Accelerator Physics

Beam Dynamics

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PHYS481: Accelerator Physics

Mid-term feedback

Only six responses received – rather disappointing engagement with feedback process.

General Comments

"I like the structure of pre recorded as i can pause and take my time. Live session also really helpful when going in more detail."

[we received] "immediate feedback in the tutorial…was helpful to help us remember the important points."

Good practice

"Linking to real world examples regularly" – consider attending Daresbury Lab visit.

"Nice clear structure of canvas for module."

"The poster task is very useful for developing report reading and presentation skills" – remember submission deadline

Suggestions for improvement

"Timetabling has not worked well" (discussed in more detail in SSLC)

- We are now releasing the material that students need to engage with before the Monday live-session already on Wednesdays rather than the Fridays before the Monday session.

"Pace was sometimes too high"

- Live sessions focus on learning outcomes – we recommend going through material again in own pace.
- (clarified in SSLC): Focus will now be put more on derivations, physics background and meaning of material, rather than description of equations and slide content.

We have adapted the content acc ording to interests expressed in week 2 and hope you continue to enjoy the module.

Please remember to engage with all sessions fully.

Learning objectives

- Concepts of emittance, dispersion, chromaticity, tune of an accelerator and resonances
- Homogenous and inhomogeneous equation of motion (off-momentum particles)
- Strong focusing with quadrupoles
- Different forms of transfer matrices (in terms of Twiss parameters, non-periodic, per turn of a periodic lattice etc.)

Emittance

Phase Space

- The Twiss (or Courant-Snyder) parameters α, β, γ have a geometric meaning:

(1) $$x(s) = \sqrt{\epsilon\beta(s)}cos(\mu(s) + \mu_0) \qquad x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}(sin(\mu(s) + \mu_0) + \alpha(s)cos(\mu(s) + \mu_0))$$

(2) $$cos(\mu(s) + \mu_0) = \frac{x(s)}{\sqrt{\epsilon}\sqrt{\beta(s)}} \qquad sin(\mu(s) + \mu_0) = \frac{\beta x' + \alpha x}{\sqrt{\beta(s)}\sqrt{\epsilon}}$$

(3) $$cos^2(\mu(s) + \mu_0) = \frac{x^2(s)}{\epsilon\beta(s)} \qquad sin^2(\mu(s) + \mu_0) = \frac{1}{\epsilon\beta}\left(\beta^2(sx'^2(s) + 2\beta(s)\alpha(s)x'(s)x(s) + \alpha^2(s)x^2(s)\right)$$

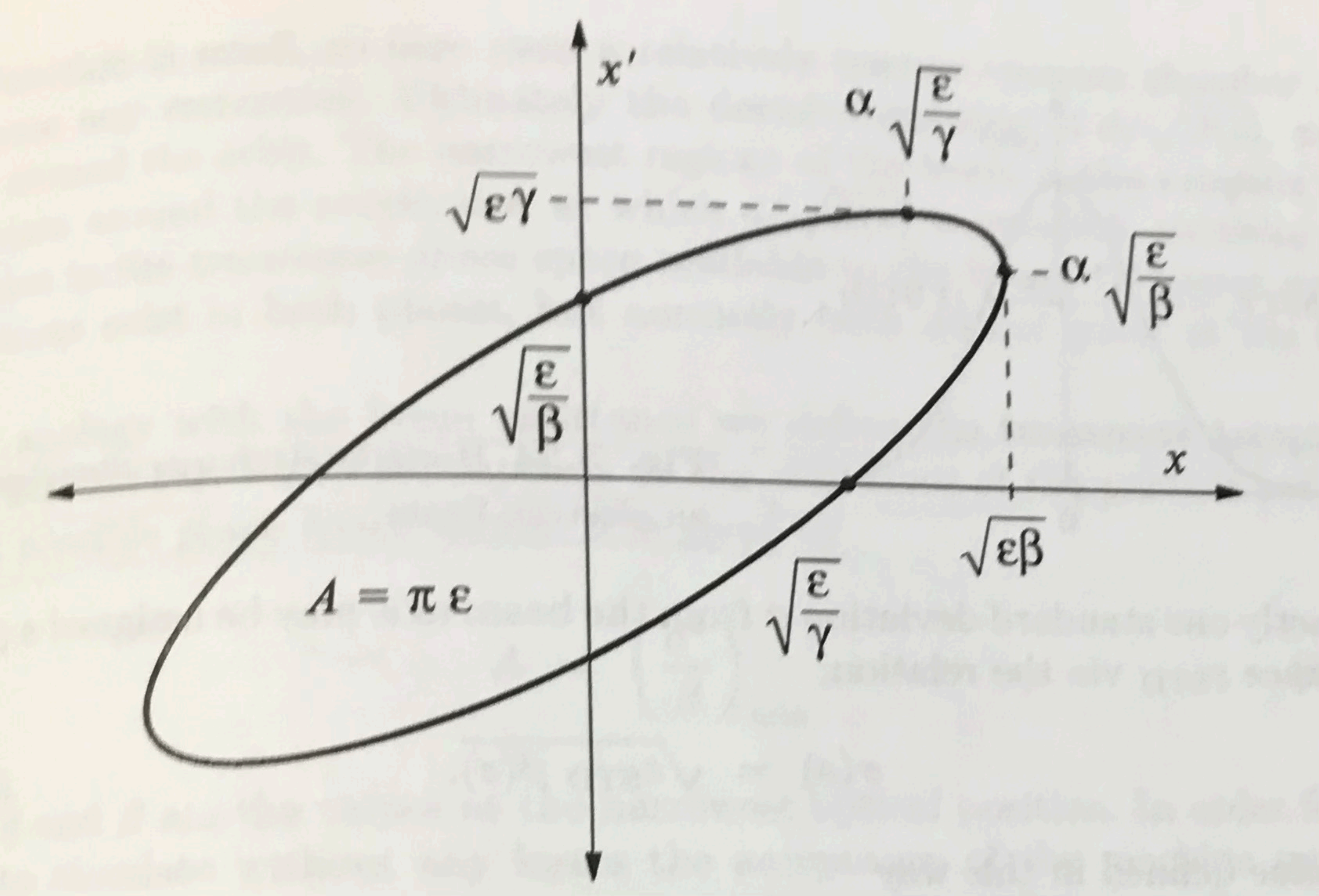
(4) $$sin^2(\mu(s) + \mu_0) + cos^2(\mu(s) + \mu_0) = 1$$

(5) $$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

- Function value of s-dependent quantities is constant.

Note: $$\alpha(s) = \frac{\beta'}{2} \qquad \gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

$$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

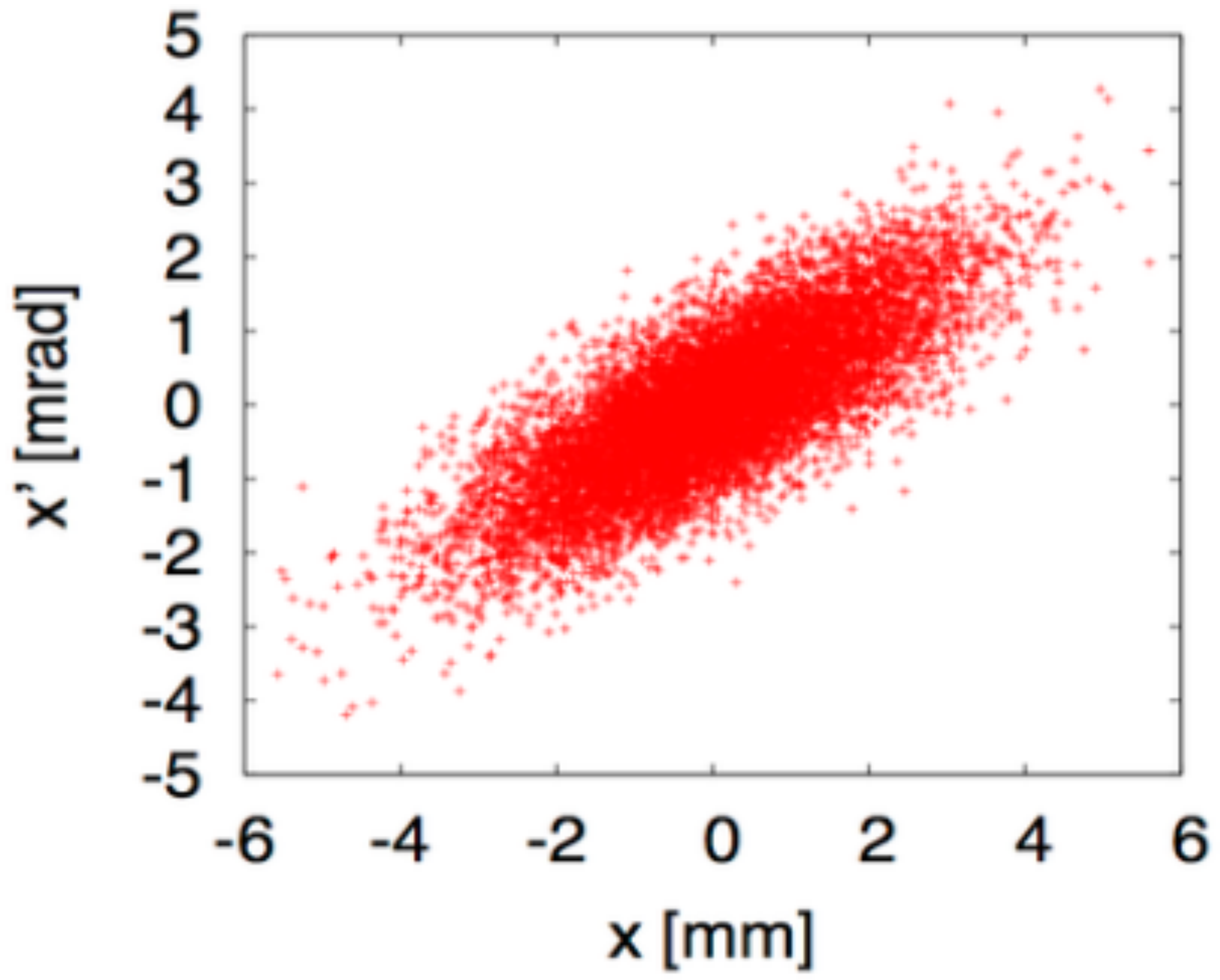


Emittance

- **Statistical definition**

Emittance characterises the spread of an ensemble of particles in a 2D (x,x') phase space.

Emittance is a measure of the phase space area occupied by a beam.



$$\epsilon_{x,rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\langle x^2\rangle = \frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x\rangle)^2$$

$$\langle x'^2\rangle = \frac{1}{N}\sum_{i=1}^{N}(x_i' - \langle x'\rangle)^2$$

$$\langle xx'\rangle = \frac{1}{N}\sum_{i\neq j}^{N}\sum_{j=1}^{N}(x_i - \langle x\rangle)(x_j' - \langle x'\rangle)$$

Emittance

Take an arbitrary particle orbit in terms of lattice Twiss parameters (general solution of Hill's equation):

