Science & Technology Facilities Council

Special Relativity

Chris Prior

ASTeC RAL

and

Trinity College Oxford

Overview

- The principle of special relativity
- Lorentz transformation and consequences
- Space-time
- 4-vectors: position, velocity, momentum, invariants, covariance.
- Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- An accelerator problem in relativity
- Photons and wave 4-vector
- Relativistic particle dynamics
- Lagrangian and Hamiltonian Formulation
- Radiation from an Accelerating Charge
- Motion faster than speed of light

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D.F. Lawden: An Introduction to Tensor Calculus and Relativity (Dover, 2003)
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)
- C.Misner, K.Thorne and J.Wheeler: Relativity (Freeman, 1973)
- C.R. Prior: Special Relativity, CERN Accelerator School (Zeegse)

Science & Technology Facilities Council

Historical Background

- Groundwork of Special Relativity laid by **Lorentz** in studies of electrodynamics, with crucial concepts contributed by **Einstein** to place the theory on a consistent footing.
- **Maxwell**'s equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3 \times 10^8$ **m/s** in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
- To avoid setting e/m apart from classical mechanics, assume
  - light has speed *c* only in frames where source is at rest
  - the ether has a small interaction with matter and is carried along with astronomical objects such as the Earth

Science & Technology Facilities Council

Contradicted by Experiment

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

Science & Technology Facilities Council

The Principle of Special Relativity

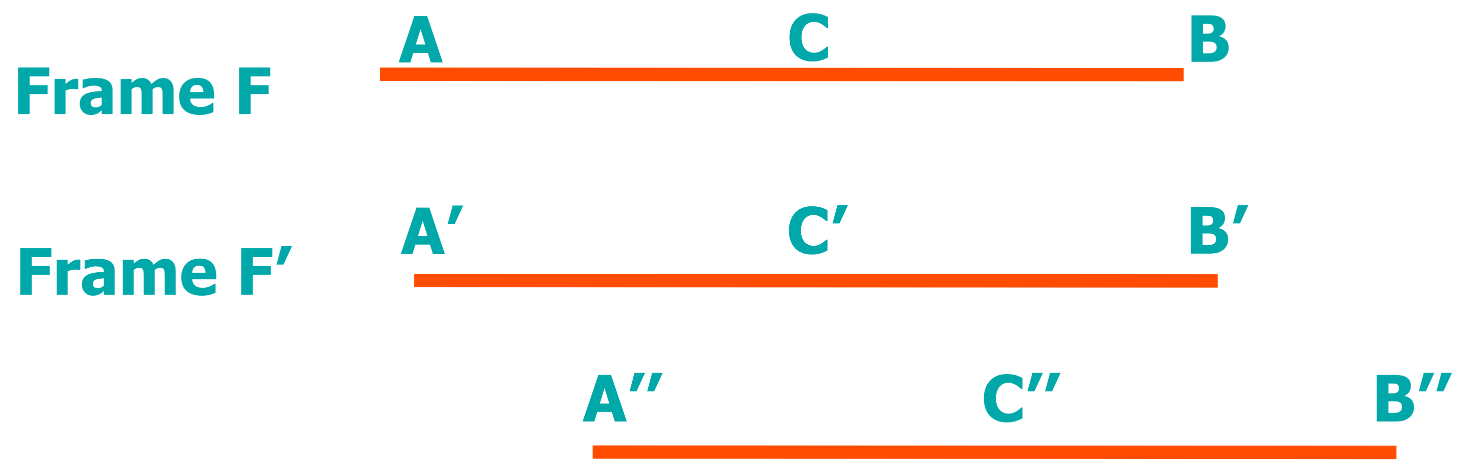
- A frame in which particles under no forces move with constant velocity is *inertial*.
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another: *related frames*.
- Assume:
  - Behaviour of apparatus transferred from F to F' is independent of mode of transfer
  - Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.

- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

Science & Technology Facilities Council

Simultaneity

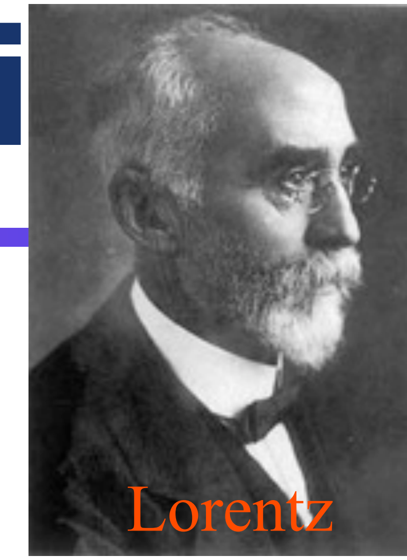
- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is **not absolute** but frame dependent.

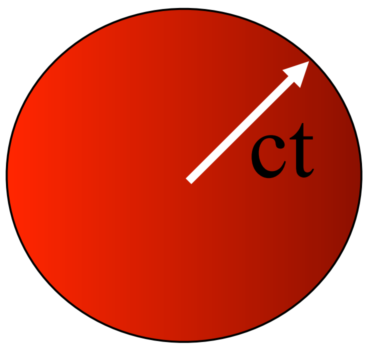
Science & Technology Facilities Council

The Lorentz Transformati



- Must be linear to agree with standard Galilean transformation in low velocity limit

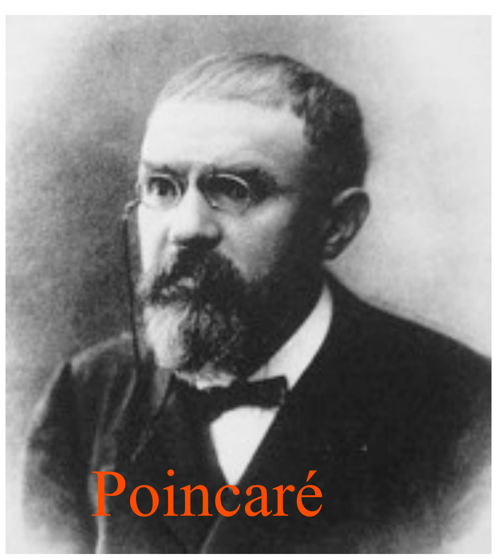
- Preserves wave fronts of pulses of light,



$$\text{i.e.} \quad P = x^2 + y^2 + z^2 - c^2t^2 = 0$$

$$\text{whenever} \quad Q = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

- Solution is the **Lorentz transformation** from frame $F(t,x,y,z)$ to frame $F'(t',x',y',z')$ moving with velocity $\mathbf{v}$ along the x-axis:



$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Science & Technology Facilities Council

Outline of Derivation

Set
$$\begin{aligned} t' &= \alpha t + \beta x \\ x' &= \gamma x + \delta t \\ y' &= \epsilon y \\ z' &= \zeta z \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are constants

$$kP = Q$$

$$\iff k(c^2t^2 - x^2 - y^2 - z^2) = c^2{t'}^2 - {x'}^2 - {y'}^2 - {z'}^2$$

$$= c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \epsilon^2 y^2 - \zeta^2 z^2$$

Equate coefficients of x, y, z, t

Impose isotropy of space $\implies k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

Apply some common sense (e.g. $\epsilon, \zeta, k = +1$ and not -1)

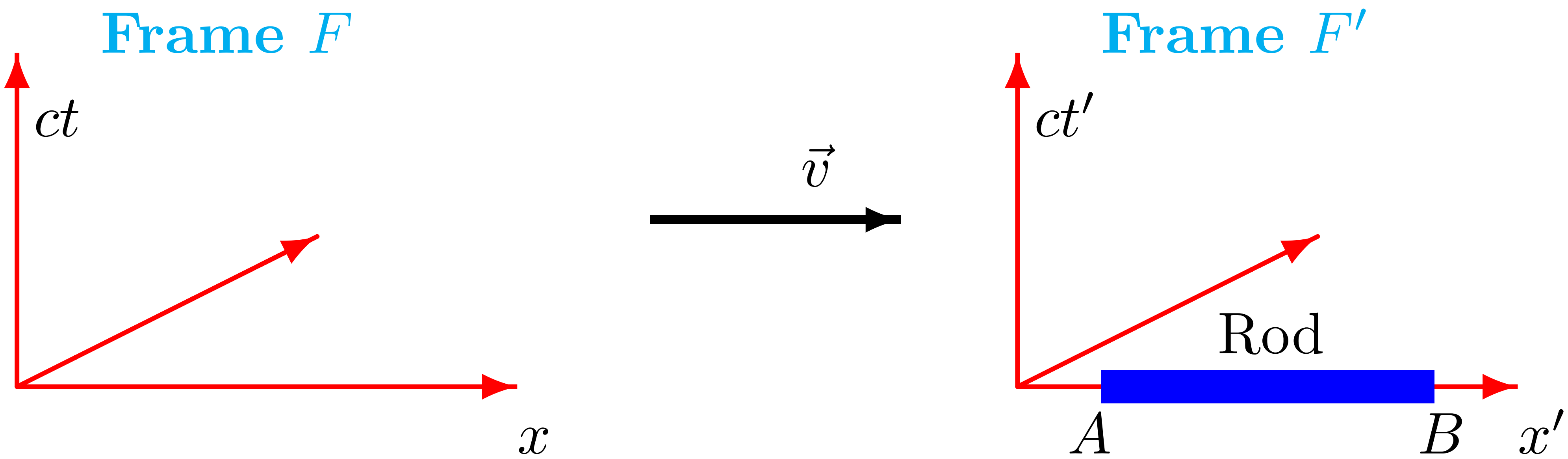
General 3D form of Lorentz Transformation:

$$\vec{x}' = \vec{x} - \vec{v}\left(\gamma t - (\gamma - 1)\frac{\vec{v}\cdot\vec{x}}{v^2}\right)$$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right)$$

Science & Technology Facilities Council

Consequences: length contraction



A rod AB of length L', fixed in frame F' at x'_A, x'_B. What is its length measured in F?

Must measure position of ends in F at same time, so events in F are (ct, x_A) and $ct, x_B)$.

