift-defocusing-drift) or triplets (DFD or FDF), etc.

4 Digression: Symplecticity of the Transfer Matrix

This is a digression, and it is not essential that you understand this section... but some of you may have noted that the determinant of the transfer matrices given in Eq. (4) are all equal to 1. This is no accident. It is a consequence of the fact that Eq. (2) derives from a Hamiltonian. To see that, let's rewrite Eq. (2) in the form of two coupled first order differential equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= x'\,, \\ \frac{\mathrm{d}x'}{\mathrm{d}s} &= -k_x\,x\,. \end{aligned} \tag{6}$$

These two differential equations derive the following Hamiltonian:

$$H(x, x'; s) = \frac{x'^2}{2} + k_x\frac{x^2}{2}\,. \tag{7}$$

[2] like in elements that are short compared with their transverse aperture, see the lecture on magnet design.

Try to take partial derivatives of H w.r.t. x and x', and you will get back to Eq. (6)[3]. The phase space variables x and x' are canonically conjugated, and as a consequence, phase-space density is exactly conserved throughout the motion. This is called **Liouville's theorem**. It is true for any Hamiltonian process, even in the presence of non-linear forces.

Looking a little closer, $x' = \frac{P_x}{P_0}$ is actually canonically conjugated to x only after normalizing the Courant-Snyder Hamiltonian [2] by the reference momentum P_0, and only when P_0 is a constant. As P_0 is not generally constant, the true canonical pair is (x, P_x), where P_x is the transverse canonical momentum. It is only, in general, the phase space density in the canonical (x, P_x) phase space that is conserved. As a consequence, the determinant of the transfer matrix in (x, x') space over a section of beamline where acceleration happens will not in general be 1, but $\frac{P_A}{P_B}$, the ratio between the initial and the final reference momentum.

In this lecture we have so far only considered the case of 1-dimensional transverse motion (2-dimensional phase space). But the particle state vector $\mathbf{X}$ will in general be a 6-dimensional vector, containing 3 coordinates of position and 3 coordinates of momentum, measured with respect to the reference particle. The transfer matrix $\mathbf{M}$ is thus in general a 6×6 matrix, with multiple constraints on the determinants of the sub-blocks of the matrix, which can be summarized as:

$$\mathbf{M}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{S} \tag{8}$$

where the symplectic matrix $\mathbf{S}$ is:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{9}$$

For a proof, and to read more on this topic, see for instance chapter 3 of Ref. [3].

5 The Beam Matrix

Let's go back to the 1-dimensional case, where the particle state vector is:

$$\mathbf{X} = \begin{pmatrix} x \\ x' \end{pmatrix} . \tag{10}$$

We have just reviewed how to transport a single particle state vector from point A to point B by using the transfer matrix $\mathbf{M}$:

$$\mathbf{X}_{\mathrm{B}} = \mathbf{M} \cdot \mathbf{X}_{\mathrm{A}} . \tag{11}$$

This allows you to track the evolution of a single particle, by repeating the operation all along a beamline, and compute trajectories like in Fig. 2. You see as k_x varies along s, the particle trajectory does not follow a simple sine wave, but an oscillation modulated by the variation of $k_x(s)$[4]. These transverse oscillations are called, for historical reasons, **betatron oscillations**.

But beams are often made up of millions, or billions of particles (up to about 100 million particles per bunch in both the TRIUMF's electron linac and TRIUMF's cyclotron). Are we going to repeat the same operation millions of times to track the evolution of our beam? Can we not think of something more 'clever' to do?

[3] Mind the signs. See https://en.wikipedia.org/wiki/Hamiltonian_mechanics if you forgot.
[4] which is the general form of the solution of Hill's equation [2].