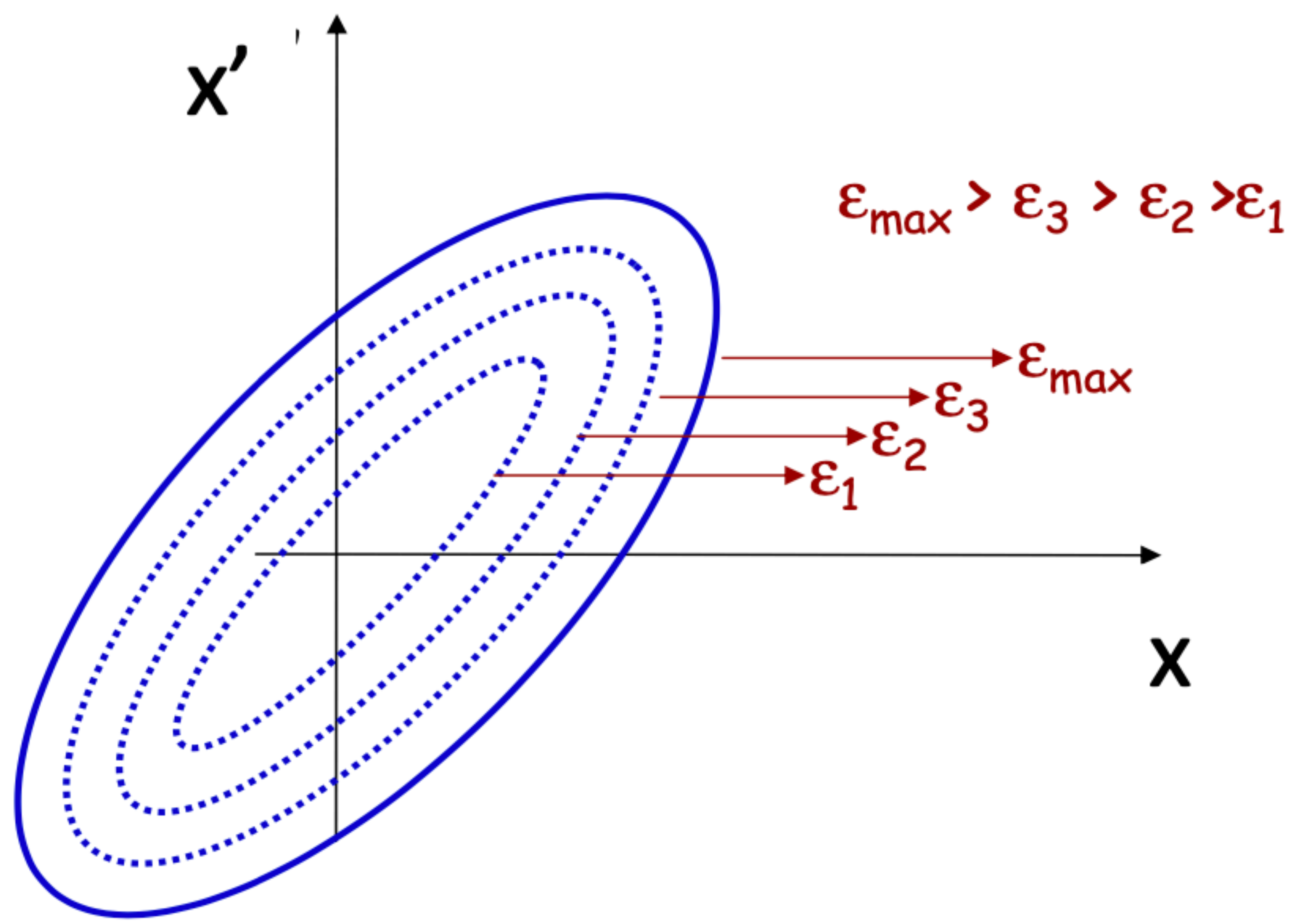
$$(i = 1 \ to \ N)\begin{cases} x_i(s) = \sqrt{\epsilon_i \beta(s)} cos(\psi(s) + \phi_i) \\ x_i{}'(s) = = \sqrt{\epsilon_i / \beta(s)} \left(\alpha(s) cos(\psi(s) + \phi_i) + sin(\psi(s) + \phi_i)\right) \end{cases}$$

$$\langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i \beta(s) cos^2 \left(\psi(s) + \phi_i\right) = \beta(s) \langle \epsilon \rangle$$

$$\langle x'^2 \rangle = \gamma(s) \langle \epsilon \rangle$$

$$\langle x x' \rangle = \alpha(s) \langle \epsilon \rangle$$

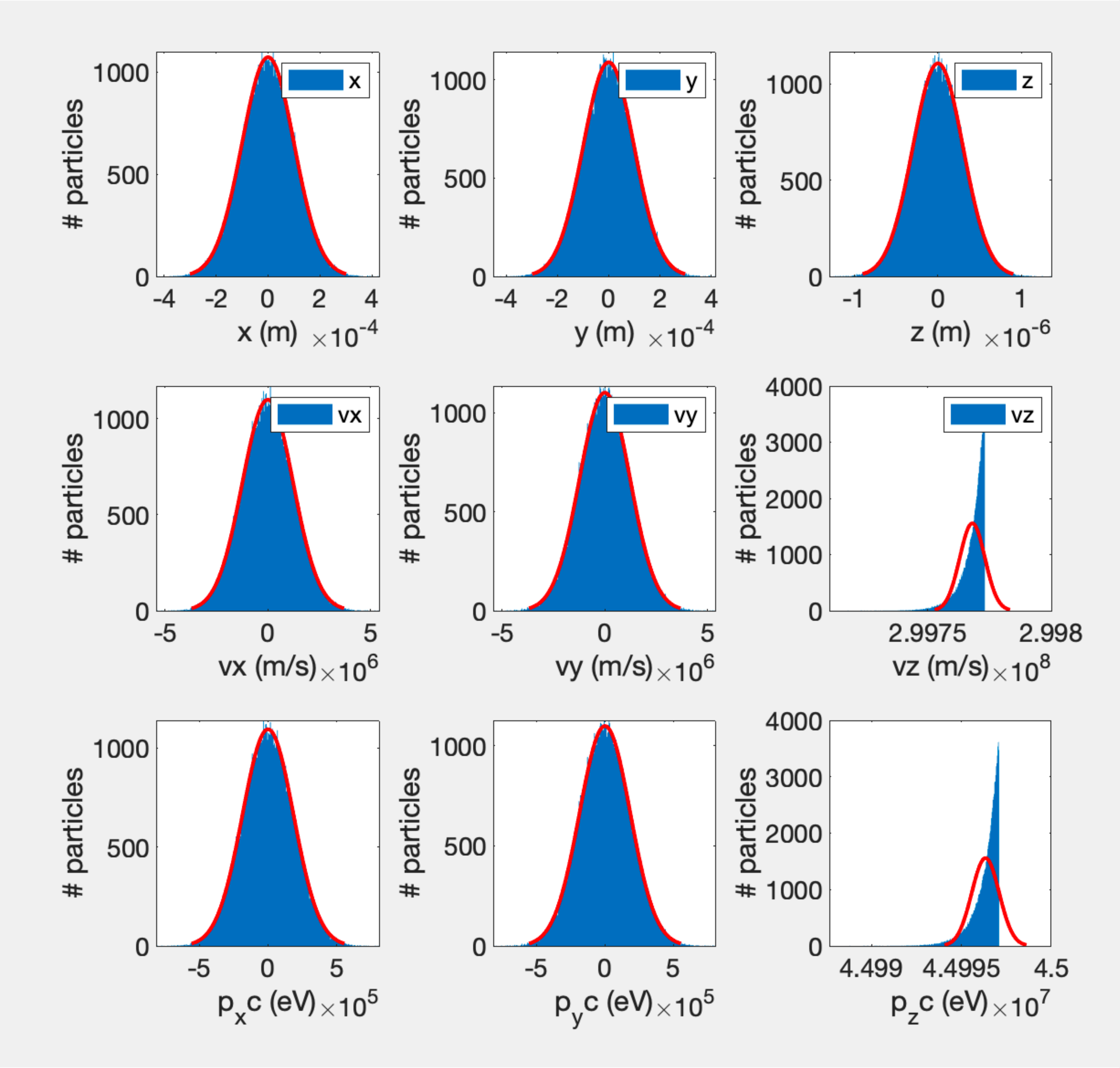
$$\epsilon_{rms} = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i = \langle \epsilon \rangle$$

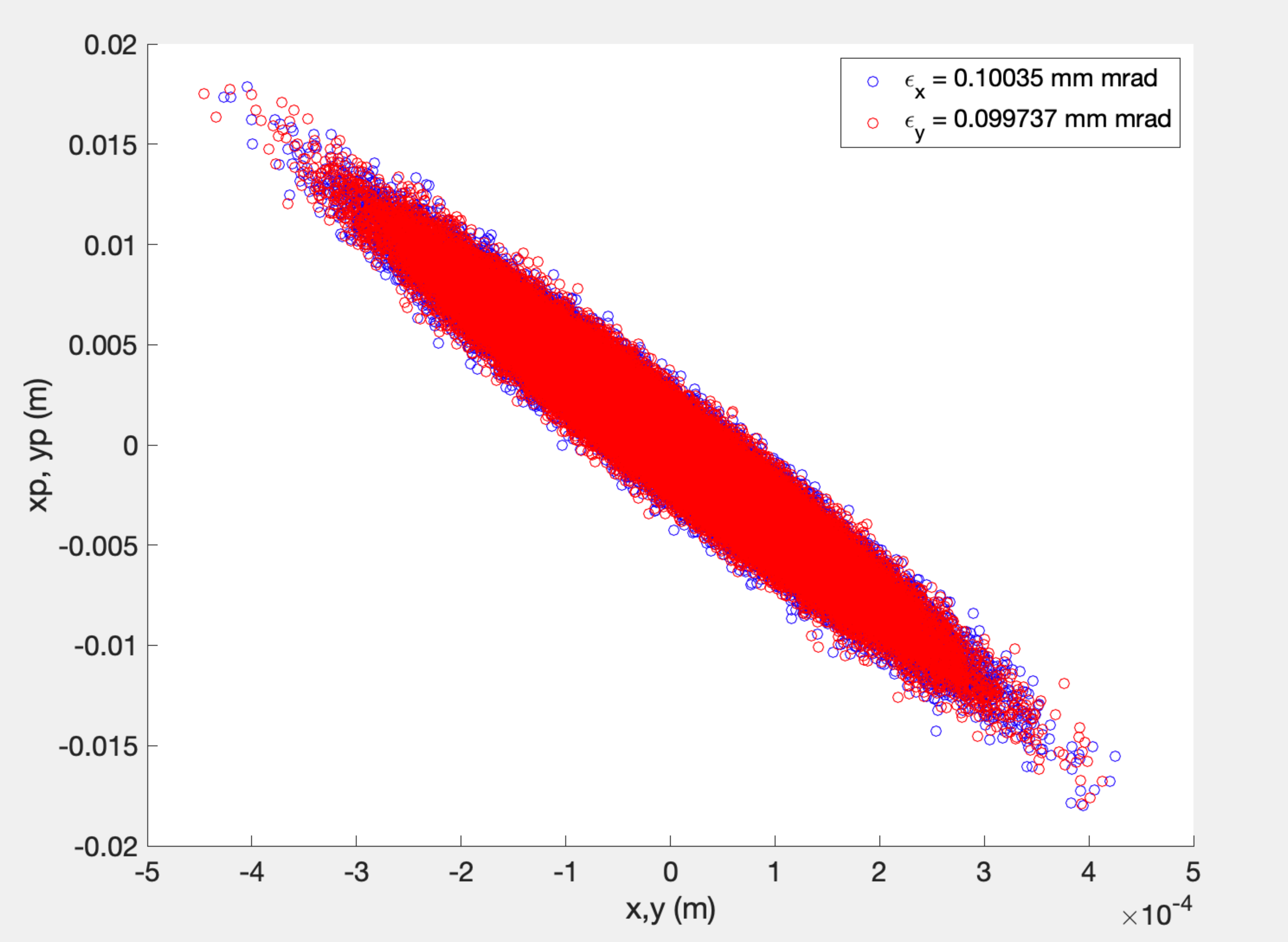


Courant-Snyder invariant is independent of s.

Emittance

- **Statistical definition (see the codes…)**





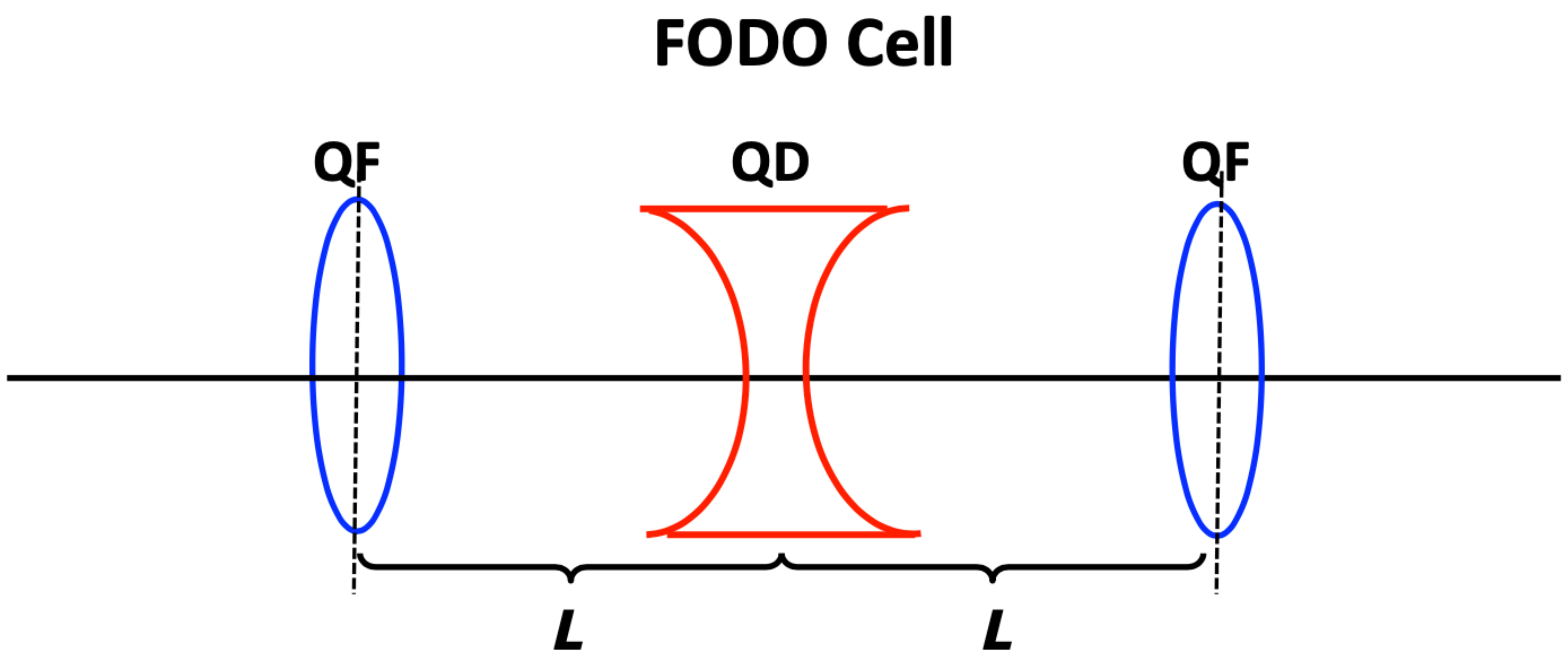
A 6D beam distribution

Corresponding transverse phase space and rms emittances

Strong focusing

Evolution of phase space in a FODO cell

Remember:

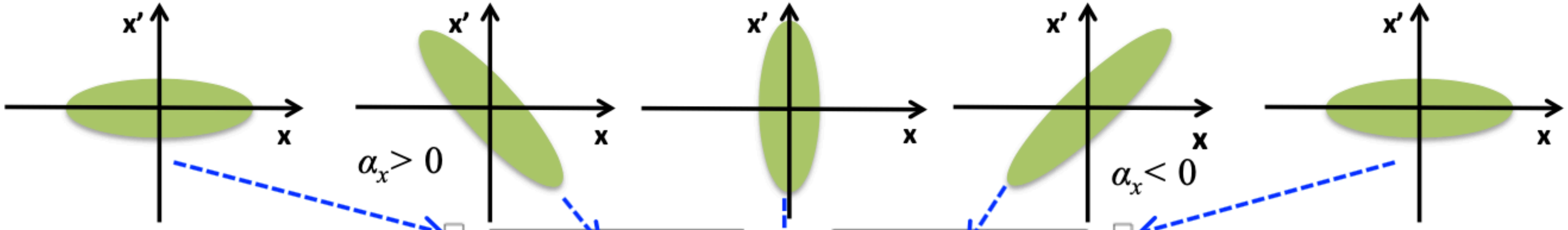


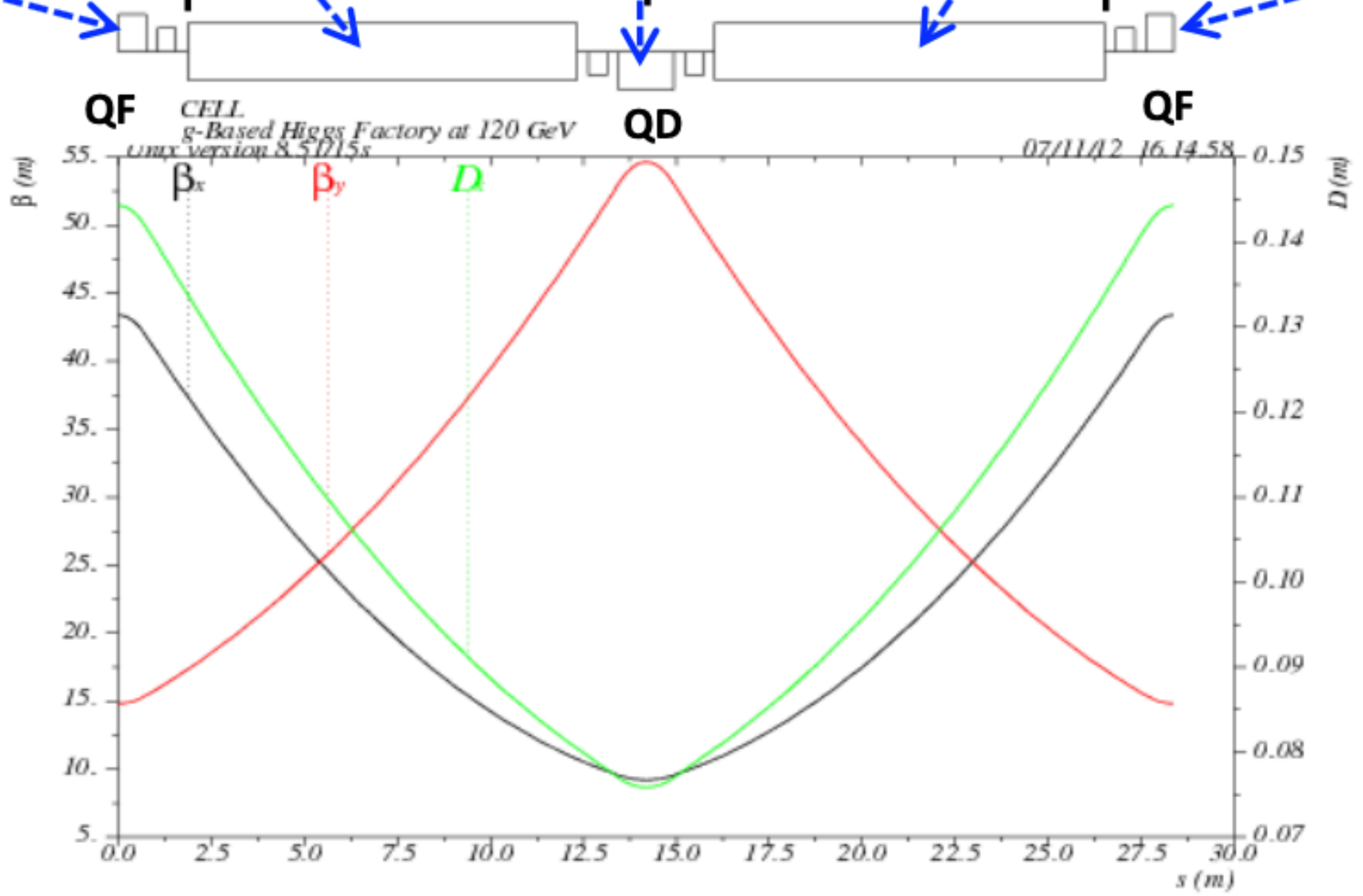
Symmetric transfer matrix for centre to centre distance between the focusing quadrupole magnets.

$$M_{FODO} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

Evolution of phase space in a FODO cell

- A large β-function corresponds to a large beam size and a small divergence.





In the middle of focusing quadrupole, QF, β_x is maximum (β_y is minimum), and $\alpha_{x,y} = 0$.

In the middle of defocusing quadrupole, QD, β_y is maximum (β_x is minimum), and $\alpha_{x,y} = 0$.

Transfer matrix in terms of Twiss parameters

Transfer matrix in terms of Twiss parameters

(1) Using trigonometric identities: $sin(a+b) = sin(a)*cos(b) + cos(a)*sin(b)$ and $cos(a+b) = cos(a)*cos(b) - sin(a)sin(b)$

(2) We can re-organise the solutions of Hill's equation:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta_s}(cos\psi_s cos\phi - sin\psi_s sin\phi)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\beta_s}(\alpha_s cos\psi_s cos\phi - \alpha_s sin\psi_s sin\phi + sin\psi_s cos\phi + cos\psi_s sin\phi)$$

(3) Let's determine the initial conditions as:

$$x(0) = x_0, \psi(0) = 0$$

Hence,

$$cos\phi = \frac{x_0}{\sqrt{\epsilon\beta_0}} \qquad sin\phi = -\frac{1}{\epsilon}(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}})$$

(4) Substituting these to eliminate the arbitrary phase:

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s)x_0 + (\sqrt{\beta_s\beta_0}sin\psi_s)x'_0$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s)x_0 + \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s)x'_0$$

(5) The transfer matrix in terms of the beta functions (or non-periodic transfer matrix):

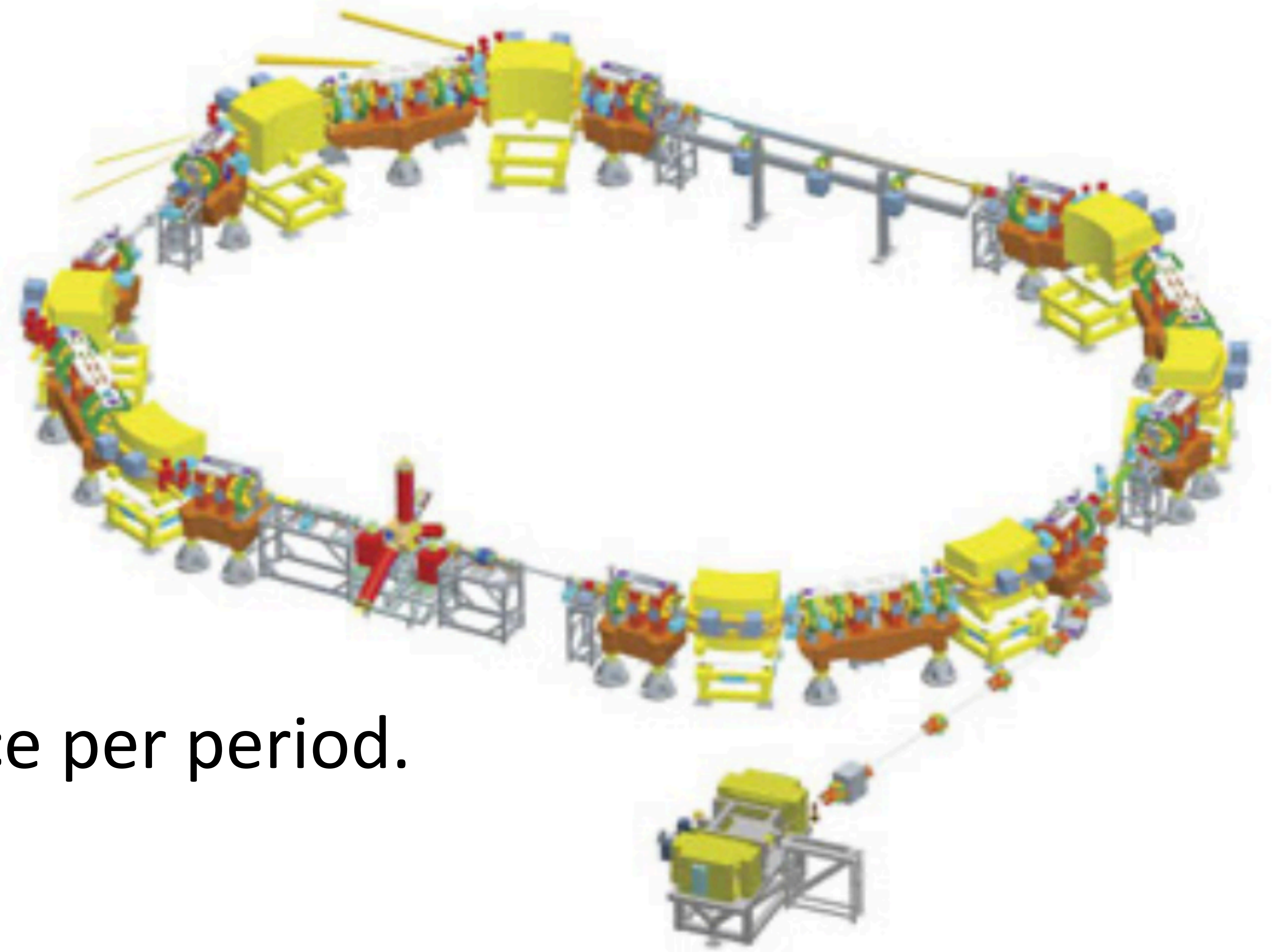
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & (\sqrt{\beta_s\beta_0}sin\psi_s) \\ \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$

Transfer matrix in terms of Twiss parameters

- In a periodic lattice Twiss parameters will have the same value as their initial values after a full turn.

$$\beta_s = \beta_{s+L} \quad \alpha_s = \alpha_{s+L} \quad \gamma_s = \gamma_{s+L} \longrightarrow \beta_0 = \beta_s, \alpha_0 = \alpha_s,$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & (\sqrt{\beta_s\beta_0}sin\psi_s) \\ \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$