By Lorentz:

$$\left.\begin{array}{rcl} x'_A &=& \gamma(x_A - vt) \\ x'_B &=& \gamma(x_B - vt) \end{array}\right\} \implies \begin{array}{rcl} L' &=& x'_B - x'_A \\ &=& \gamma(x_B - x_A) \\ &=& \gamma L > L \end{array}$$

Moving objects appear contracted in the direction of the motion

Consequences: time dilation

- Clock in frame *F* at point with coordinates *(x,y,z)* at different times t_A and t_B
- In frame *F′* moving with speed *v*, Lorentz transformation gives

$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right) \qquad t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

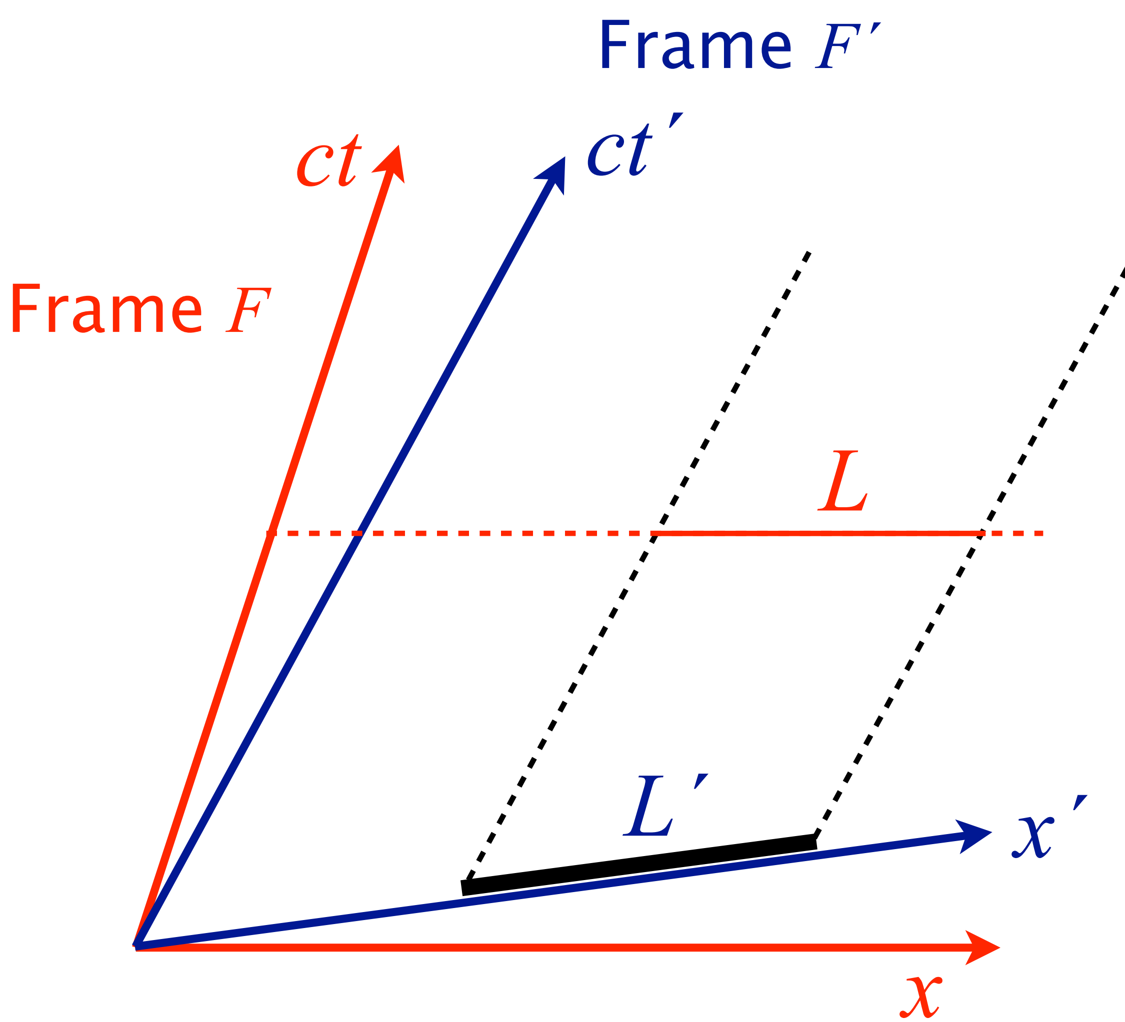
- So

$$\Delta t' = t'_B - t'_A = \gamma\Big(t_B - t_A\Big) = \gamma \Delta t > \Delta t$$

Moving clocks appear to run slow

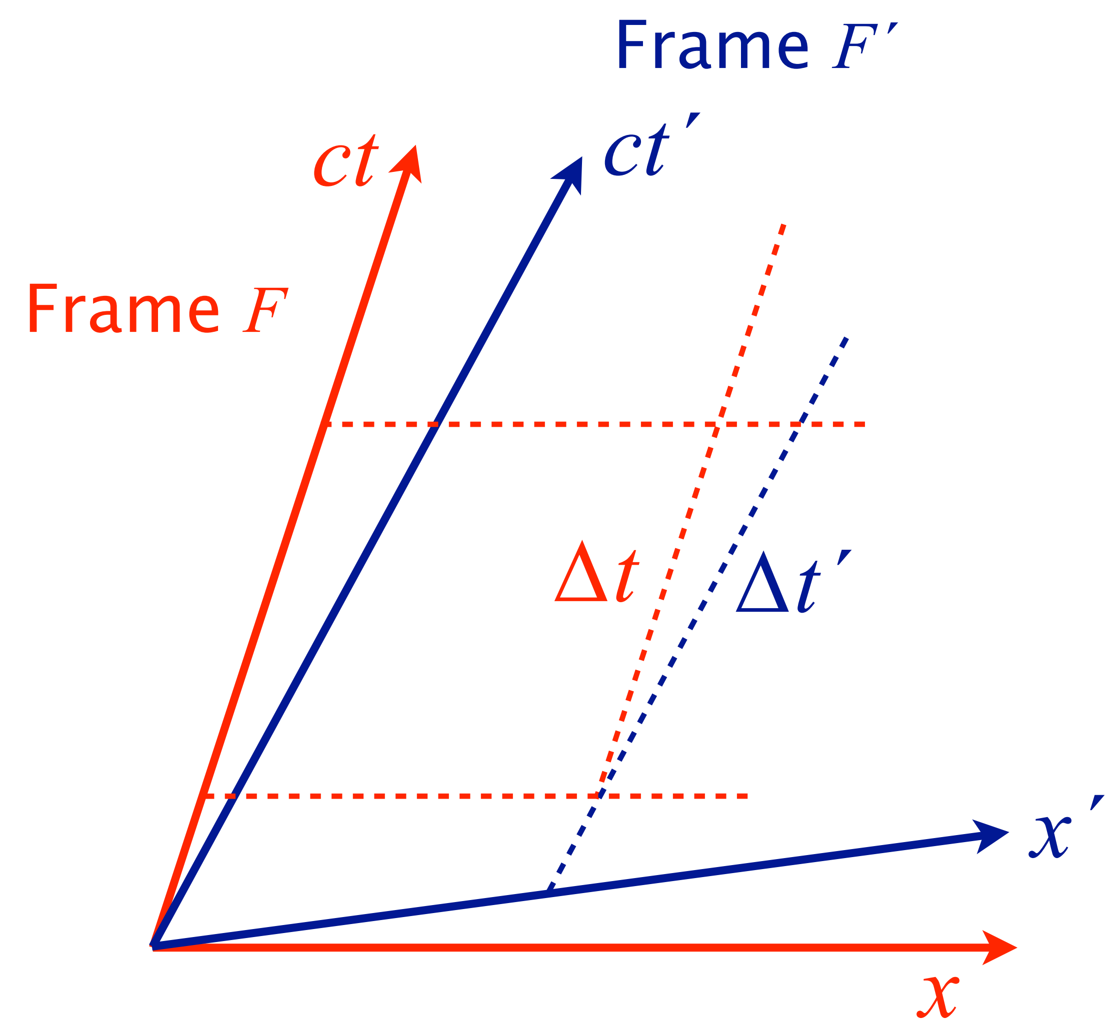
Science & Technology Facilities Council

Schematic Representation



Length contraction $L < L'$

Rod at rest in F'. Measurements in F at a fixed time t, along a line parallel to x-axis

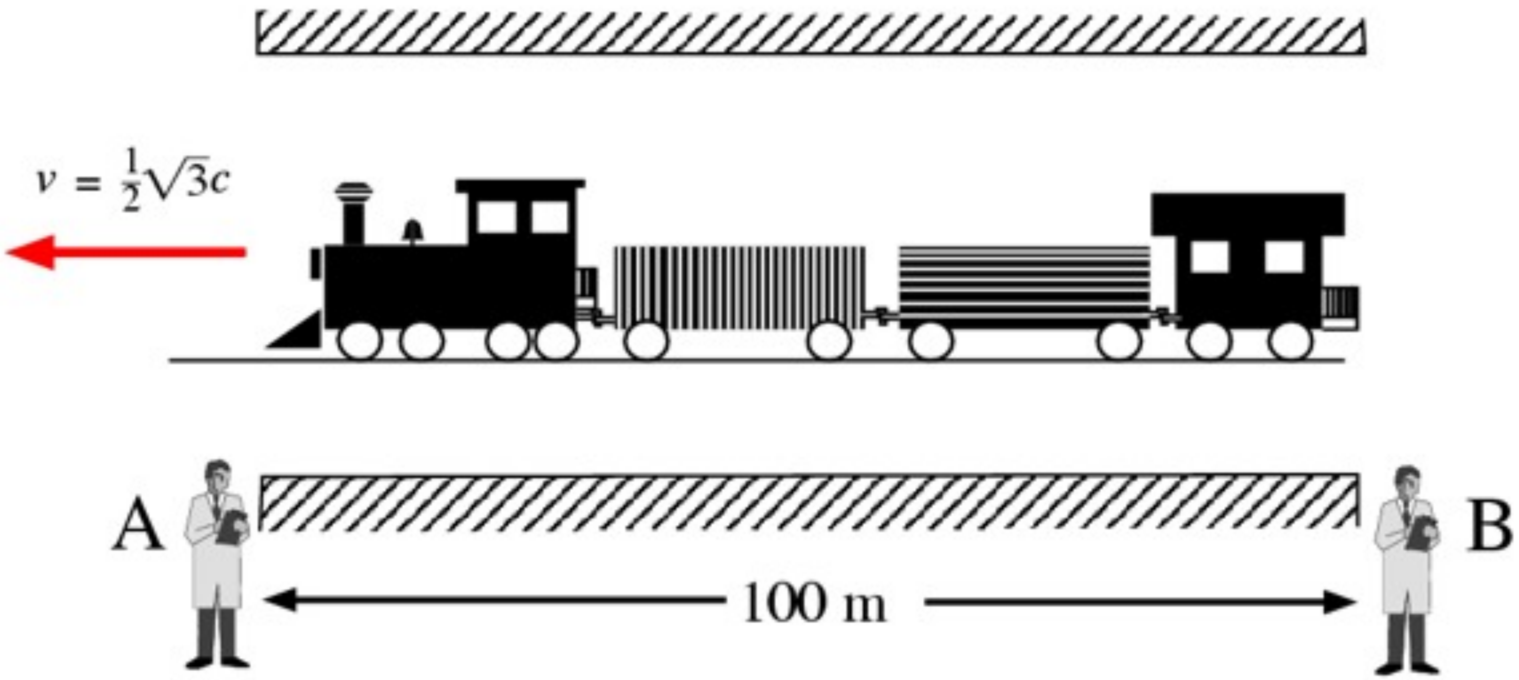


Time dilation $\Delta t < \Delta t'$

Clock at rest in F. Time difference in F' from line parallel to t'-axis

Science & Technology Facilities Council

Example: High Speed Train



All clocks synchronised.

A's clock and driver's clock read 0 as front of train emerges from tunnel.