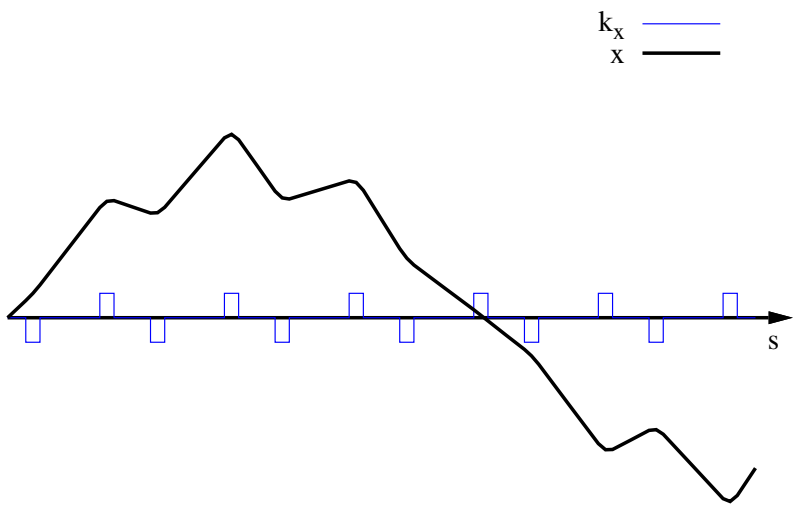

Figure 2: Example of a single particle trajectory along a periodic FODO focusing channel.

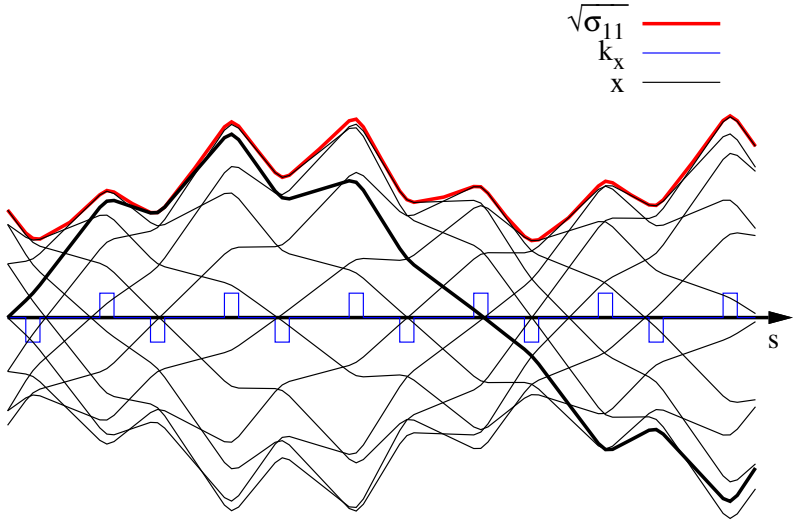

Figure 3: The trajectories of a dozen individual particles is shown in black. The single particle trajectory shown in Fig. 2 is shown with a thicker black line. The focal strength of the periodic struture (in blue) alternates in sign between focusing and defocusing in what is commonly called a FODO lattice. The corresponding beam envelope is shown in red.

In most cases, we only need to know the evolution of the beam size, to answer questions such as: is the beam going to make it through this aperture? If we detune this quadrupole by such amount, is the beam going to scrape the beampipe, etc. So how to define the 'size' of an ensemble of particles: with statistical quantities. Here is what statisticians call the

covariance matrix:

$$\mathbf{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{X}\cdot\mathbf{X}^{\mathrm{T}} \tag{12}$$

Where N is the total number of particles in the beam. With rather less flourish, accelerator physicists call this the **beam matrix**, or the 'sigma' matrix. Note that $\mathbf{\Sigma}$ is a symmetric matrix, which dimension is given by the size of $\mathbf{X}$. The elements of $\mathbf{\Sigma}$ are called the **second moments** of the particle distribution: the diagonal elements are the square of the RMS beam size in the corresponding dimension (position or momentum), and the off-diagonal terms are related to the statistical correlations of the particle distribution between the different dimensions. More specifically, in 1-dimension, the coefficients of the transfer matrices $\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ are such that:

- $\sqrt{\sigma_{11}}$ is the RMS beam size,
- $\sqrt{\sigma_{22}}$ is the RMS beam divergence, and
- $\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ is the statistical correlation between the two phase space variables.