The transfer matrix for a turn in a regular lattice is given as below.

$$M = \begin{pmatrix} cos\mu_L + \alpha_s sin\mu_L & \beta_s sin\mu_L \\ -\gamma_s sin\mu_L & cos\mu_L - \alpha_s sin\mu_L \end{pmatrix}$$

where μ_L is the phase advance per period.

$$\mu_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

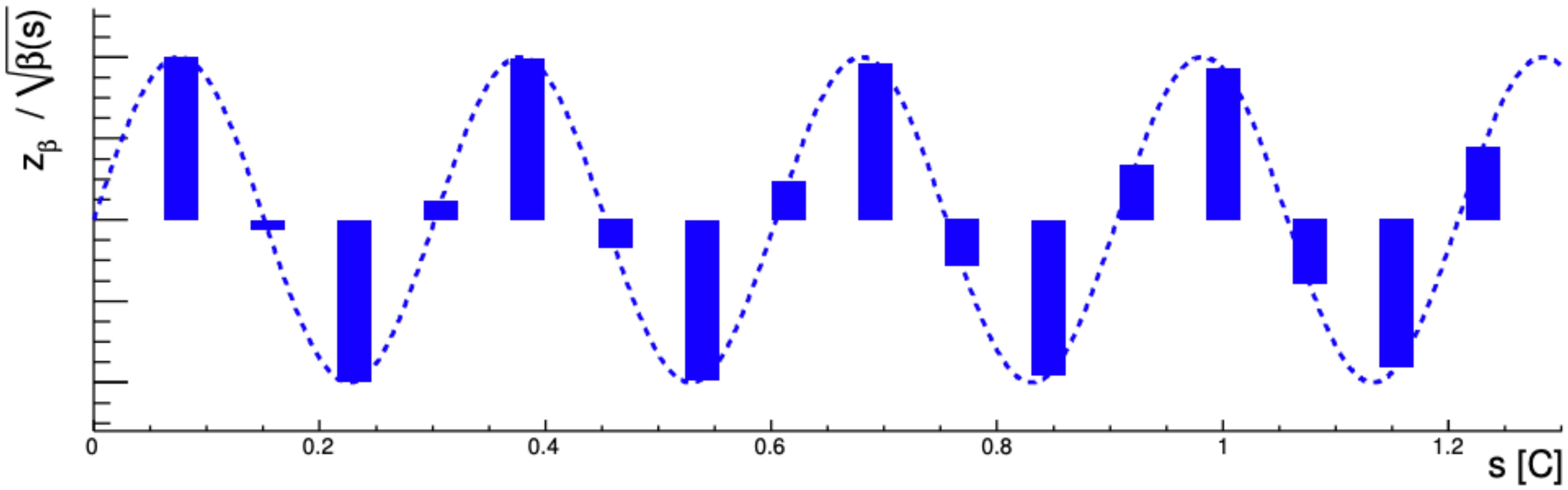


Fig. 2: Normalized betatron motion (continuous curve) as seen in the local co-rotating coordinate system. The motion has been normalized by the square root of the betatron function to remove amplitude modulation dependence. An exemplary sampling by beam position monitors distributed throughout the ring and indicated by bars, is shown. The tune of $Q = 3.31$ is visible, counting the number of full oscillation periods N_o for a given path length L (here: $N_o = 4$ at $L \approx 1.21$ C, $Q = \frac{N_o}{L}$).

R. J. Steinhagen, https://cds.cern.ch/record/1213281/files/p317.pdf

Break-out time!

- Consider the values of the matrix elements for a turn and calculate the tune of the accelerator.

$$M = \begin{pmatrix} cos\mu_L + \alpha_s sin\mu_L & \beta_s sin\mu_L \\ -\gamma_s sin\mu_L & cos\mu_L - \alpha_s sin\mu_L \end{pmatrix} = \begin{pmatrix} 0.866 & 50 \\ 3.6e-4 & 0.866 \end{pmatrix}$$

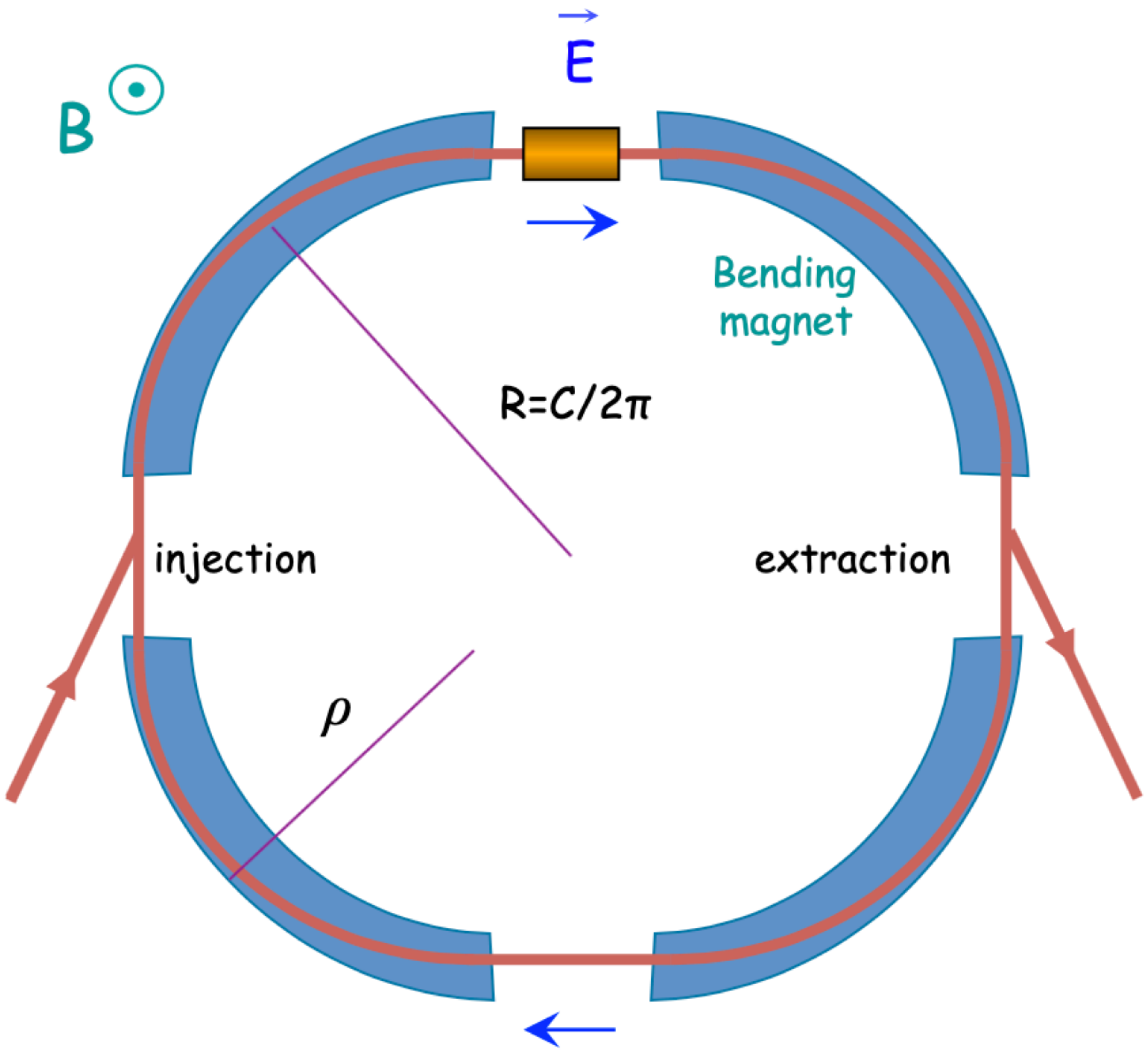
$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

- Do you expect any resonance driven beam losses?

Synchrotron

Synchrotron

- Synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each tune. That implies the following operating conditions.



$e\hat{V}sin\phi$ Energy gain per turn

$\phi = \phi_s = constant$ Synchronous particle's phase

$\omega_{RF} = h\omega_r$ RF synchronism (h-harmonic number)

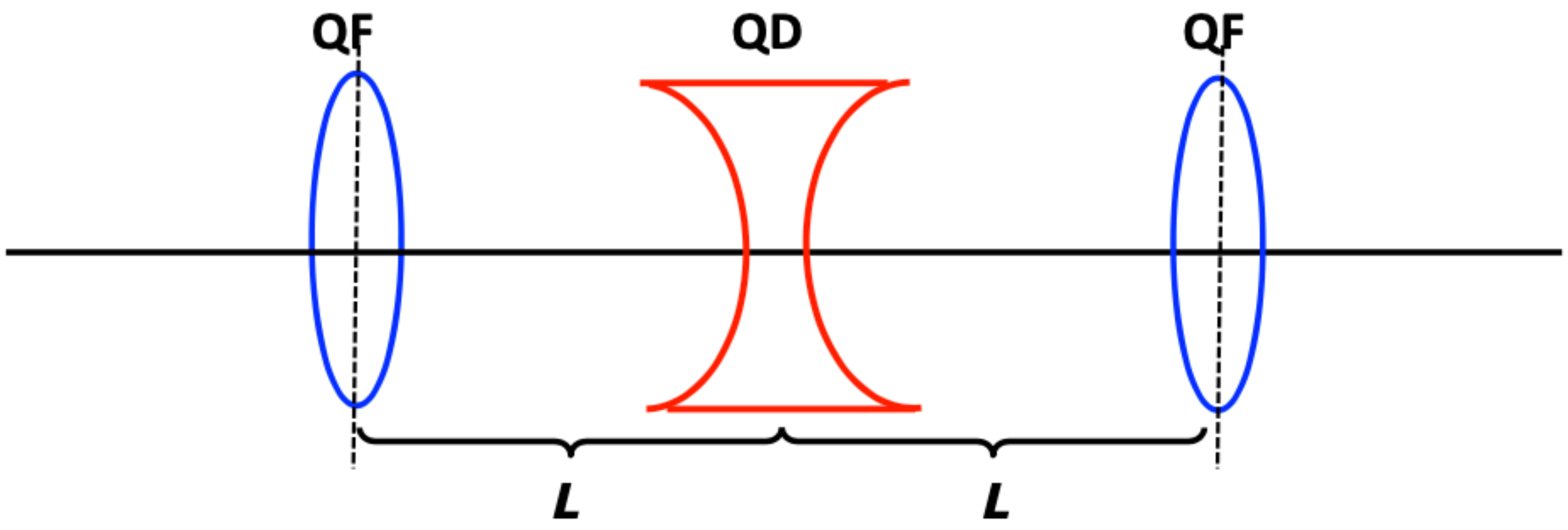
$2\pi R = h\lambda_{RF}$

$\rho = constant,\ R = constant$ Constant orbit

$B\rho = p/e \rightarrow B$ Variable magnetic field

- We can vary the magnetic field for changing beam energy to maintain the same orbit.
- We can vary RF frequency to maintain synchronism.

FODO cell



Symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\text{FODO}} = \mathcal{M}_{\text{HQF}} \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{QD}} \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{HQF}}$$

$$\mathcal{M}_{\text{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} , \quad \mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} , \quad \mathcal{M}_{\text{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$\mathcal{M}_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

FODO cell

If we compare the previous matrix with the Twiss representation over one period,

$$\mathcal{M}_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

we can derive interesting properties.