- Observers A and B at exit and entrance of tunnel say the train is moving, has contracted and has length

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 100 \times \left(1 - \frac{3}{4}\right)^{1/2} = 50\text{m}$$

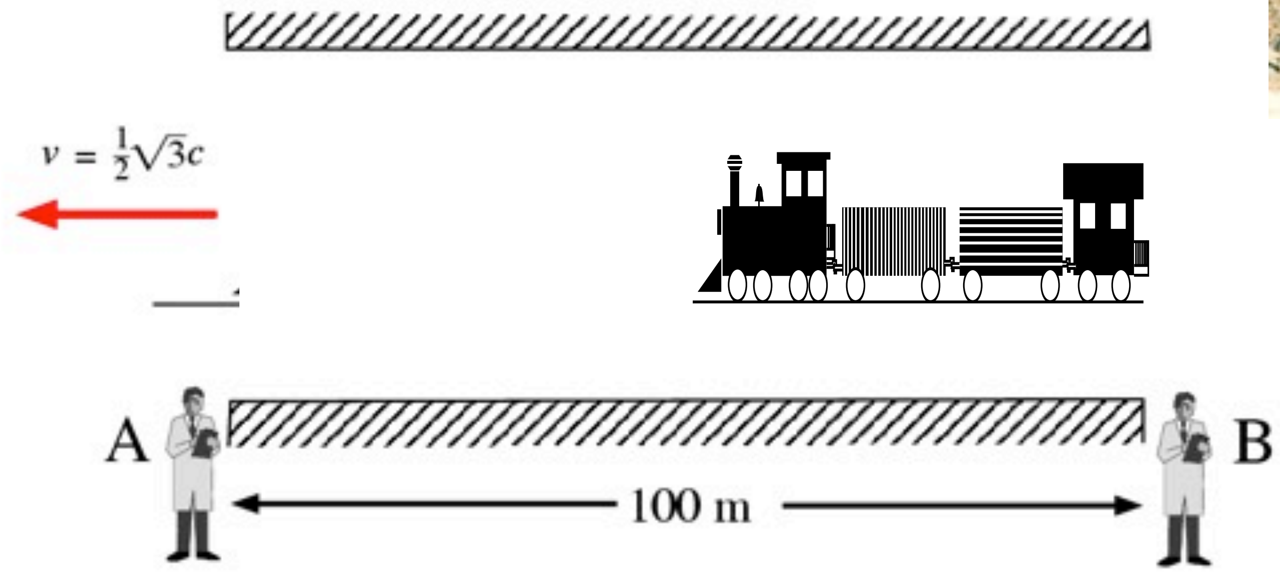
- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

Science & Technology Facilities Council

Question 1

A’s clock and the driver's clock read zero as the driver exits tunnel.

What does B’s clock read when the guard goes in?



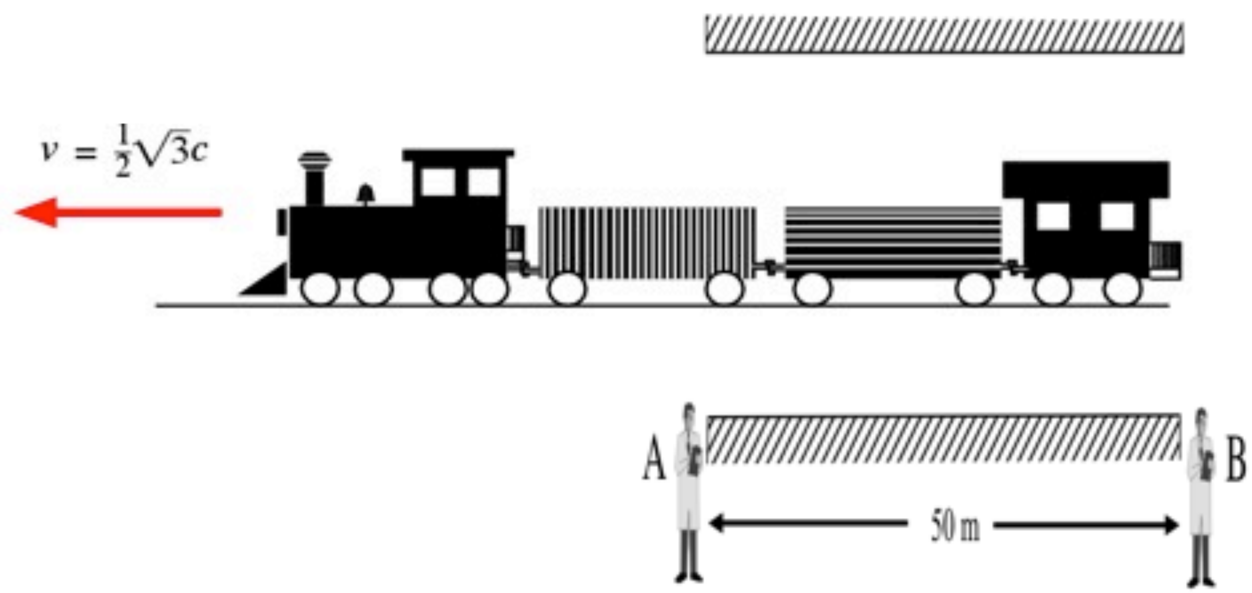
Moving train length 50m, so driver has still 50m to travel before he exits and his clock reads 0. A's clock and B's clock are synchronised. Hence the reading on B's clock is

$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200\,\text{ns}$$

Science & Technology Facilities Council

Question 2

What does the guard's clock read as he goes in?



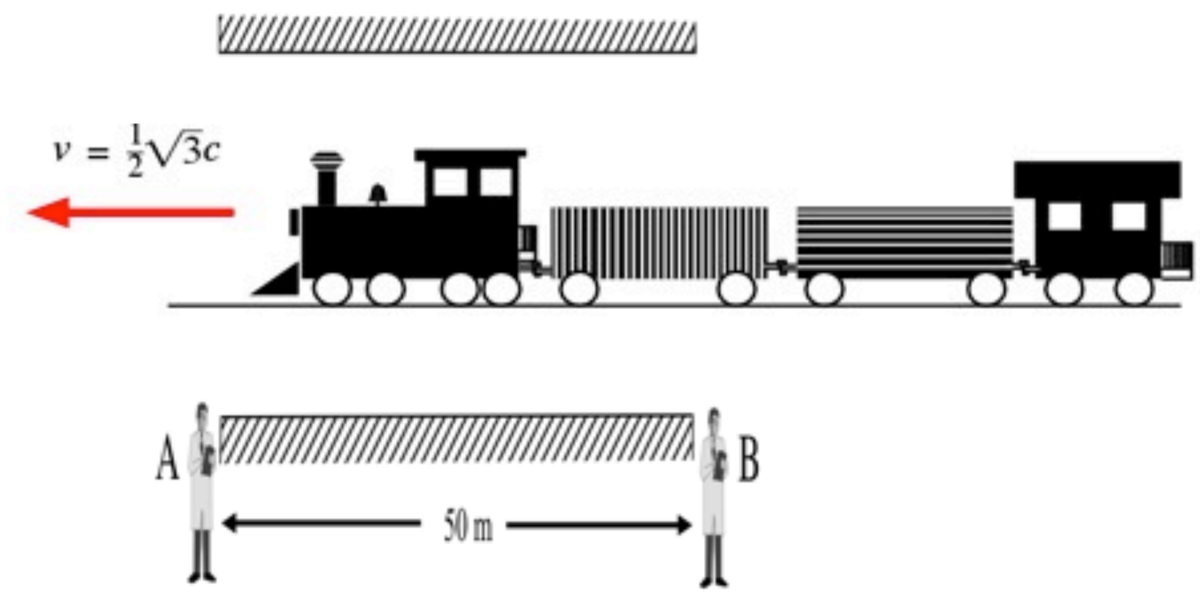
To the guard, tunnel is only 50m long, so driver is 50m past the exit as guard goes in. Hence clock reading is

$$+\frac{50}{v} = +\frac{100}{\sqrt{3}c} \approx +200\,\text{ns}$$

Science & Technology Facilities Council

Question 3

Where is the guard when his clock reads 0?



Guard's clock reads 0 when driver's clock reads 0, which is as driver exits the tunnel. To guard and driver, tunnel is 50m, so guard is 50m from the entrance in the train's frame, or 100m in tunnel frame.

So the guard is 100m from the entrance to the tunnel when his clock reads 0.

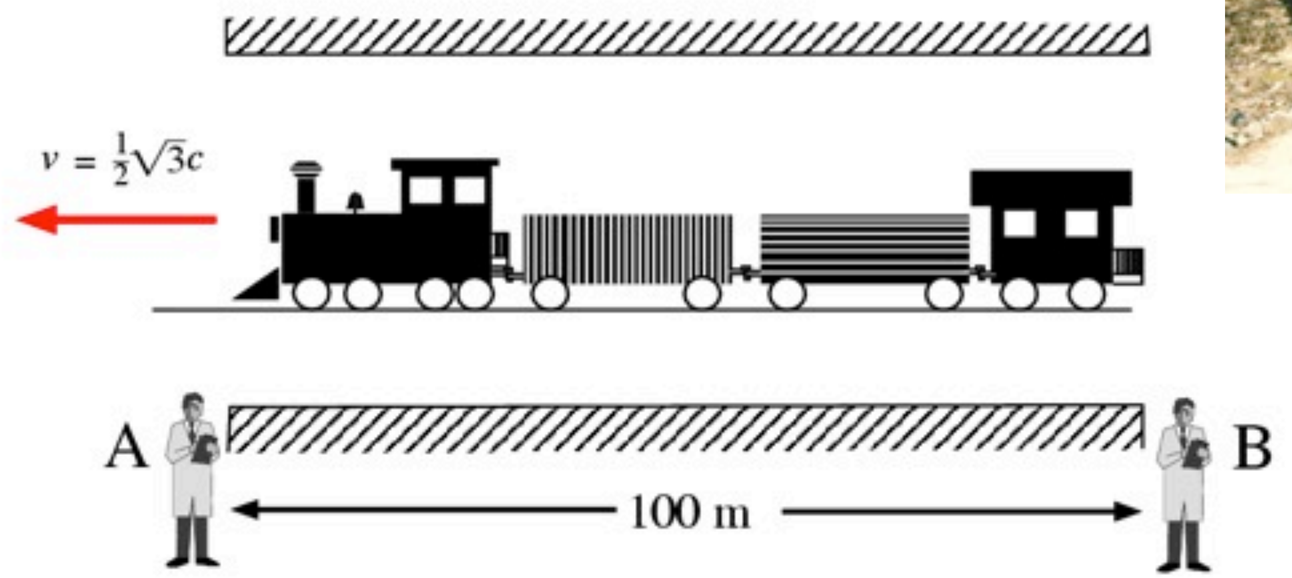
Science & Technology Facilities Council

Repeat within framework of Lorentz transformation

Question 1

A's clock and the driver's clock read zero as the driver exits tunnel.

What does B's clock read when the guard goes in?



F(t,x) is frame of A and B, F'(t',x') is frame of driver and guard.