Combining together Eq. (11) and Eq. (12) leads to:

$$\mathbf{\Sigma}_{\mathrm{B}} = \mathbf{M}\cdot\mathbf{\Sigma}_{\mathrm{A}}\cdot\mathbf{M}^{\mathrm{T}} \tag{13}$$

That's how you can transport the beam parameters from point A to point B without having to track every single particle in the beam. This is how you can track the red envelope in Fig. 3 without having to track individual particles, and thus however many individual particles there are in the beam. This technique works independently of the precise shape of the particle distribution. It can also be extended to take into account the effect of the Coulomb interaction of the particles within the beam, but this is beyond the scope of this lecture.

6 Beam Emittance

The RMS beam emittance is the square root of the determinant of the beam:

$$\epsilon = \sqrt{|\mathbf{\Sigma}|}\,. \tag{14}$$

This is also referred to as the **geometrical** RMS emittance of the beam, and is sometime written more explicitly ϵ_{rms}. Following from Eq. (13), the change of emittance from point A to point B is given:

$$\epsilon_B = |M|\epsilon_A\,. \tag{15}$$

As discussed in Section 4, the determinant $|M|$ of the transfer matrix is 1 for sections of beamline with no acceleration, and scales with the ratio of momentum $\frac{P_A}{P_B}$ otherwise. This implies that the geometrical emittance of the beam decreases when the beam is accelerated. Given this scaling, the quantity:

$$\epsilon_n = \frac{P_0}{mc}\epsilon = \beta\gamma\epsilon\,, \tag{16}$$

which is called the **normalized** RMS emittance, is constant throughout the acceleration process. The formula that people often remember is $\epsilon_n = \beta\gamma\epsilon$ where β and γ are the relativistic factors and NOT the Twiss/Courant-Snyder parameters that we will talk about

in Section 9.1. The mc factor is just here so that both geometrical and normalized are homogenous to a length. Typical values of the normalized RMS beam emittance is of the order of 1 µm, although some secondary particle beams (pions, muons, etc) may have orders of magnitude larger emittances.

Unlike the local phase-space density that is conserved through any conservative processes, the RMS normalized emittance is only conserved when Eq. (13) applies, i.e. with purely linear 1-dimensional motion. Non-linear forces and coupling to other dimensions (longitudinal or transverse) can changes the beam emittance, often leading to a degradation of the 'beam quality', i.e. an increase of the RMS emittance.

7 Beam Ellipse

The beam emittance is related to the area occupied by the beam in phase space. Try to take any 2-dimensional phase-space distribution of particle and draw around it the ellipse defined by the equations:

$$\sigma_{22}x^2 - 2\sigma_{12}xP_x + \sigma_{11}P_x = (4\,\epsilon)^2\,. \tag{17}$$

You will find that most particles fall within this ellipse, see for instance Fig. 4. The factor 4

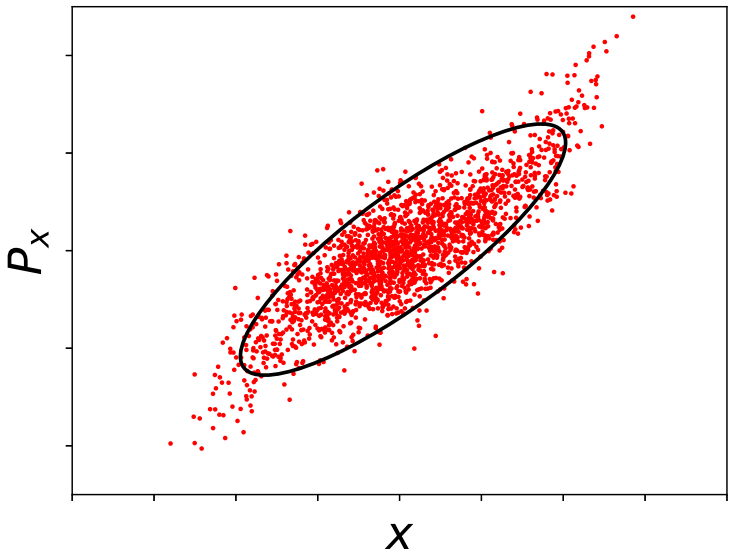

Figure 4: Example of beam distribution with the '4 RMS' ellipse drawn around it. Note that the area of this ellipse is $4\pi\epsilon_{\mathrm{rms}}$.

is somewhat arbitrary: it is such that the ellipse exactly matches the edge of an homogenous elliptical distribution. For other types of distributions the fractions of particles falling inside the '4 RMS' ellipse varies, but it is typically of the order of 90%, see Table 1.