- Phase advance

$$\cos\mu = \frac{1}{2}\text{trace}(M) = 1 - \frac{L^2}{2f^2}$$

remembering that $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$

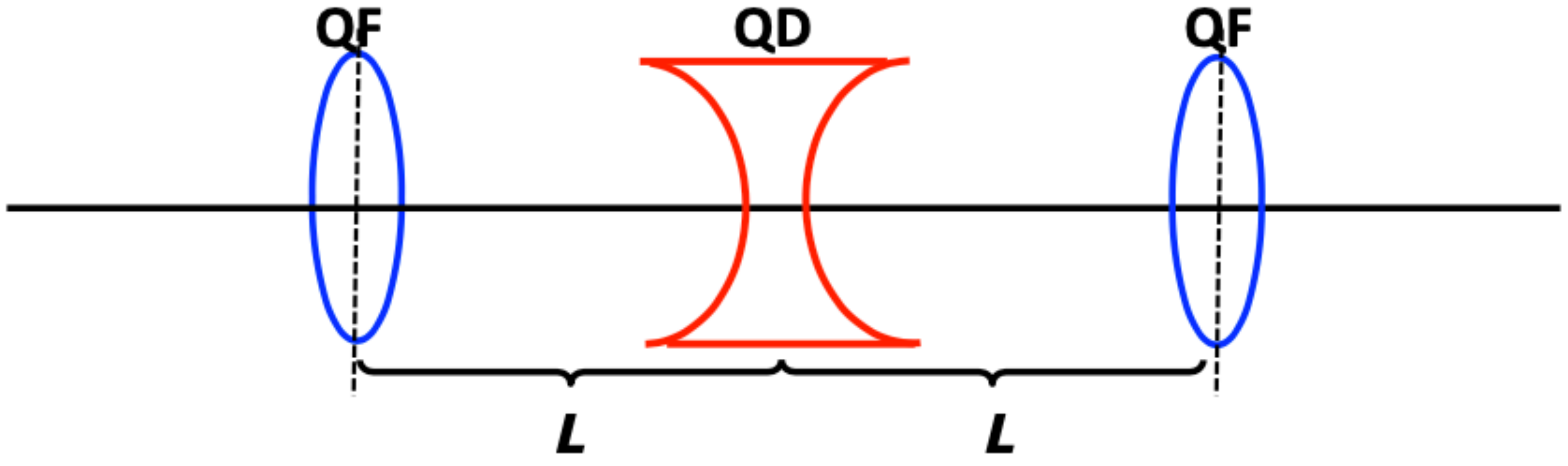
$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{2f}$$

This equation allows to compute the phase advance per cell from the cell length and the focal length of the quadrupoles.

FODO cell

- Maximum and minimum Twiss parameters

For a FODO cell like in figure, with two thin quadrupoles separated by length *L*



one has

$$f = \frac{L}{2\sin\frac{\mu}{2}}$$

$$\beta^{\pm} = \frac{2L\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu}$$

$$\alpha^{\pm} = \frac{\mp 1 - 1\sin\frac{\mu}{2}}{\cos\frac{\mu}{2}}$$

β^{+}: maximum $\beta(s)$
β^{-}: minimum $\beta(s)$

α^{+}: maximum $\alpha(s)$
α^{-}: minimum $\alpha(s)$

POP QUIZ: In a FODO cell, α=0 in the middle of a quadrupole, β is maximum (minimum) in middle of a QF (QD), prove the expressions for $\beta_{max} = \hat{\beta} = \beta^{+}$ and $\beta_{min} = \check{\beta} = \beta^{-}$.

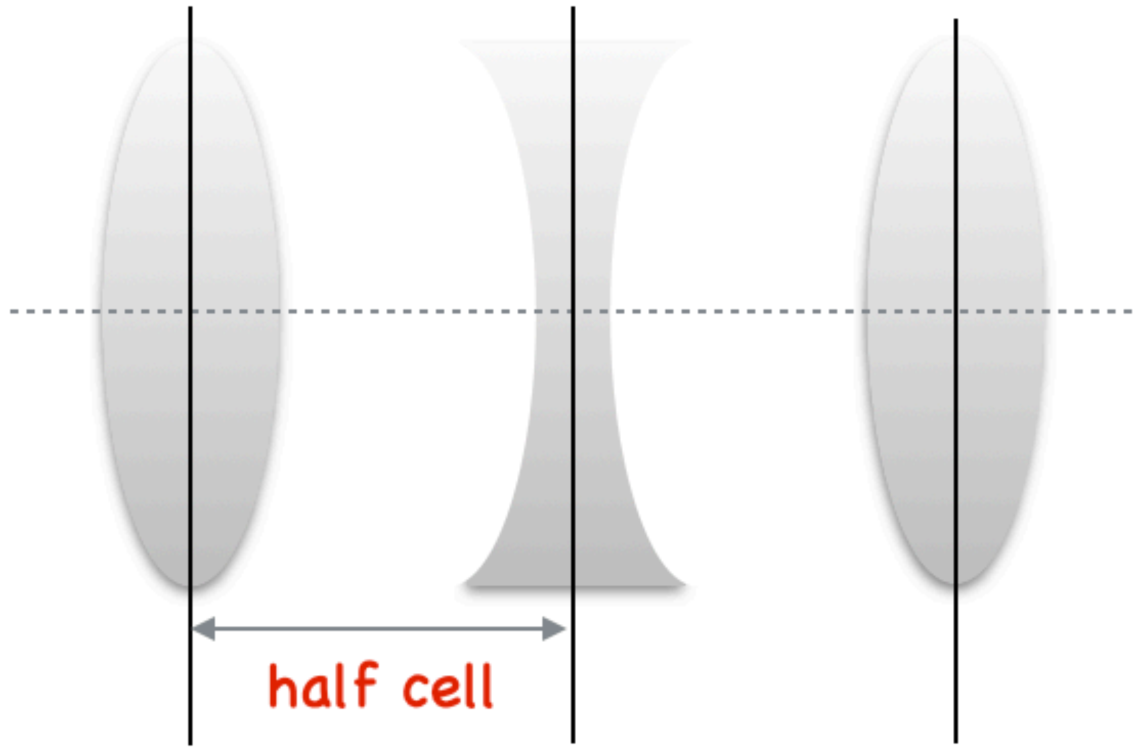
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & (\sqrt{\beta_s\beta_0} sin\psi_s) \\ \frac{1}{\sqrt{\beta_s\beta_0}}((\alpha_0 - \alpha_s)cos\psi_s - (1 + \alpha_0\alpha_s)sin\psi_s) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$

- **In a FODO cell, α=0 in the centre of a focusing quadruple.**
- **Therefore, the beta functions evolve from βmax to βmin along the first half of the cell.**

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}} cos\mu/2 & (\sqrt{\check{\beta}\hat{\beta}} sin\mu/2) \\ -\frac{1}{\sqrt{\hat{\beta}\check{\beta}}} sin\mu/2) & \sqrt{\frac{\hat{\beta}}{\check{\beta}}} cos\mu/2 \end{pmatrix}$$

Let's move from the first focusing magnet to defocusing magnet in a FODO cell…

$$M_{FoDo} = \begin{pmatrix} 1 - \frac{l^2}{2f^2} & 2l(1 - \frac{l}{2f}) \\ -\frac{l}{2f^2}(1 + \frac{l}{2f}) & 1 - \frac{l^2}{2f^2} \end{pmatrix}$$



From QF to QD

$$M = \begin{pmatrix} 1 - \frac{l_d}{2f} & l_d \\ -\frac{l_d}{4f^2} & 1 + \frac{l_d}{2f} \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}} cos\mu/2 & (\sqrt{\check{\beta}\hat{\beta}} sin\mu/2) \\ -\frac{1}{\sqrt{\hat{\beta}\check{\beta}}} sin\mu/2) & \sqrt{\frac{\hat{\beta}}{\check{\beta}}} cos\mu/2 \end{pmatrix}$$

$$\frac{S'}{C} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + \frac{l_d}{2f}}{1 - \frac{l_d}{2f}} = \frac{1 + sin\mu/2}{1 - sin\mu/2}$$

$$\frac{S}{C'} = \hat{\beta}\check{\beta} = 4f^2 = \frac{l_d^2}{sin^2\mu/2}$$

$$\hat{B} = \frac{(1 + sin\frac{\mu}{2})L_{Cell}}{sin\mu} \qquad \check{B} = \frac{(1 - sin\frac{\mu}{2})L_{Cell}}{sin\mu}$$

Dispersion

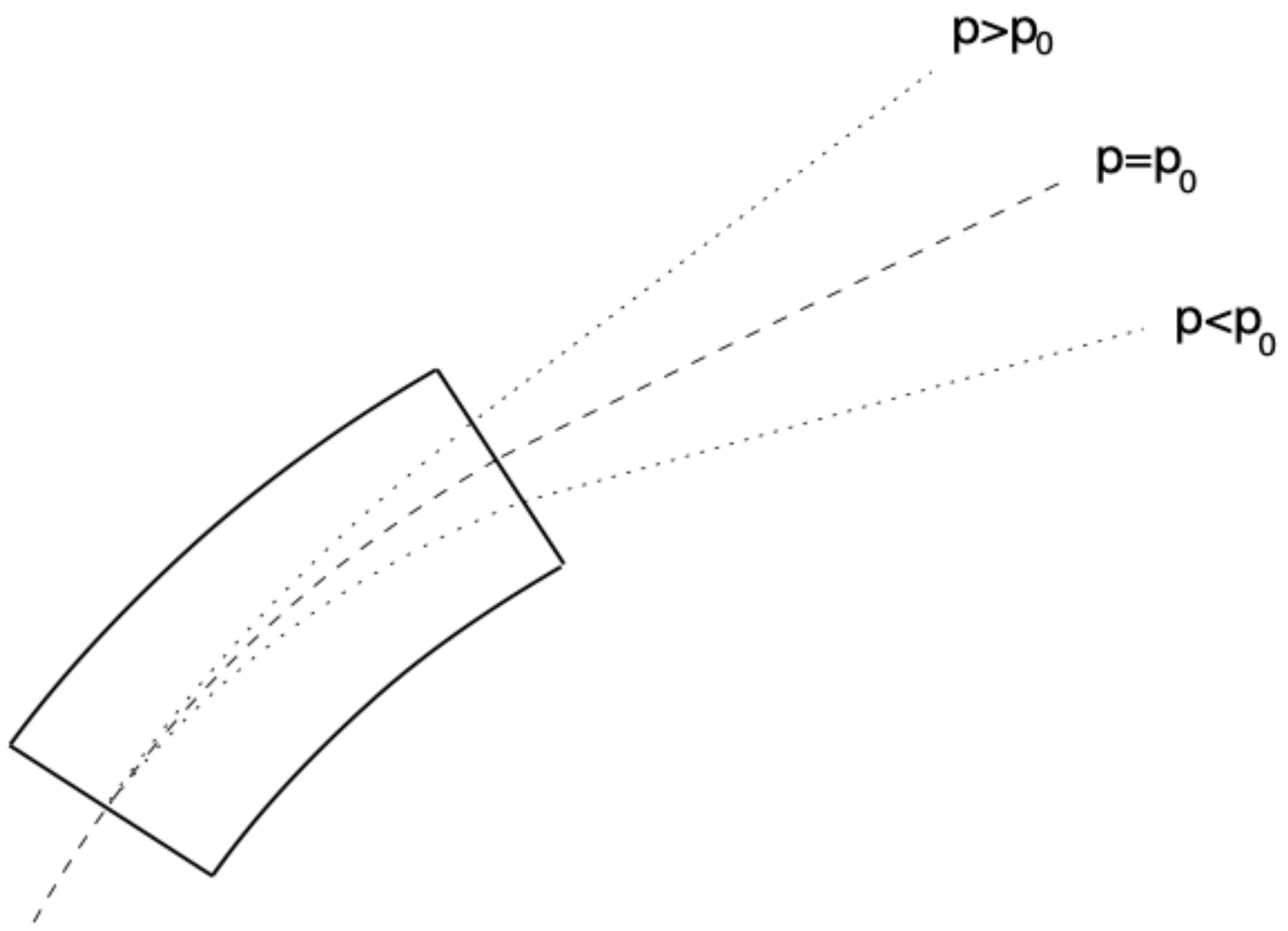
Off-momentum particles and dipoles

So far we have studied monochromatic beams of particles, but this is slightly unrealistic:

We usually have some (small?) momentum spread among all particles:

Consider three particles with momentum *p*, respectively: less than, greater than, and equal to (nominal) p_0, traveling through a dipole. Remembering $\Delta p = p - p_0$:

$$B\rho = \frac{p}{q}$$



The system introduces a correlation of momentum with transverse position. This correlation is known as **dispersion** (an intrinsic property of the dipole magnets)

Inhomogeneous Hill's equation

Derivation of the equations of motion for off-momentum particles:

(From Lecture 1 - Part 3)

$$x'' - \frac{1}{\rho}(1 - \frac{x}{\rho}) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$mv \leftarrow p = p_0 + \Delta p$

Repeat the calculation taking into account a small momentum variation.

$$\Delta p << p_0 \rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{eB_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{exg}{p_0}}_{k * x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0 \quad (x, \Delta p \rightarrow small)}$$

$$x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

Inhomogeneous Hill's equation

The equations of motion for off-momentum particles:

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The solution is a sum of the homogeneous equation (on-momentum) and inhomogeneous (off-momentum).