We know $x_A = 0$, $x_B = 100$, $x'_D = 0$, $x'_G = 100$. We want t_B when G and B coincide.

$$x = \gamma(x' - vt') \quad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$

$$x' = \gamma(x + vt) \quad t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

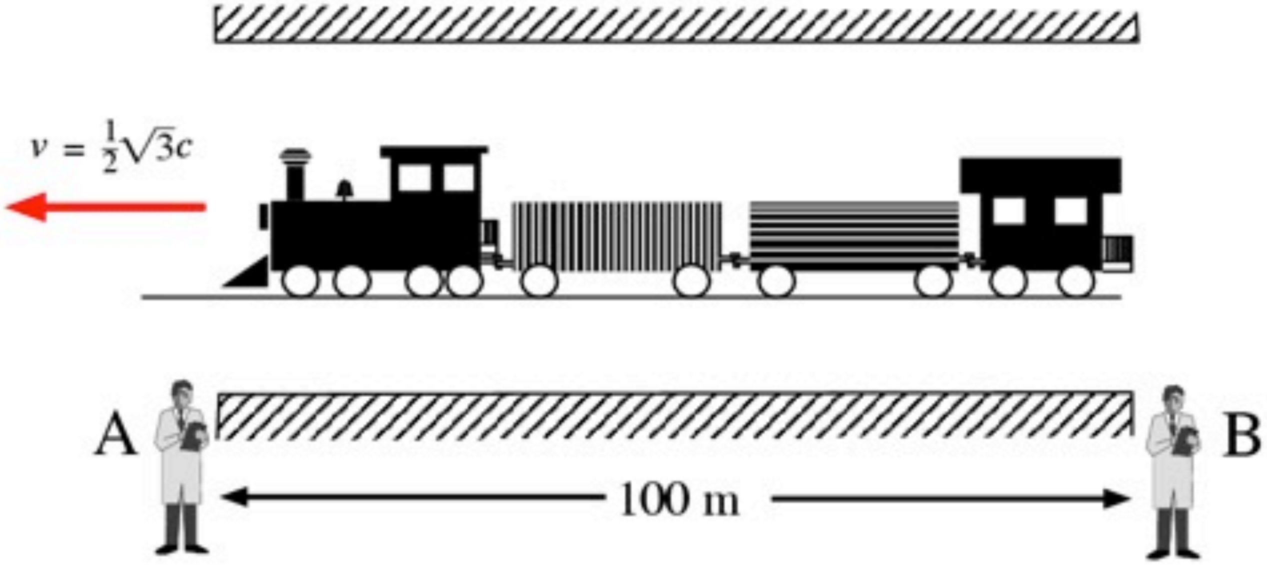
$$100 = \gamma(100 + vt_B)$$

$$\implies t_B = 100\frac{1-\gamma}{\gamma v} = -\frac{50}{v}$$

Science & Technology Facilities Council

Question 2

What does the guard's clock read as he goes in?



F(t,x) is frame of A and B, F'(t',x') is frame of driver and guard.

We know $x_A = 0$, $x_B = 100$, $x'_D = 0$, $x'_G = 100$. We want t'_G when B and G coincide.

$$x = \gamma(x' - vt') \qquad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$

$$x' = \gamma(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

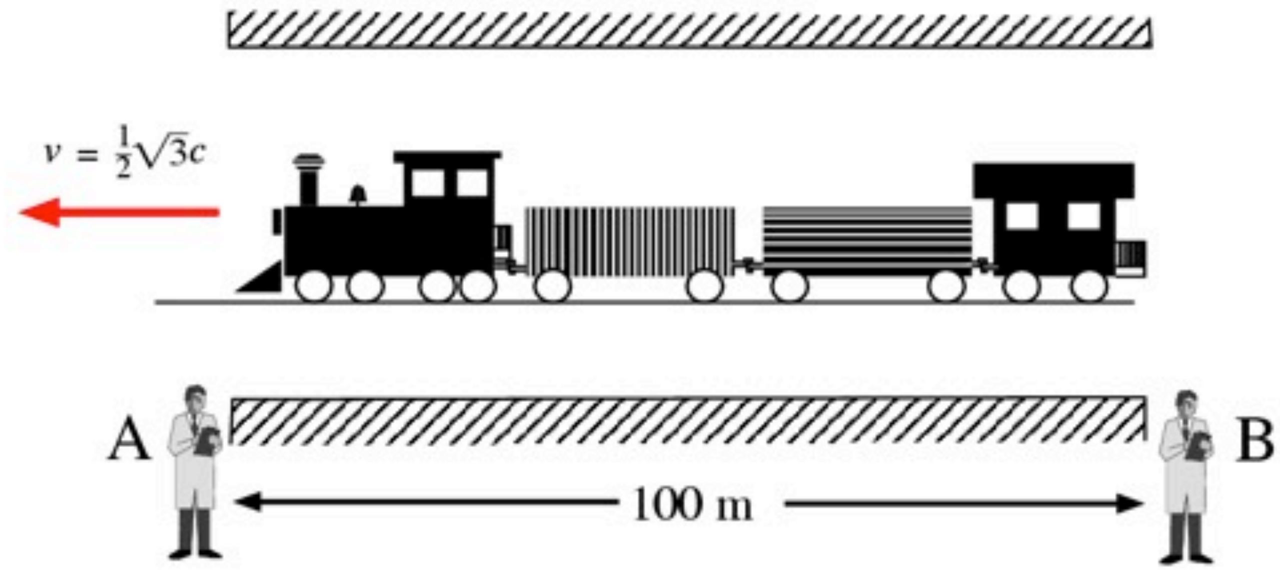
$$100 = \gamma(100 - vt'_G)$$

$$\implies t_G = 100\frac{\gamma - 1}{\gamma v} = +\frac{50}{v}$$

Science & Technology Facilities Council

Question 3

Where is the guard when his clock reads 0?



F(t,x) is frame of A and B, F'(t',x') is frame of driver and guard.

We know $x_A = 0$, $x_B = 100$, $x'_D = 0$, $x'_G = 100$.

We want x corresponding to $x'_G = 100$ given $t'_G = 0$.

$$x = \gamma(x' - vt') \qquad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$

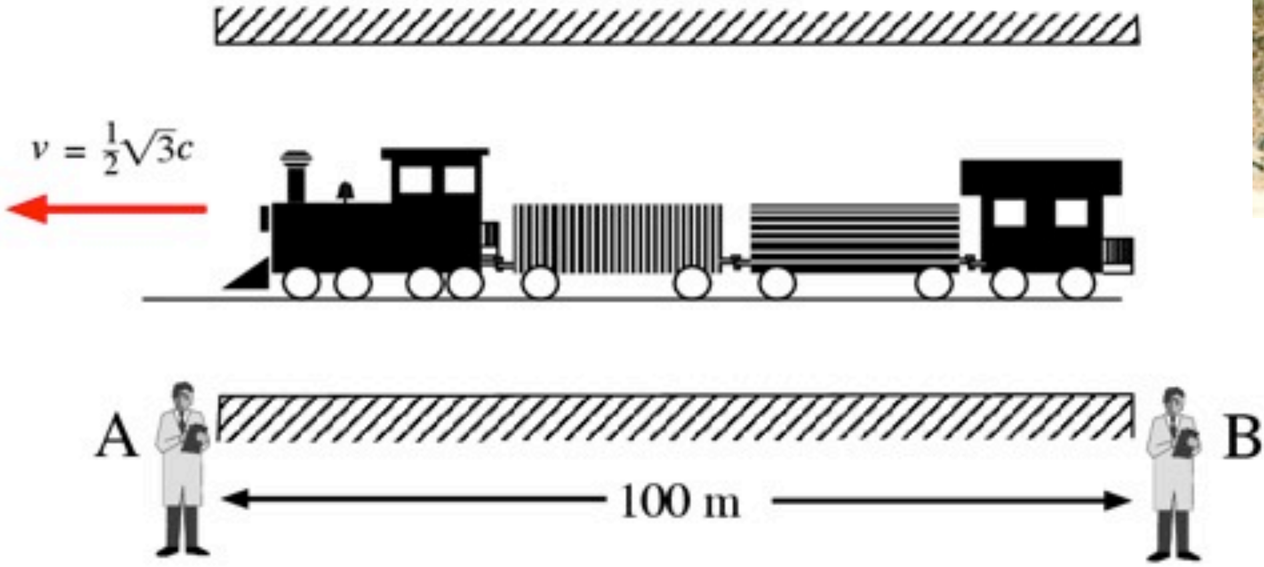
$$x = \gamma(100 - v \times 0) = 200\,\mathrm{m}$$

$$x' = \gamma(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

Or 100m from the entrance to the tunnel

Question 4

Where was the driver when his clock reads the same as the guard's when he enters the tunnel?



F(t,x) is frame of A and B, F'(t',x') is frame of driver and guard.

We know $x_A = 0$, $x_B = 100$, $x'_D = 0$, $x'_G = 100$. We want x_D when $t'_D = \dfrac{50}{v}$.

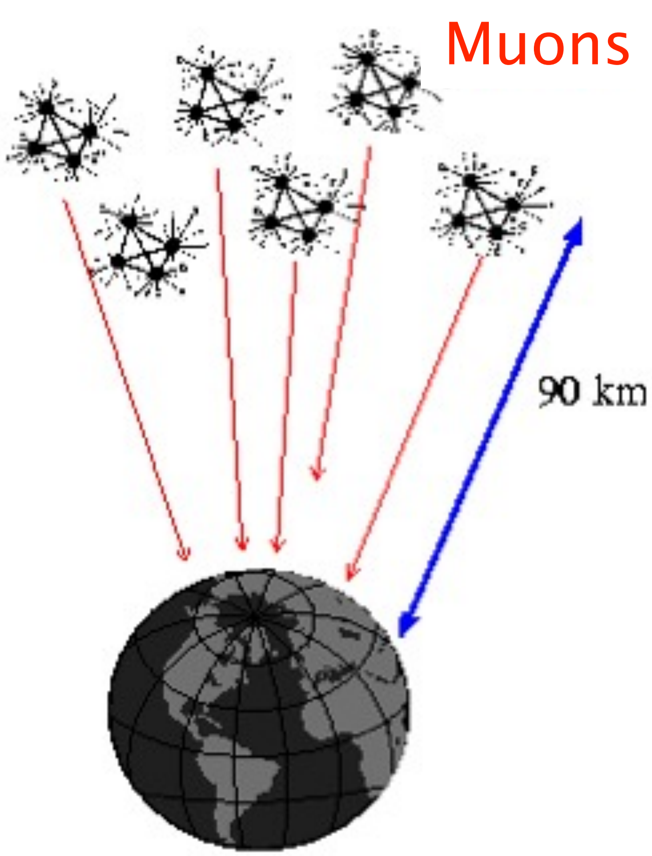
$$x = \gamma(x' - vt') \qquad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$

$$x' = \gamma(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

$$\begin{aligned} x =& \gamma(x'_D - vt'_D) \\ =& -50\gamma = -100\,\mathrm{m} \end{aligned}$$

The driver is already out of the tunnel by 100m

Example: Cosmic Rays



- Muons are created in the upper atmosphere, 90km from earth. Their half life is τ=2 μs, so they can travel at most 2 $\times 10^{-6}c$=600 m before decaying. So how do more than 50% reach the earth's surface?