Phase-space area	Gaussian	Uniform	Maxwellian
$\pi\epsilon_{\rm rms}$	39%	25%	35%
$4\pi\epsilon_{\rm rms}$	86%	100%	93%
$\zeta\pi\epsilon_{\rm rms}$	$1-\exp(-\zeta/2)$	$\zeta/4$	$\mathrm{erf}(\pi\zeta/\sqrt{96})$

Table 1: Fraction of particles contained within a given area in phase space for different types of 2-dimensional distributions with elliptical symmetry [4].

8 The factor π

You will sometimes see in scientific publications emittance numbers given like this:

$$\epsilon = 10\,\pi\,\mathrm{mm\,mrad} \tag{18}$$

with the extra π factor. This notation is archaic and confusing [5, 4]. Don't use it. Write instead:

$$\epsilon = 10\,\mu\mathrm{m}\,. \tag{19}$$

And write down explicitly whether this is an RMS or a **4**RMS emittance. You can use the notations '$\epsilon_{\rm rms}$' for RMS emittances and '$4\epsilon_{\rm rms}$' for 4RMS emittances.

9 Periodic Structures

Let's now consider a periodic focusing structure. It can be a periodic transport section in a beamline or a circular accelerator, such as a synchrotron. The beam size at any location along the beamline depends on the initial conditions following Eq. (13). There is an infinite number of possible beam envelope through the structure, see for instance Fig. 3. But there is a unique solution (see Section 9.2) for which the envelopes have the same periodicity as the focusing structure itself, such as in Fig. 5. This is what is called the **matched** beam envelope.

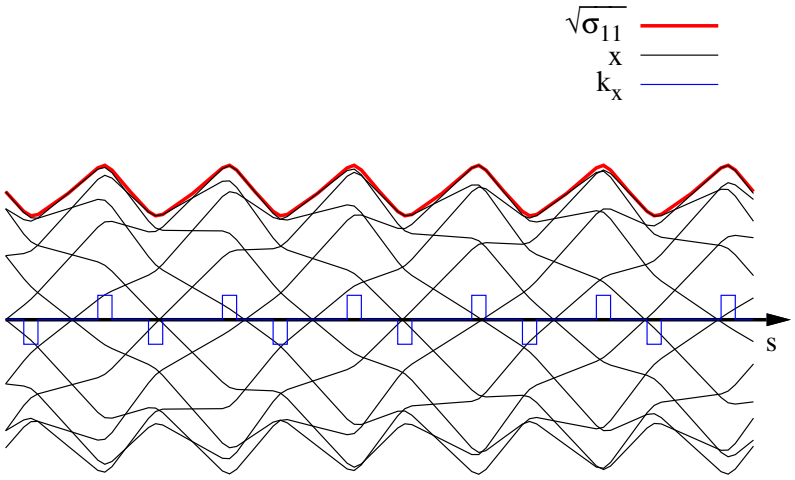

Figure 5: Example of a 'matched' beam envelope (in red), where, contrary to in Fig. 3, the beam envelope has the same periodicity as the lattice (in blue). The trajectories of a dozen individual particles are shown in black.