$$x(s) = x_\beta(s) + x_D(s)$$

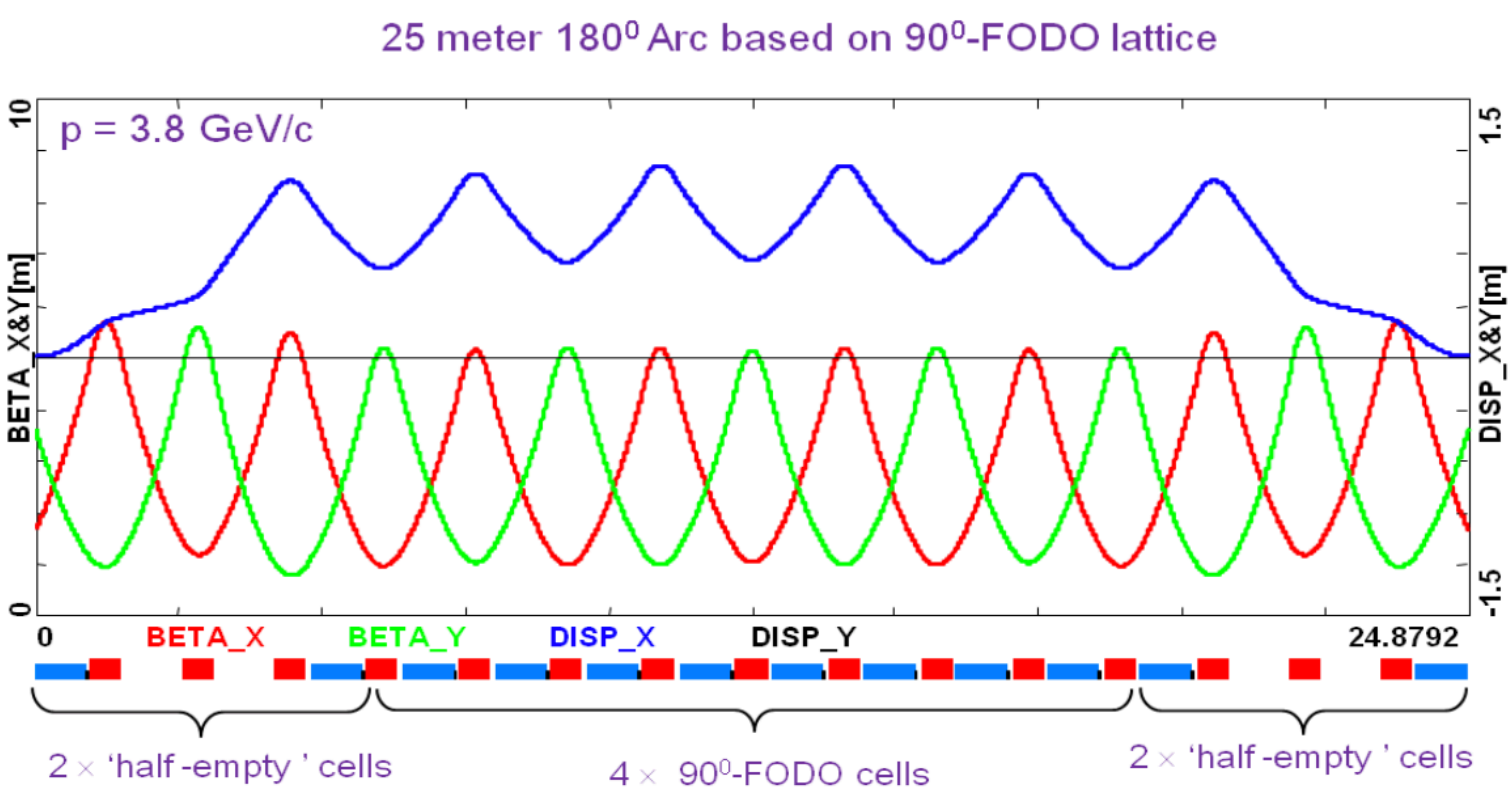
$$x''_\beta + K(s)x_\beta = 0$$

$$x''_D + K(s)x_D = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

We may define the dispersion function:

$$D(s) = \frac{x_D(s)}{\Delta p/p_0}$$

Dispersion function in a FODO lattice



Aperture radius: r = 15 cm

12 × Dipoles:	field: 3.9 Tesla	length: 85 cm
15 × Quads:	gradient : 25 Tesla/m (3.8 Tesla at the pole)	length: 50 cm

Dispersion and beam size

How does dispersion change the beam size?

Quadratic sum

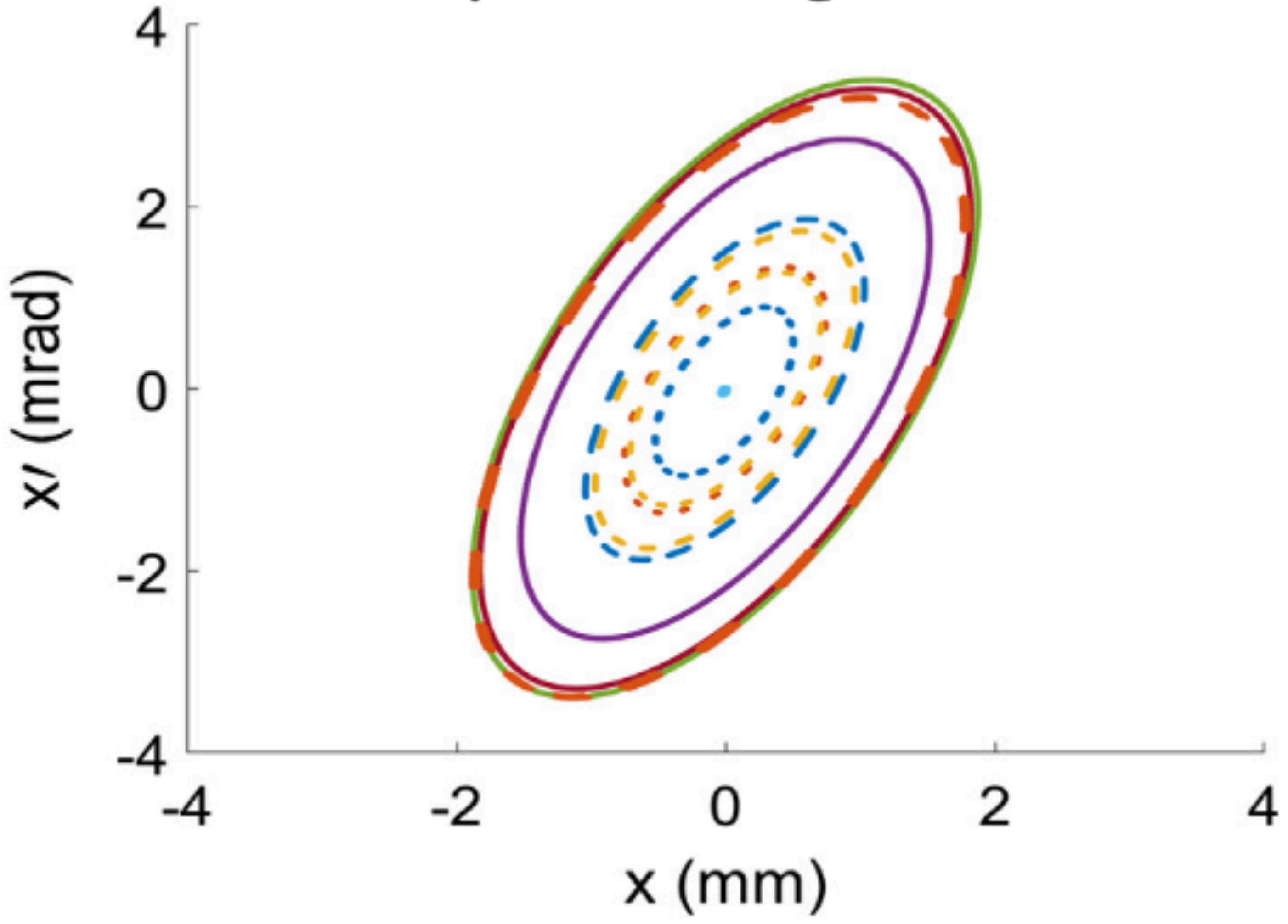
$$\sigma_x = \sqrt{\beta_x \epsilon_x + D^2 \left(\frac{\sigma_p}{p_0}\right)^2}$$

Linear sum

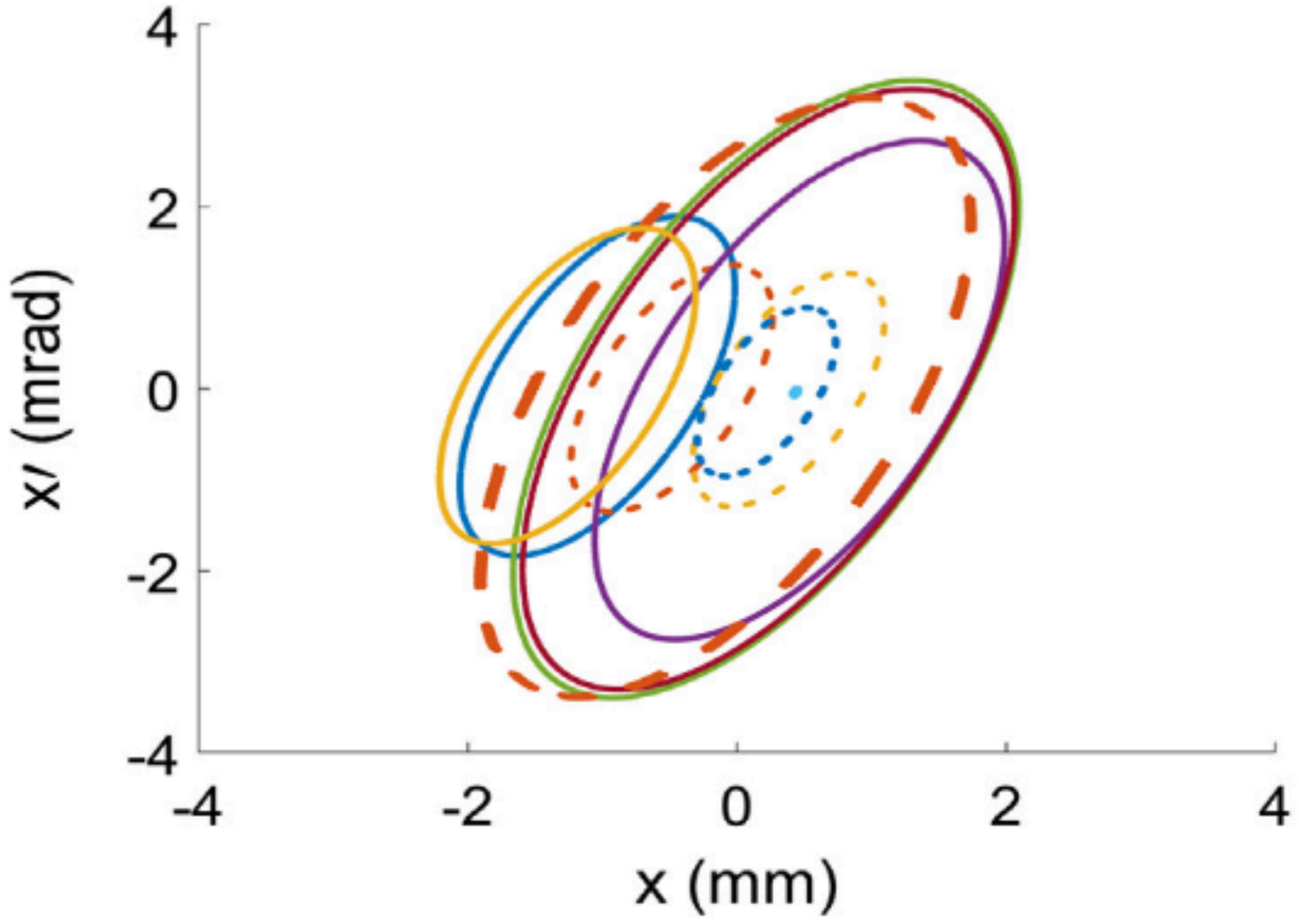
$$\sigma_x = \sqrt{\beta_x \epsilon_x} + D\frac{\sigma_p}{p_0}$$

(more conservative for aperture studies)