  Muons see distance contracted by γ, so

$$v\tau \approx \frac{90}{\gamma}\,km$$

  Earthlings say muons' clocks run slow so their half-life is $\gamma\tau$ and

$$v(\gamma\tau) \approx 90\,km$$

- Both give

$$\frac{\gamma v}{c} = \frac{90\,km}{c\tau} = 150, \quad v \approx c, \quad \gamma \approx 150$$

Science & Technology Facilities Council

Space-time

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of

$$c^2t^2 - (x^2 + y^2 + z^2) = (ct)^2 - \vec{x}^2$$

- 4D-space with coordinates *(t,x,y,z)* is called space-time and the point *(t,x,y,z)=(t,**x**)* is called an event.
- Fundamental invariant (preservation of speed of light):

$$\begin{aligned}\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 &= c^2\Delta t^2\left(1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2\Delta t^2}\right)\\ &= c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) = c^2\left(\frac{\Delta t}{\gamma}\right)^2\end{aligned}$$

$$\tau = \int \frac{\mathrm{d}t}{\gamma}$$

is called proper time, the time in the instantaneous rest-frame and an invariant. Δs is called the separation between two events.

Science & Technology Facilities Council

4-Vectors

The Lorentz transformation can be written in matrix form as

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \implies \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz matrix L

Position 4-vector X

$$X' = LX$$

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

Science & Technology Facilities Council

4-Vector Invariants

Basic invariant:

$$c^2t^2 - x^2 - y^2 - z^2 = (ct, x, y, z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X^t g X = X \cdot X$$

Inner product of two four vectors $A = (a_0, \vec{a})$, $B = (b_0, \vec{b})$:

$$A \cdot B = A^T g B = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \vec{a} \cdot \vec{b}$$

Invariance:

$$A' \cdot B' = (LA)^T g (LB) = A^T (L^T g L) B = A^T g B = A \cdot B$$

In particular $A \cdot A = a_0^2 - \vec{a}^2$.

Science & Technology Facilities Council

4-Vectors in S.R. Mechanics

- Velocity: $$V = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}X}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}(ct, \vec{x}) = \gamma(c, \vec{v})$$
- Note invariant: $$V \cdot V = \gamma^2(c^2 - \vec{v}^2) = \frac{c^2 - \vec{v}^2}{1 - \vec{v}^2/c^2} = c^2$$
- Momentum: $$P = m_0 V = m_0\gamma(c, \vec{v}) = (mc, \vec{p})$$

$m = m_0\gamma$ is the relativistic mass

$p = m_0\gamma\vec{v} = m\vec{v}$ is the relativistic 3-momentum

Science & Technology Facilities Council

4-Force

From Newton's 2nd Law expect 4-Force given by

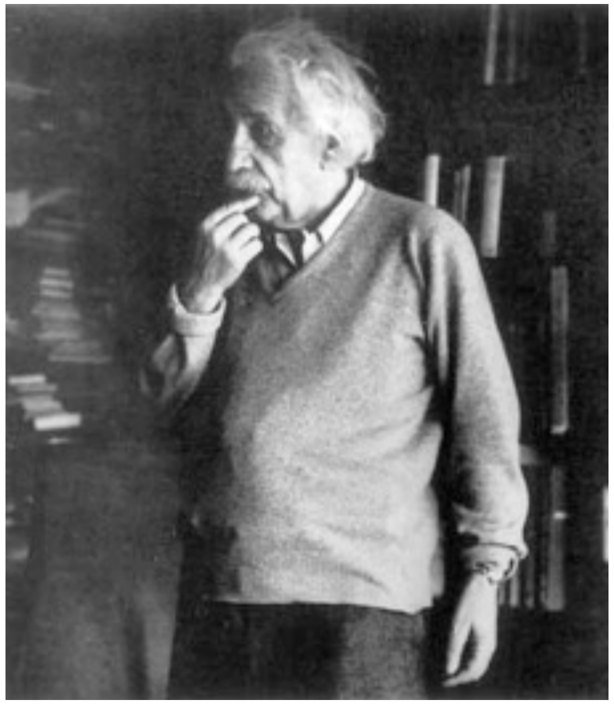
$$\begin{aligned} F &= \frac{\mathrm{d}P}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}P}{\mathrm{d}t} \\ &= \gamma \frac{\mathrm{d}}{\mathrm{d}t}(mc, \vec{p}) = \gamma \left(c\frac{\mathrm{d}m}{\mathrm{d}t}, \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \right) \\ &= \gamma \left(c\frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f} \right) \end{aligned}$$

Note: 3-force equation: $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \vec{v})$

Science & Technology Facilities Council

Einstein's Relation: Energy and Mass

- Momentum invariant $P \cdot P = m_0^2 V \cdot V = m_0^2 c^2$
- Differentiate $P \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot F = 0$



$$\implies \gamma(c, \vec{v}) \cdot \gamma \left(c\frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f} \right) = 0$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) - \vec{v} \cdot \vec{f} = 0$$

$$\begin{aligned} \vec{v} \cdot \vec{f} &= \text{rate at which force does work} \\ &= \text{rate of change of kinetic energy} \end{aligned}$$

Therefore kinetic energy is

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E=mc^2$ is total energy

Science & Technology Facilities Council

Summary of 4-Vectors

Position	X	$=$	$(ct, \vec{x})$
Velocity	V	$=$	$\gamma(c, \vec{v})$
Momentum	P	$=$	$m_0 V = m(c, \vec{v}) = \left(\frac{E}{c}, \vec{p}\right)$
Force	F	$=$	$\gamma\left(c\frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f}\right) = \gamma\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \vec{f}\right)$

Science & Technology Facilities Council

Example: Addition of Velocities

An object has velocity $\vec{u} = (u_x, u_y)$ in frame F', which moves with velocity $\vec{v} = (v, 0)$ with respect to frame F.

The 4-velocity $U = \gamma_u(c, u_x, u_y)$ has to be Lorentz transformed to F, resulting in a 4-velocity $W = \gamma_w(c, w_x, w_y)$:

$$\begin{pmatrix} c\gamma_w \\ \gamma_w w_x \\ \gamma_w w_y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c & 0 \\ \gamma v/c & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\gamma_u \\ \gamma_u u_x \\ \gamma_u u_y \end{pmatrix}$$

$$\begin{aligned} \gamma_w &= \gamma\gamma_u \left(1 + \frac{vu_x}{c^2}\right) \\ \gamma_w w_x &= \gamma\gamma_u \left(v + u_x\right) \\ \gamma_w w_y &= \gamma_u u_y \end{aligned}$$

$$\begin{aligned} w_x &= \frac{v + u_x}{\left(1 + \frac{vu_x}{c^2}\right)} \\ w_y &= \frac{u_y}{\gamma\left(1 + \frac{vu_x}{c^2}\right)} \end{aligned}$$

Using Invariants

A neater way of finding the speed of the particle as measured in frame F:

An observer in F has 4-velocity V and the object has 4-velocity U

$$\boxed{U \cdot V \quad \text{is invariant}}$$

Evaluated in frame F' : $\qquad U = \gamma_u(c, u_x, u_y), \quad V = \gamma(c, -v, 0)$

$$\Longrightarrow U \cdot V = \gamma\gamma_u(c^2 + vu_x)$$

Evaluated in frame F' : $\qquad U = (c, 0, 0), \quad V = \gamma_w(c, w_x, w_y)$

$$\Longrightarrow U \cdot V = c^2\gamma_w$$

Hence

$$\gamma_w = \gamma\gamma_u\left(1 + \frac{vu_x}{c^2}\right)$$

Basic Quantities used in Accelerator Calculations

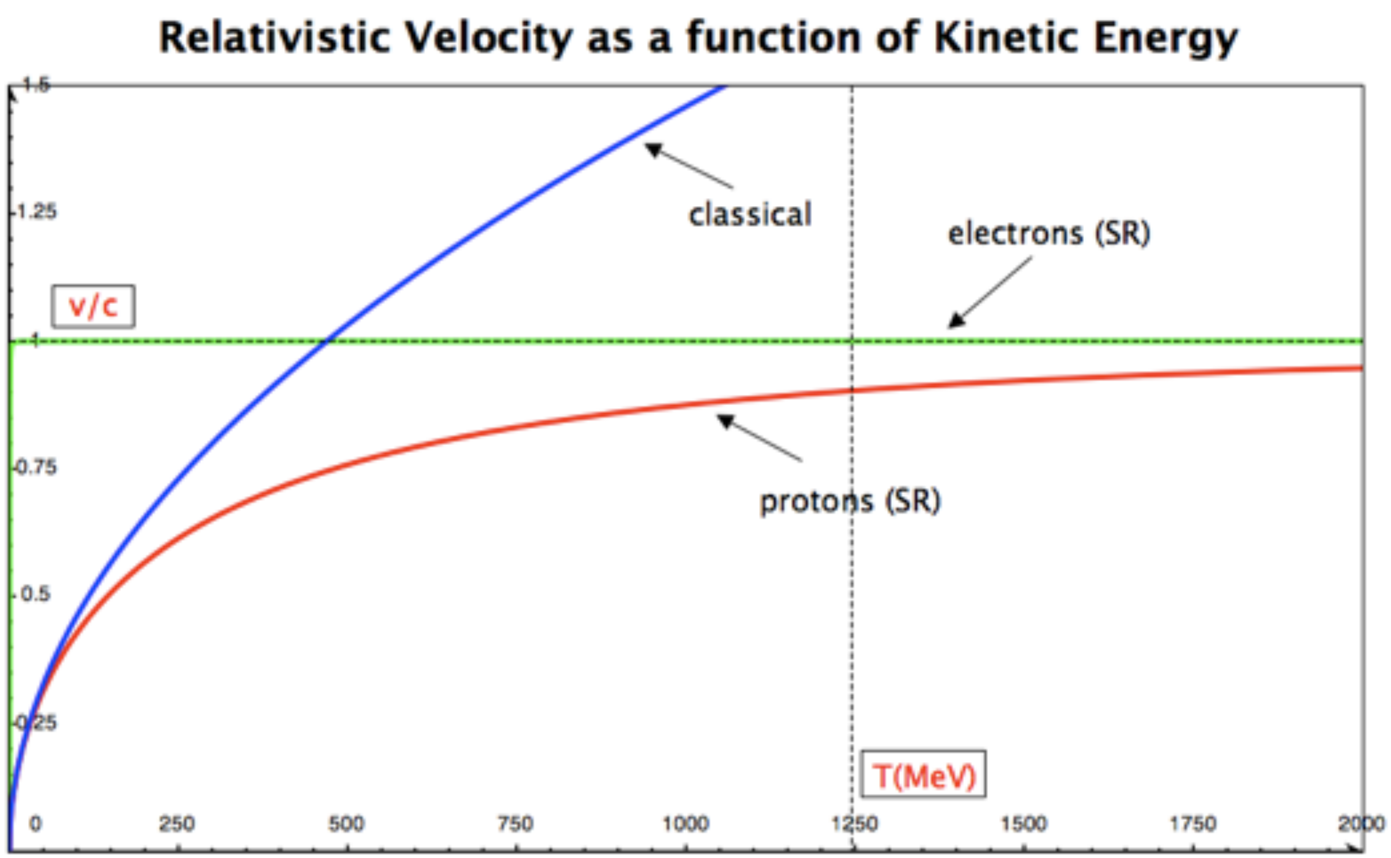
Relative velocity	$\beta = \frac{v}{c}$
Velocity	$v = \beta c$
Momentum	$p = mv = m_0\gamma v = m_0\gamma\beta c$
Kinetic energy	$T = mc^2 - m_0c^2 = (\gamma - 1)m_0c^2 = (\gamma - 1)E_0$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Longrightarrow \quad (\beta\gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Longrightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Science & Technology Facilities Council