9.1 The Twiss/Courant-Snyder Parameters

The shape of the matched beam envelope is an intrinsic property of a given periodic lattice. To describe it in a beam-independent manner, accelerator physicists often use the so-called Twiss/Current-Snyder parameters [6, 2] α_x, β_x, and γ_x defined as:

$$\epsilon_{\text{rms}} \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}_{\text{matched}} . \tag{20}$$

Similarly α_y, β_y, and γ_y are defined as:

$$\epsilon_{\text{rms}} \begin{pmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \gamma_y \end{pmatrix} = \begin{pmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{34} & \sigma_{44} \end{pmatrix}_{\text{matched}} . \tag{21}$$

Be careful to not confuse the Twiss parameters β_x, and γ_x with the Lorentz factors: they have nothing to do with each other! The Twiss parameters are like the second moments of the beam distribution (the 'sigmas'), only scaled by the beam emittance ϵ_{rms}. Because of this scaling the determinant of the $\begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$ matrix is 1, leading to the first relation between the 3 Twiss parameters:

$$\gamma_x = \frac{1 + \alpha_x^2}{\beta_x} . \tag{22}$$

There is another relation between the slope $\frac{\mathrm{d}\beta_x}{\mathrm{d}s} = \beta_x'$ and α given by:

$$\beta_x' = -2\alpha_x . \tag{23}$$

For a proof, see Ref. [2]. A summary of relations between the various parameters introduced in this lecture and the shape/size of the beam ellipse is Fig. 6. Note that β_x and γ_x are positive quantities, while α_x may be either positive or negative.

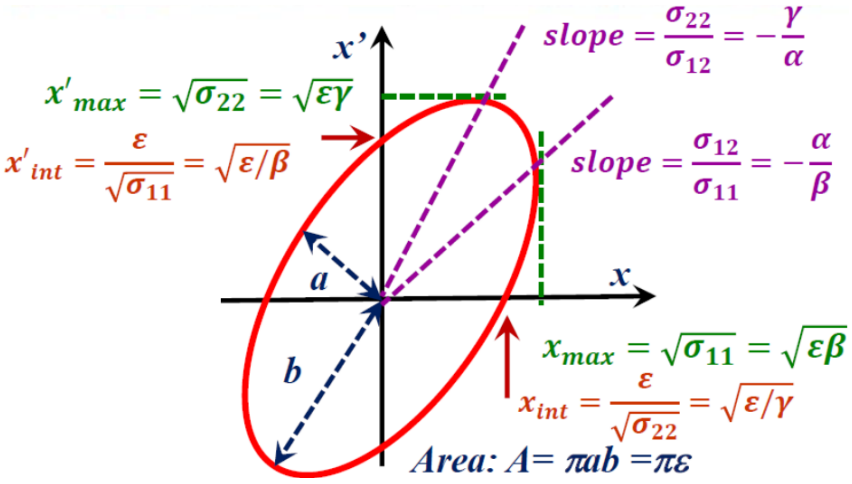

Figure 6: Summary of the relations between the different parameters introduced in this lectures.

9.2 Stability Condition

There does not always exist a matched envelope. If $\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ is the transfer matrix through one period of the lattice, considering Eqs. (13) and (21),the periodic twiss parameters satisfy:

$$\begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \cdot \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{\mathrm{T}} . \tag{24}$$

I encourage you to try to solve Eq. (26), and you will find that a real solution exists if and only if:

$$\frac{|m_{11} + m_{22}|}{2} \leq 1 . \tag{25}$$

This is the **condition of stability**: a periodic solution exists in a periodic transport system if and only if the half trace of the lattice transfer matrix is smaller than 1 and larger than -1. When it exists the solution is unique[5] and given by:

$$\mathbf{M} = \mathbf{I} \cos\mu + \mathbf{J} \sin\mu , \tag{26}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{J} = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}$, and:

$$\cos\mu = \frac{m_{11} + m_{22}}{2} . \tag{27}$$

The angle μ is called the phase advance. You can now see that the particle advances around an ellipse by an angle μ per cell. In a circular accelerator the phase advance per turn, divided by 2π, is called the tune:

$$\nu = \frac{\mu_{\mathrm{turn}}}{2\pi} . \tag{28}$$

The tune is the number of betatron oscillations that a particle makes around the reference orbit each turn. The values of the horizontal tune ν_x and vertical tune ν_y play a central role in the study of betatron resonances in circular accelerators, but this is beyond the scope of this lecture.

[5] Remember: both β and γ are defined positive.

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