Phase space ellipses for **on-momentum** particles in a dispersive region



Phase space ellipses for **off-momentum** particles in a dispersive region

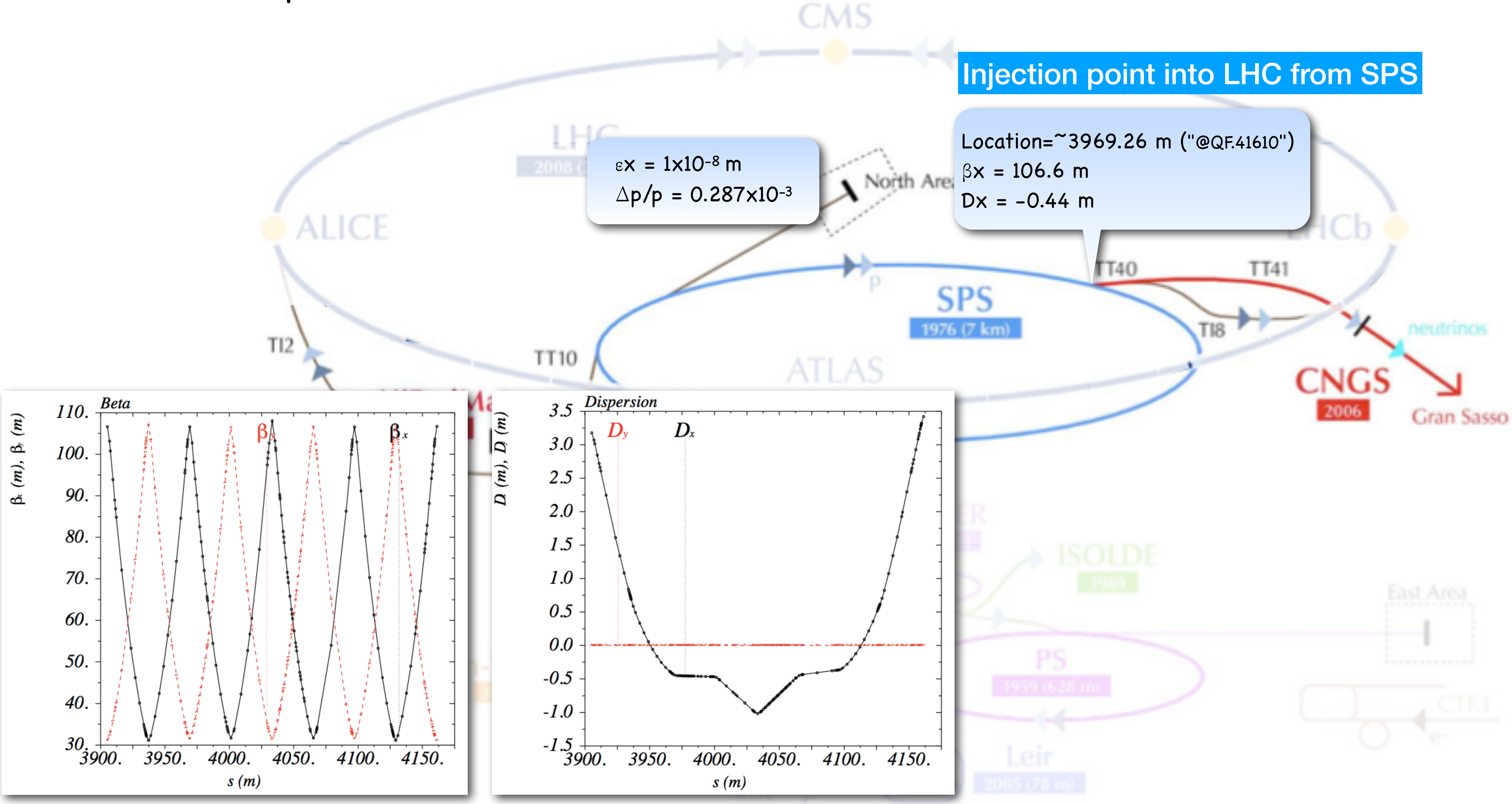


Relative *rms* momentum spread.

$$\frac{\sigma_p}{p_0} = \sqrt{\left\langle \left(\frac{\Delta p}{p_0}\right)^2 \right\rangle}$$

Break-out time!

- Calculate the proton beam size at injection into LHC.



Chromaticity

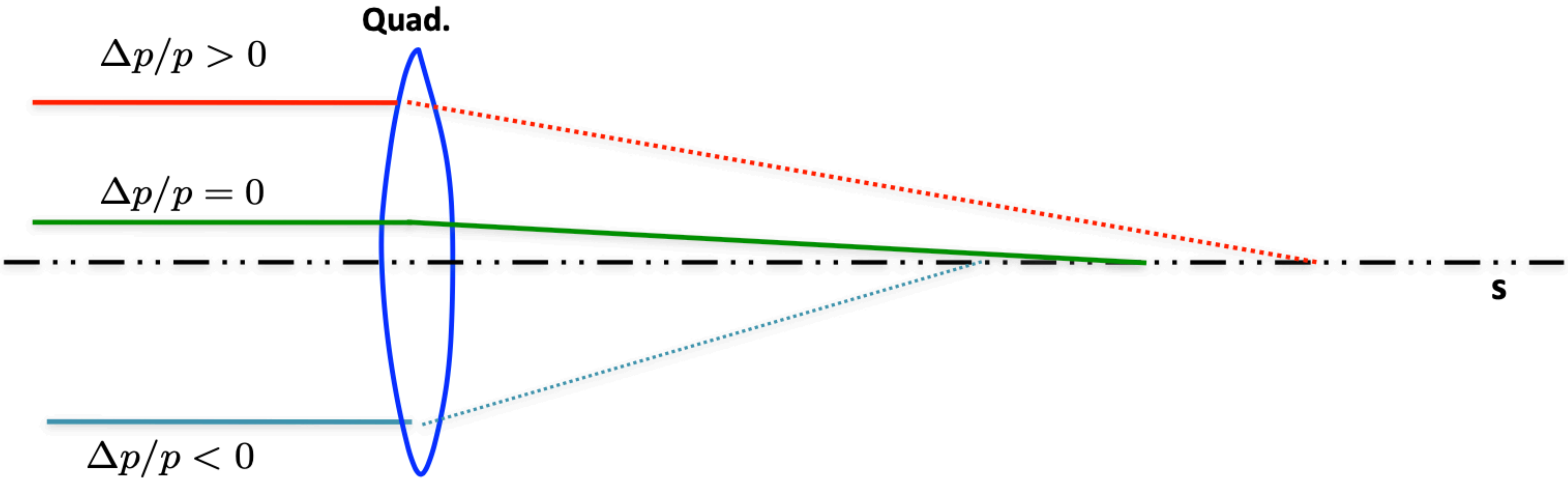
Chromaticity

- Tune: frequency of the betatron oscillations
- Chromaticity: its dependence on particle momentum
- Therefore in presence of beam momentum spread chromaticity can cause unwanted tune spread
- Unlike beam trajectory, current, radius, orbit, tune and chromaticity are the first non-trivial parameters that can not be measured directly.

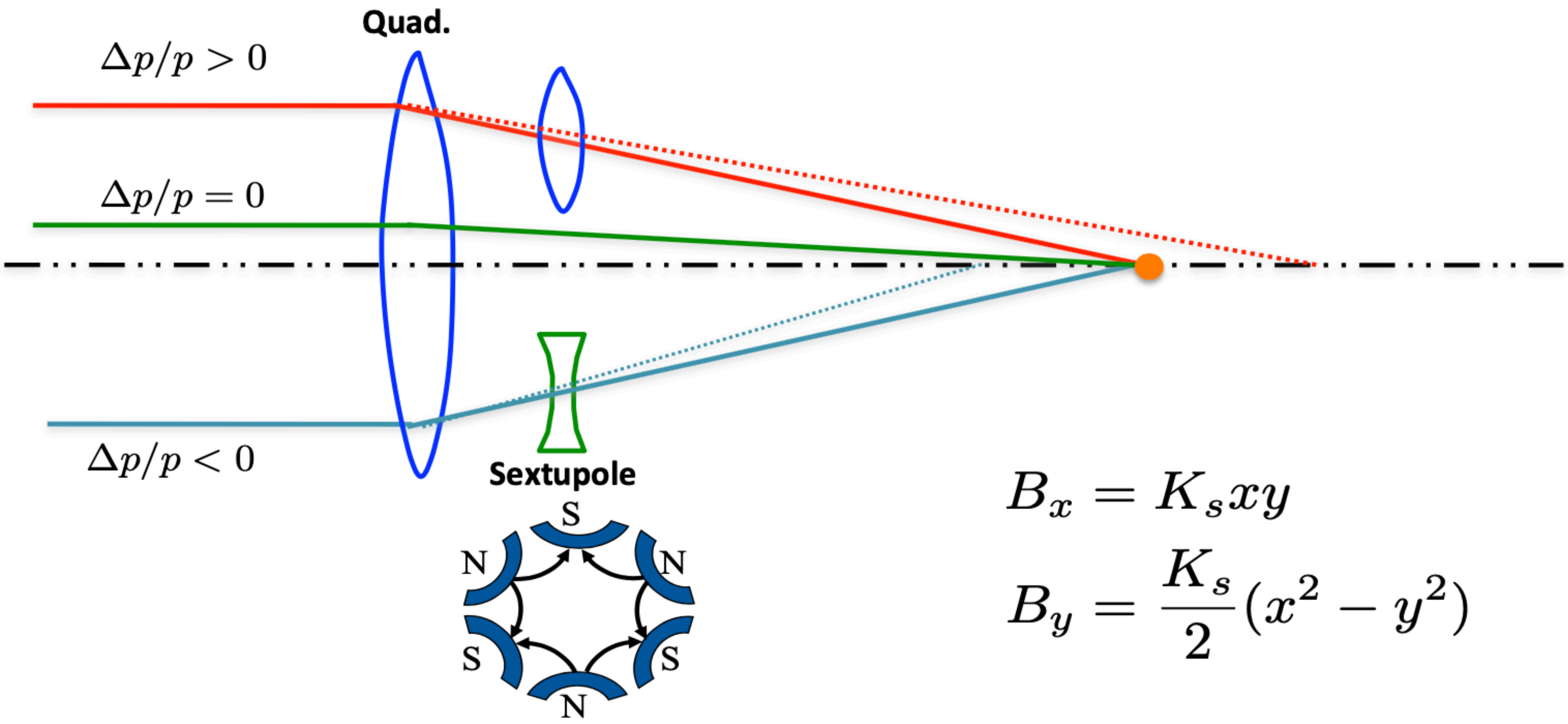
Chromaticity

How do off-momentum particles behave in quadrupoles?

Off-momentum particle gets different focusing



Chromaticity correction

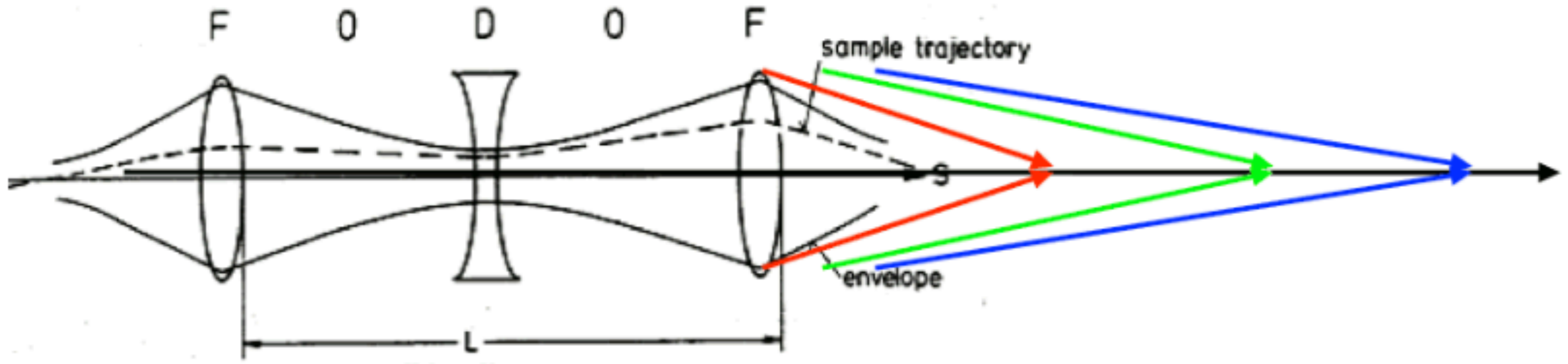


$$B_x = K_s xy$$

$$B_y = \frac{K_s}{2}(x^2 - y^2)$$

Chromaticity

Is an error (optical aberration) that happens in quadrupoles when $\Delta P/P_0 \neq 0$:



The chromaticity ξ is the variation of tune ΔQ with the relative momentum error:

$$\Delta Q = \xi \frac{\Delta P}{P_0} \quad \Rightarrow \quad \boxed{\xi = \frac{\Delta Q}{\Delta P/P_0}}$$

Remember the quadrupole strength:

$$k = \frac{g}{P/q} \qquad \text{with } P = P_0 + \Delta P = P_0\,(1+\delta)$$

then

$$k = \frac{qg}{P_0 + \Delta P} = \frac{k_0}{1+\delta} \approx \frac{q}{P_0}\left(1 - \frac{\Delta P}{P_0}\right) g = k_0 + \Delta k$$

$$\boxed{\Delta k = -\frac{\Delta P}{P_0} k_0}$$

Perturbed particle motion described by Hill's equation:

$$\begin{cases} x'' + (k(s) + \frac{1}{\rho^2(s)}) \cdot x = \frac{1}{\rho(s)}\frac{\Delta p}{p} + f_x(s,t) \\ y'' - k(s) \cdot y = 0 + f_y(s,t) \end{cases}$$

→ Perturbation terms

Notice the form:

$$F\frac{\Delta p}{p} \sim k \cdot x$$

Chromaticity correction is done by means of a sextupole magnet at a location with non-vanishing dispersion where particles are sorted according to their momentum.