Velocity and Energy



$$T = m_0 c^2(\gamma - 1) = E_0(\gamma - 1)$$

$$\gamma = 1 + \frac{T}{m_0 c^2} = 1 + \frac{T}{E_0}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$p = m_0 \beta \gamma c = \frac{E_0}{c}\beta\gamma$$

For $v \ll c$, $$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$$

so $$T = m_0 c^2(\gamma - 1) \approx \frac{1}{2} m_0 v^2$$

Energy-Momentum

- Important invariant:

$$P = (E/c, \mathbf{p}) \implies P \cdot P = E^2/c^2 - \mathbf{p}^2$$

$$\text{and} \quad P = m_0 V \implies P \cdot P = m_0^2 V \cdot V = m_0^2 c^2 = E_0^2/c^2$$

$$\frac{E^2}{c^2} = \mathbf{p}^2 + m_0^2 c^2 \qquad \text{or} \qquad E^2 = \mathbf{p}^2 c^2 + E_0^2$$

$$\implies \qquad p^2 c^2 = E^2 - E_0^2 = (E - E_0)(E + E_0) = T(T + 2E_0)$$

Example: ISIS at RAL accelerates protons ($E_0 = 938\,\text{MeV}$) to $800\,\text{MeV}$

$$\implies pc = \sqrt{800 \times (800 + 2 \times 938)}\,\text{MeV}$$

$$= 1.463\,\text{GeV}$$

$$\beta\gamma = \frac{m_0 \beta\gamma c^2}{m_0 c^2} = \frac{pc}{E_0} = 1.56$$

$$\gamma^2 = (\beta\gamma)^2 + 1 \implies \gamma = 1.85$$

$$\beta = \frac{\beta\gamma}{\gamma} = 0.84$$

Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$\begin{aligned} (\beta\gamma)^2 &= \gamma^2 - 1 \\ \Longrightarrow \quad \beta\gamma\Delta(\beta\gamma) &= \gamma\Delta\gamma \\ \Longrightarrow \quad \beta\Delta(\beta\gamma) &= \Delta\gamma \qquad (1) \end{aligned}$$

$$\begin{aligned} \frac{1}{\gamma^2} &= 1 - \beta^2 \\ \Longrightarrow \quad \frac{1}{\gamma^3}\Delta\gamma &= \beta\Delta\beta \qquad (2) \end{aligned}$$

$$\begin{aligned} \frac{\Delta p}{p} &= \frac{\Delta(m_0\beta\gamma c)}{m_0\beta\gamma c} = \frac{\Delta(\beta\gamma)}{\beta\gamma} \\ &= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E} \\ &= \gamma^2\frac{\Delta\beta}{\beta} \\ &= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)} \end{aligned}$$

Note: valid to first order only

Science & Technology Facilities Council

<table>
<tr><th></th><th>$\frac{\Delta\beta}{\beta}$</th><th>$\frac{\Delta p}{p}$</th><th>$\frac{\Delta T}{T}$</th><th>$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$</th></tr>
<tr><td rowspan="2">$\frac{\Delta\beta}{\beta} =$</td><td rowspan="2">$\frac{\Delta\beta}{\beta}$</td><td>$\frac{1}{\gamma^2}\frac{\Delta p}{p}$</td><td rowspan="2">$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$</td><td>$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$</td></tr>
<tr><td>$\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$</td><td>$\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$</td></tr>
<tr><td>$\frac{\Delta p}{p} =$</td><td>$\gamma^2\frac{\Delta\beta}{\beta}$</td><td>$\frac{\Delta p}{p}$</td><td>$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$</td><td>$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$</td></tr>
<tr><td>$\frac{\Delta T}{T} =$</td><td>$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$</td><td>$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$</td><td>$\frac{\Delta T}{T}$</td><td>$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$</td></tr>
<tr><td>$\frac{\Delta E}{E} =$</td><td>$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$</td><td>$\beta^2\frac{\Delta p}{p}$</td><td rowspan="2">$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$</td><td rowspan="2">$\frac{\Delta\gamma}{\gamma}$</td></tr>
<tr><td>$\frac{\Delta\gamma}{\gamma} =$</td><td>$(\gamma^2-1)\frac{\Delta\beta}{\beta}$</td><td>$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$</td></tr>
</table>

Table 1: Incremental relationships between energy, velocity and momentum.

4-Momentum Conservation

- Equivalent expression for 4-momentum

$$P = m_0\gamma(c, \vec{v}) = (mc, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right)$$

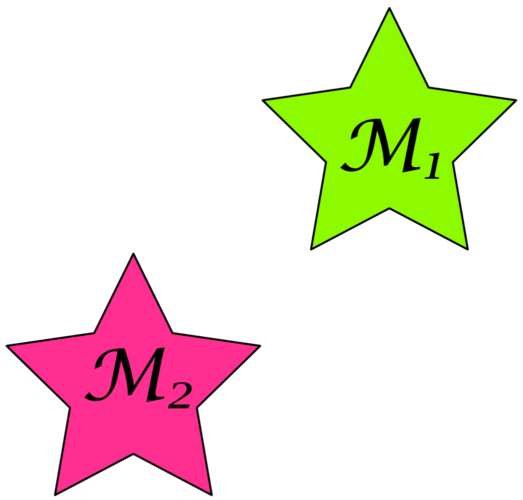
- Invariant $\quad m_0^2c^2 = P \cdot P = \frac{E^2}{c^2} - \vec{p}^2 \quad \Longrightarrow \quad \boxed{\frac{E^2}{c^2} = \vec{p}^2 + m_0^2c^2}$

- Classical conservation laws:
  - conservation of total mass
  - conservation of total 3-momentum
- In relativity, mass equivalent to energy (incl. rest energy)

$$\Longrightarrow \sum_{\text{particles } i} E_i \quad \text{and} \quad \sum_{\text{particles } i} \mathbf{p}_i \text{ constant}$$

$$\Longrightarrow \text{ total 4-momentum } \boxed{\sum_{\text{particles } i} P_i \text{ constant}}$$

Problem



A body of mass M disintegrates while at rest into two parts of rest masses M_1 and M_2.

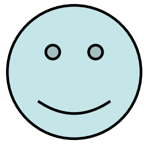
Show that the energies of the parts are given by

$$E_1 = c^2 \frac{M^2 + M_1^2 - M_2^2}{2M}, \quad E_2 = c^2 \frac{M^2 - M_1^2 + M_2^2}{2M}$$

Science & Technology Facilities Council

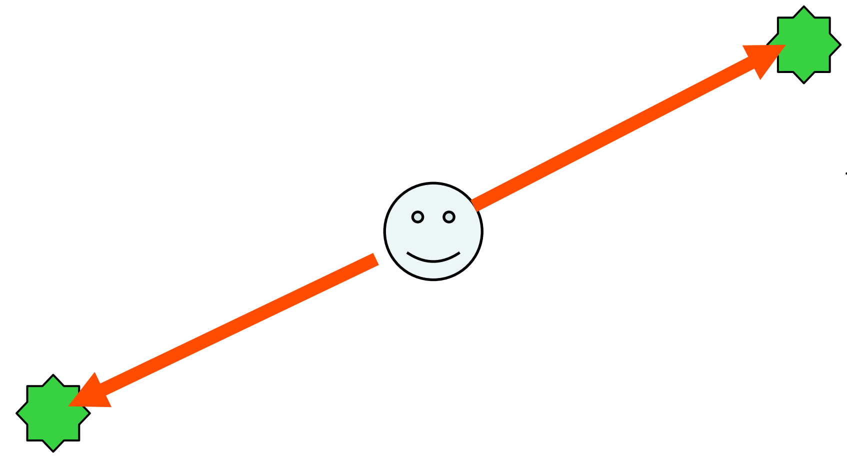
Solution

Before:



$$P = \left(Mc, \vec{0}\right)$$

After:



$$P_1 = \left(\frac{E_1}{c}, \vec{p}\right)$$

$$P_2 = \left(\frac{E_2}{c}, -\vec{p}\right)$$

Conservation of 4-momentum:

$$P = P_1 + P_2 \quad \Rightarrow \quad P - P_1 = P_2$$

$$\Rightarrow \quad (P - P_1)\cdot(P - P_1) = P_2 \cdot P_2$$

$$\Rightarrow \quad P \cdot P - 2P \cdot P_1 + P_1 \cdot P_1 = P_2 \cdot P_2$$

$$\Rightarrow \quad M^2c^2 - 2ME_1 + M_1^2c^2 = M_2^2c^2$$

$$\Rightarrow \quad E_1 = \frac{M^2 + M_1^2 - M_2^2}{2M}c^2$$

Science & Technology Facilities Council

Example of use of invariants

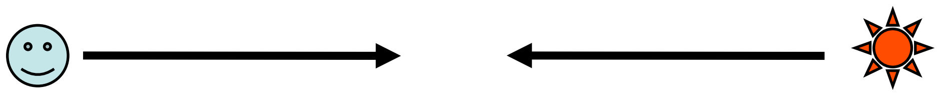
- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2

$$P_1 = \left(\frac{E_1 - m_0c^2}{c}, \vec{p}\right) \qquad P_2 = \left(m_0 c, \vec{0}\right)$$

Total energy E_1

(Fixed target experiment)



$$P_1 = \left(\frac{E_2}{2c}, \vec{p'}\right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p'}\right)$$

Total energy E_2

(Colliding beams experiment)

Invariant: $P_2 \cdot (P_1 + P_2)$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

$$\Rightarrow \qquad 2m_0c^2E_1 = E_2^2$$

Science & Technology Facilities Council

Collider Problem

In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal rest mass $M_0=100m_0$

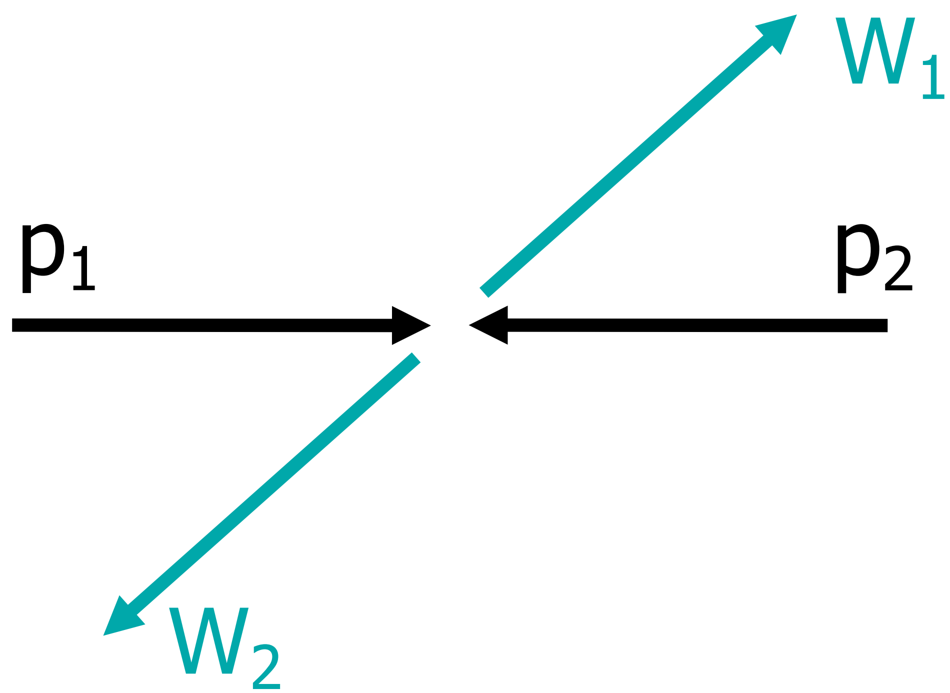
- Experiment 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.
- **Experiment 2:** in the rest frame of p_1, find the minimum energy E' of p_2 in order for W_1 and W_2 to be produced.

Science & Technology Facilities Council

Experiment 1

$$\frac{E^2}{c^2} = \vec{p}^{\,2} + m_0^2 c^2 \quad \Longrightarrow$$

Particles with same rest-mass and same momentum have same energies.



Total 3-momentum is zero before collision, so is zero afterwards

4-momenta before collision:

$$P_1 = \left(\frac{E}{c}, \vec{p}\right) \quad P_2 = \left(\frac{E}{c}, -\vec{p}\right)$$

4-momenta after collision:

$$P_1 = \left(\frac{E'}{c}, \vec{q}\right) \quad P_2 = \left(\frac{E'}{c}, -\vec{q}\right)$$

Total energy is conserved $\Longrightarrow 2E = 2E'$

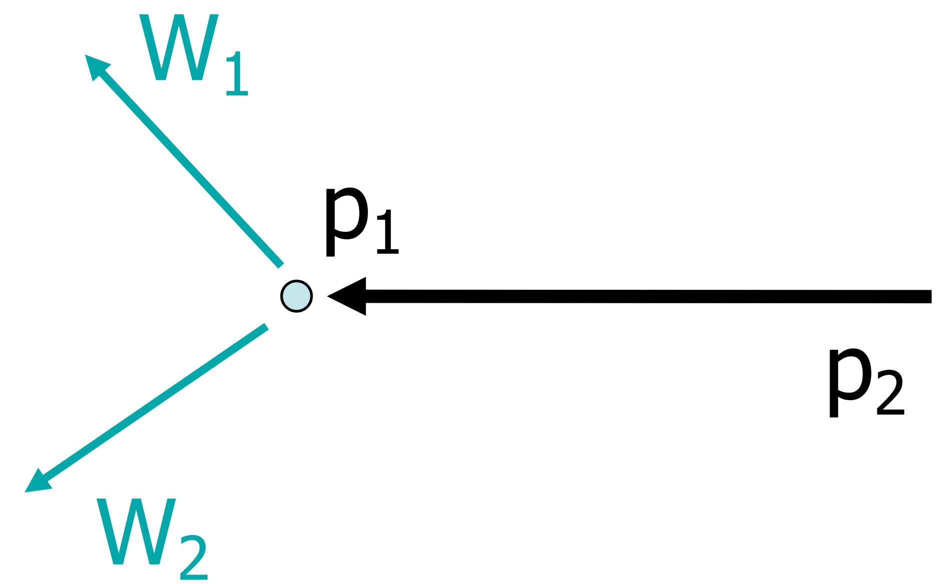
$\Longrightarrow E = E' >$ **rest energy** $= M_0 c^2 = 100 m_0 c^2$

Experiment 2

Before collision

$$P_1 = (m_0 c, \vec{0}), \quad P_2 = \left(\frac{E'}{c}, \vec{p}\right)$$

Total energy is $E_1 = E' + m_0 c^2$



Use previous result $\boxed{2m_0c^2 E_1 = E_2^2}$ to relate E_1 to total energy E_2 in the centre of mass frame

$$2m_0c^2 E_1 = E_2^2$$

$$\Longrightarrow \; 2m_0c^2 (E' + m_0c^2) = (2E)^2 > (200 m_0 c^2)^2$$

$$\Longrightarrow \; E' > (2 \times 10^4 - 1) m_0 c^2 \approx 20,000\, m_0 c^2$$

Photons and Wave 4-Vectors

- Monochromatic plane wave: $\sin(\omega t - \vec{k}\cdot\vec{x})$

- $\vec{k}$ is the *wave vector*, $|\vec{k}| = \dfrac{2\pi}{\lambda}$; ω is the *angular frequency*, $\omega = 2\pi\nu$

- The phase $\dfrac{1}{2\pi}(\omega t - \vec{k}\cdot\vec{x})$ is the number of wave crests passing an observer

- Invariant: $$\omega t - \vec{k}\cdot\vec{x} = (ct, \vec{x}) \cdot \left(\frac{\omega}{c}, \vec{k}\right)$$

  Position 4-vector   Wave 4-vector

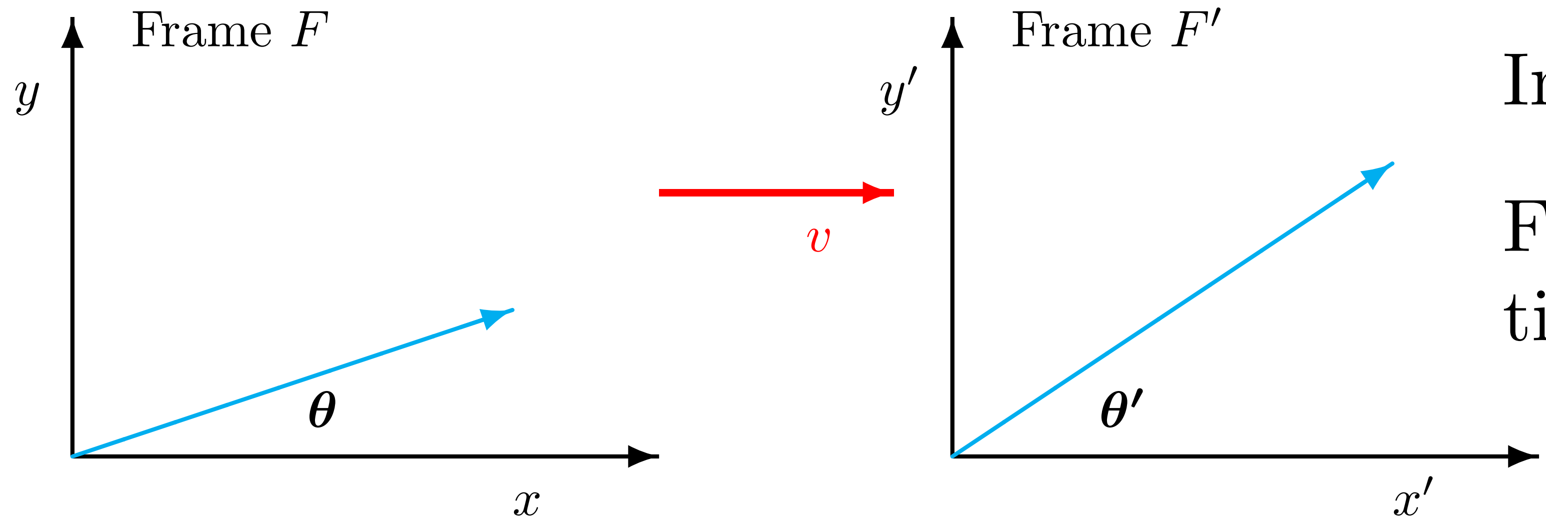
- 4-momentum $P = \left(\dfrac{E}{c}, p\right) = \hbar\left(\dfrac{\omega}{c}, k\right) = \hbar K$

Science & Technology Facilities Council

Relativistic Doppler Shift

For light rays $\omega = c|\vec{k}|$ so $K = \left(\frac{\omega}{c}, \vec{k}\right)$ is *a null vector*

and can be written $K = \frac{\omega}{c}(1, \vec{n})$ where $|\vec{n}| = 1$.



In F, $K = \frac{\omega}{c}(1, \cos\theta, \sin\theta)$

For F', use Lorentz transformation: $K' = LK$

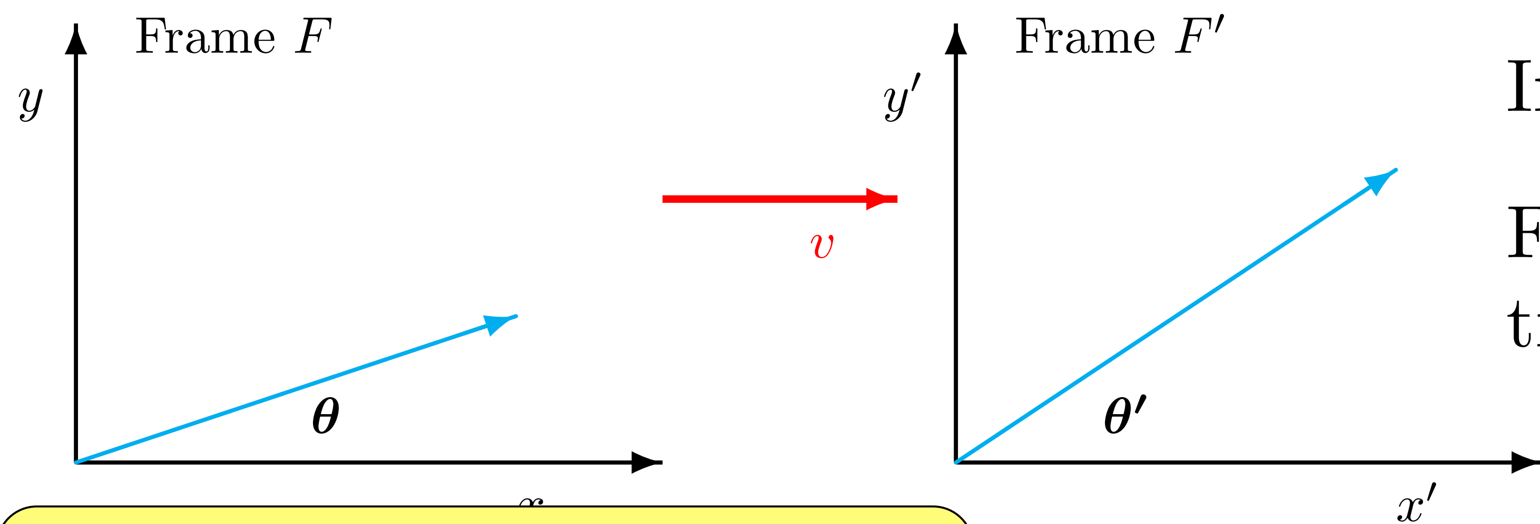
$$\begin{pmatrix} \omega'/c \\ (\omega'/c)\cos\theta' \\ (\omega'/c)\sin\theta' \end{pmatrix} = \begin{bmatrix} \gamma & -\gamma v/c & 0 \\ -\gamma v/c & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \omega/c \\ (\omega/c)\cos\theta \\ (\omega/c)\sin\theta \end{pmatrix}$$

Science & Technology Facilities Council

Relativistic Doppler Shift

For light rays $\omega = c|\vec{k}|$ so $K = \left(\frac{\omega}{c}, \vec{k}\right)$ is *a null vector*

and can be written $K = \frac{\omega}{c}(1, \vec{n})$ where $|\vec{n}| = 1$.