Driving terms for a sextupole:

$$\begin{cases} f_x(s) = +\frac{1}{2}m(s) \cdot (x^2 + y^2) \\ f_y(s) = +m(s) \cdot xy \end{cases}$$

$$m(s) = \frac{q}{p}\frac{\partial^2 B}{\partial x^2}$$

x → unperturbed coordinate

$D(s) \cdot \frac{\Delta p}{p} + x_\beta(s)$ → perturbed coordinate

$$f_x(s) = \frac{1}{2} m(s) \left[\left(D(s) \frac{\Delta p}{p} + x_\beta \right)^2 \right]$$

$$= \frac{1}{2} m(s) \left[\left(D(s) \frac{\Delta p}{p} \right)^2 + 2D(s) \frac{\Delta p}{p} x_{\beta(s)} + x_\beta^2(s) \right]$$

$$= m(s) D(s) \frac{\Delta p}{p} x_\beta(s) + \ldots$$ → first order perturbation $\quad m(s)D(s) \sim k(s)$

Total chromaticity in a lattice including the contribution from sextupoles.

$$Q' = \mp \frac{1}{4\pi} \oint \beta(s) \; [k(s) \; + \; m(s)D(s)] \; ds$$

Resonances

In reality, some lattice have significant higher order terms

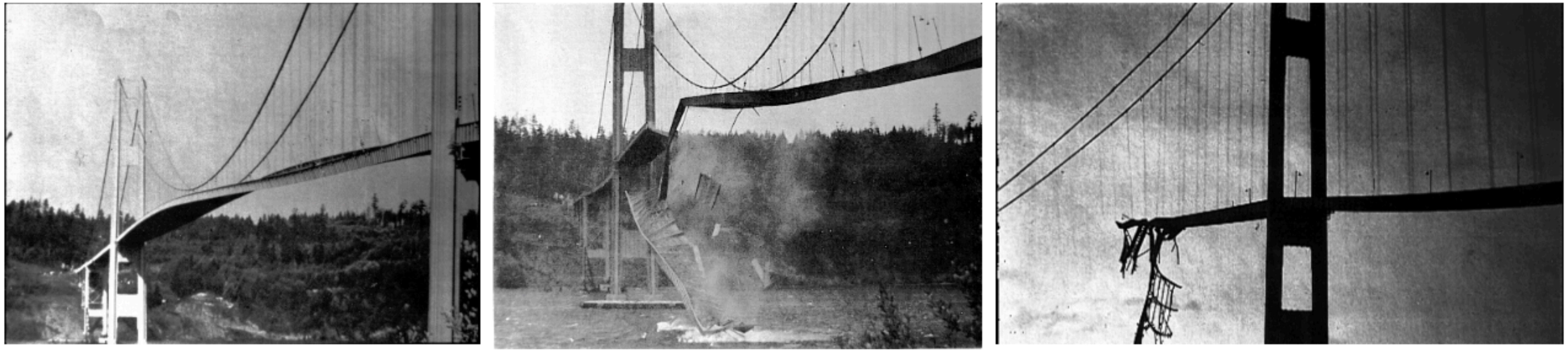
$$x'' + K(s)x = O(x^2) + \ldots$$

and magnetic imperfections, e.g. dipole errors

$$x'' + K(s)x = \delta(s - s_0)\theta_{\mathrm{error}}$$

which can drive **resonances**

Tacoma Narrow bridge 1940



(Excitation by strong wind on the eigenfrequencies)

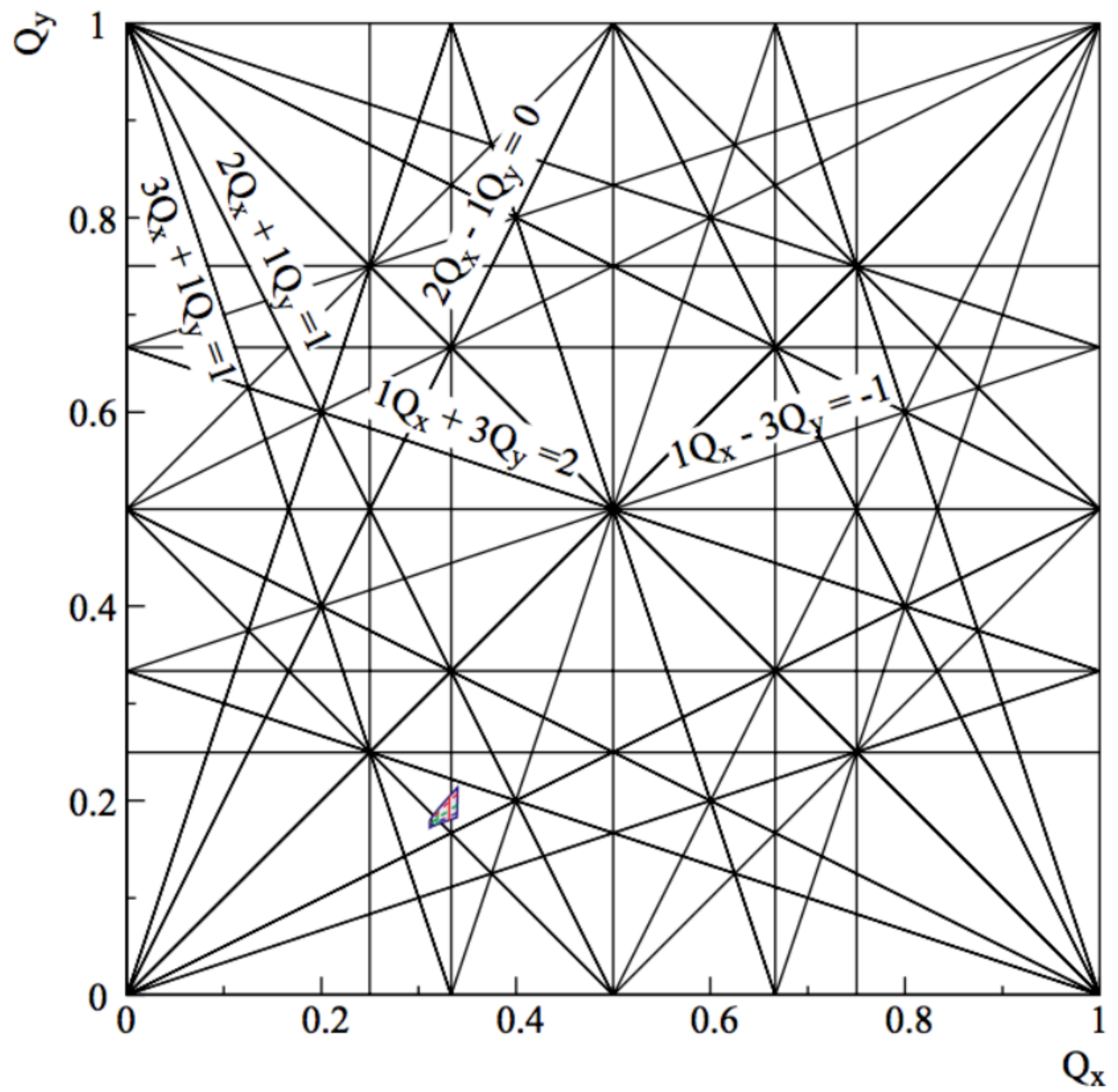
- A horizontal and a vertical tune value are defined for accelerator rings: Qx and Qy.
- For high order magnets, the field strength in one plane is related to the field strength in the other transverse plane. Hence, the betatron oscillations are coupled in these two planes.

$$pQ_x + qQ_y = m$$

m, p, q: integer numbers

- The degree of the resonance is given as |p| + |q|.

- Qx and Qy pair defined for an accelerator is called the working point of that machine.
- As the strength of a resonance significantly decreases by its degree, generally, only resonances up to 5th degree are considered.



$$pQ_x + qQ_y = m$$

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Tune combinations that cause unwanted resonances can be shown in a tune diagram. The area occupied in the tune space by a beam is called the **"tune footprint"** of that beam.

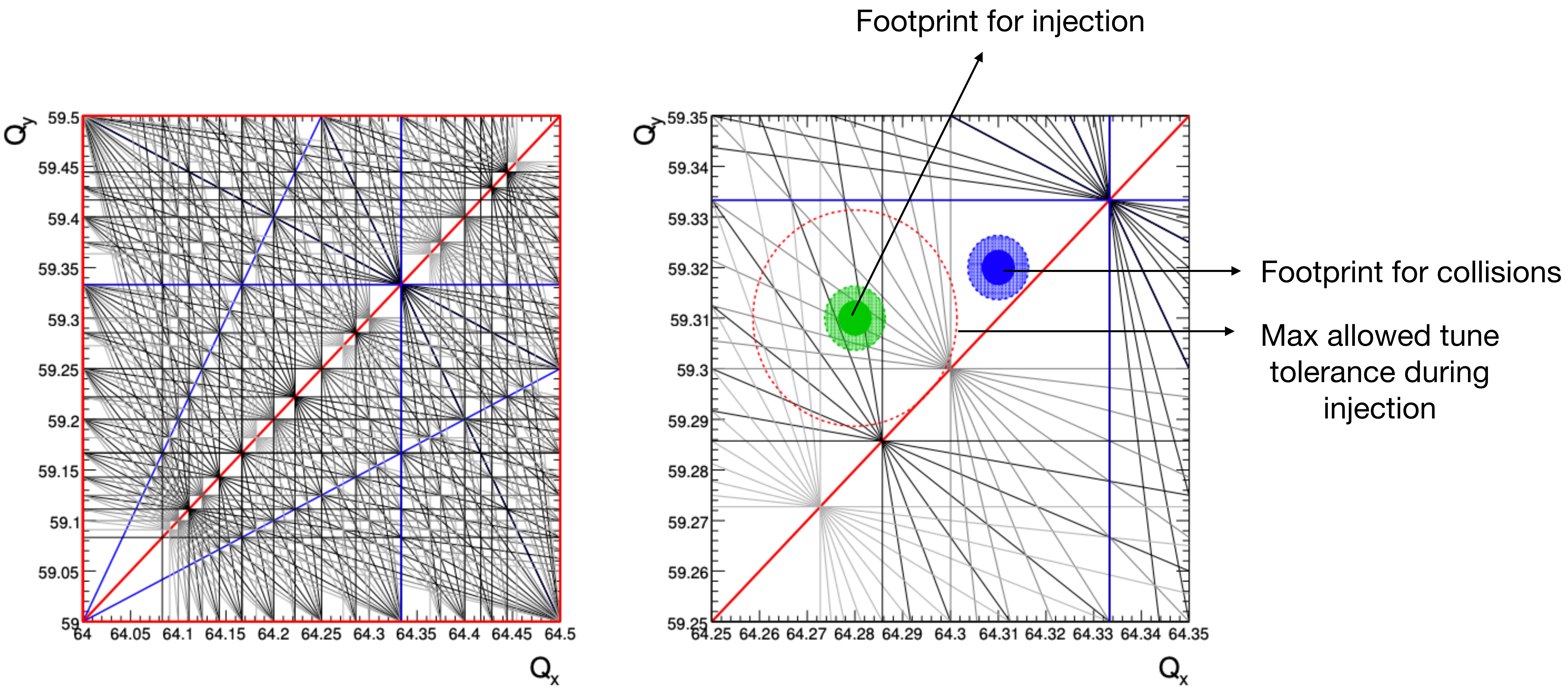
Performance of an accelerator and the particle background in a collider are related to the tune footprint of that accelerator.

Figure 7: Illustration of a tune diagram for resonances up to 4th order. The typical tune area, occupied by a colliding beam at LEP1 is also shown as shaded area ($Q_x \approx 0.31 - 0.34$ and $Q_y \approx 0.17 - 0.214$).

CERN-SL-2000-037-DI

https://jwenning.web.cern.ch/jwenning/documents/lepmain_sl.pdf

LHC working points in tune diagram



(a) Full tune diagram

(b) Zoom around LHC Q working points

https://cds.cern.ch/record/1213281/files/p317.pdf