In F, $K = \frac{\omega}{c}(1, \cos\theta, \sin\theta)$

For F', use Lorentz transformation: $K' = LK$

$$\begin{aligned} \omega' &= \gamma\left(\omega - \frac{v\omega\cos\theta}{c}\right) \\ \omega' cos\theta' &= \gamma\left(\omega\cos\theta - v\frac{\omega}{c}\right) \\ \omega'\sin\theta' &= \omega\sin\theta \end{aligned}$$

$$\begin{aligned} \omega' &= \gamma\omega\left(1 - \frac{v}{c}\cos\theta\right) \\ \tan\theta' &= \frac{\sin\theta}{\gamma\left(\cos\theta - \frac{v}{c}\right)} \end{aligned}$$

Note there is a transverse Doppler effect even when $\theta = \frac{1}{2}\pi$

4-Acceleration

- 4-Acceleration=rate of change of 4-Velocity

$$A = \frac{\mathrm{d}V}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\Big(\gamma c, \gamma \vec{v}\Big)$$

- Use $\frac{1}{\gamma^2} = 1 - \frac{\vec{v}\cdot\vec{v}}{c^2} \implies \frac{1}{\gamma^3}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\vec{v}\cdot\dot{\vec{v}}}{c^2} = \frac{\vec{v}\cdot\vec{a}}{c^2}$

$$A = \gamma\left(\gamma^3 \frac{\vec{v}\cdot\vec{a}}{c}, \gamma\vec{a} + \gamma^3\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v}\right)$$

- In instantaneous rest-frame $A = (0, \vec{a}), \quad A\cdot A = -|\vec{a}|^2$

Science & Technology Facilities Council

Radiation from an accelerating charge

- Rate of radiation, *R*, known to be invariant and proportional to $|\vec{a}|^2$ in instantaneous rest frame.
- But in instantaneous rest-frame $A \cdot A = -|\vec{a}|^2$
- Deduce $R \propto A \cdot A = -\gamma^6 \left(\left(\frac{\vec{v} \cdot \vec{a}}{c} \right)^2 + \frac{1}{\gamma^2} \vec{a}^2 \right)$
- Rearranged: $$R = \frac{e^2}{6\pi\epsilon_0 c^3} \gamma^6 \left[|\vec{a}|^2 - \frac{(\vec{a} \times \vec{v})^2}{c^2} \right]$$

Relativistic Larmor Formula

If $\vec{a} \parallel \vec{v}$, $R \propto \gamma^6$, but if $\vec{a} \perp \vec{v}$, $R \propto \gamma^4$

Science & Technology Facilities Council

Motion under constant acceleration; world lines

- Introduce *rapidity* ρ defined by

$$\beta = \frac{v}{c} = \tanh\rho \quad \Longrightarrow \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\rho$$

- Then $V = \gamma(c, v) = c(\cosh\rho, \sinh\rho)$

- And $A = \frac{\mathrm{d}V}{\mathrm{d}\tau} = c(\sinh\rho, \cosh\rho)\frac{\mathrm{d}\rho}{\mathrm{d}\tau}$

- So constant acceleration satisfies

$$a^2 = |\vec{a}|^2 = -A \cdot A = c^2\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2 \quad \Longrightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{a}{c}, \text{ so } \rho = \frac{a\tau}{c}$$

Science & Technology Facilities Council

Particle Paths

$$\frac{dx}{d\tau} = \gamma \frac{dx}{dt} = \gamma v = c \sinh \rho = c \sinh \frac{a\tau}{c}$$

$$\implies x = x_0 + \frac{c^2}{a}\left(\cosh \frac{a\tau}{c} - 1\right)$$

$$\frac{dt}{d\tau} = \gamma = \cosh \rho = \cosh \frac{a\tau}{c}$$

$$\implies t = \frac{c}{a} \sinh \frac{a\tau}{c}$$

$$\cosh^2 \rho - \sinh^2 \rho = 1$$

$$\implies \left(x - x_0 + \frac{c^2}{a}\right)^2 - c^2 t^2 = \frac{c^4}{a^2}$$

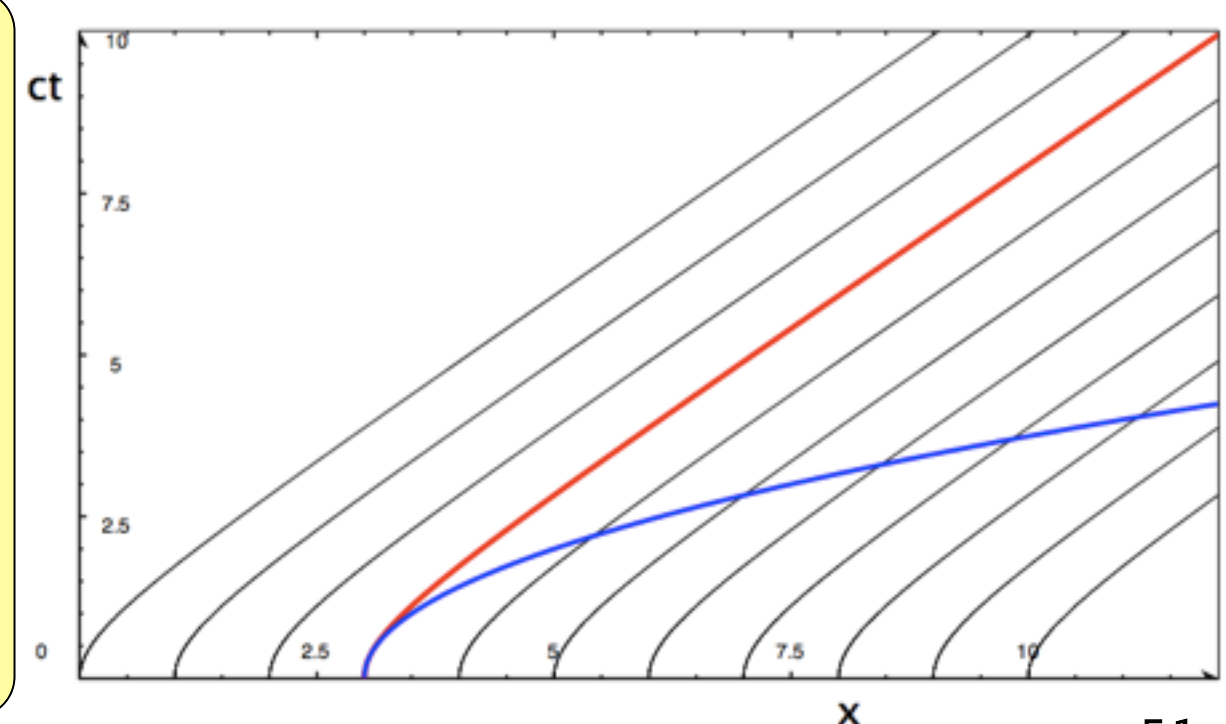
Relativistic paths are hyperbolic

$$x = x_0 + \frac{c^2}{a}\left(1 + \frac{1}{2}\frac{a^2\tau^2}{c^2} + \ldots - 1\right)$$

$$\approx x_0 + \frac{1}{2} a\tau^2$$

$$t \approx \frac{c}{a} \times \frac{a\tau}{c} = \tau$$

Non-relativistic paths are parabolic



Relativistic Lagrangian and Hamiltonian Formulation

4-force equation of motion under a potential V:

$$\vec{f} = -\nabla V = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \vec{v})$$

$$\Longrightarrow \quad m_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{1 - v^2/c^2}}\right) = -\frac{\partial V}{\partial x} \quad \text{etc., where } v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

Compare with standard Lagrangian formulation: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}.$

Deduce Relativistic Langangian:

$$\mathcal{L} = -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - V = -\frac{m_0 c^2}{\gamma} - V.$$

Note: $\mathcal{L} \neq T - V$

Science & Technology Facilities Council

Hamiltonian $\mathcal{H}$, by definition:

$$\mathcal{H} = \sum_{x,y,z} \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = \frac{m_0}{\sqrt{1 - v^2/c^2}}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \mathcal{L}$$

$$= m_0\gamma v^2 + \frac{m_0 c^2}{\gamma} + V = m_0\gamma c^2 + V$$

$$= T + V, \quad \text{Total Energy}$$

Since $E^2 = \vec{p}^{\,2}c^2 + m_0^2 c^4$,

$$\mathcal{H} = c\left(\vec{p}^{\,2} + m_0^2 c^2\right)^{\frac{1}{2}} + V$$

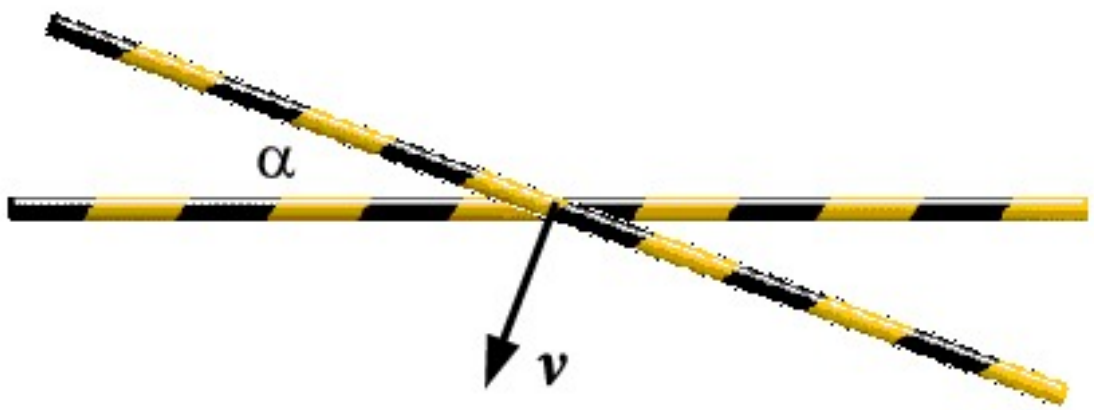
Then Hamilton's equations of motion are:

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}, \quad \dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}$$

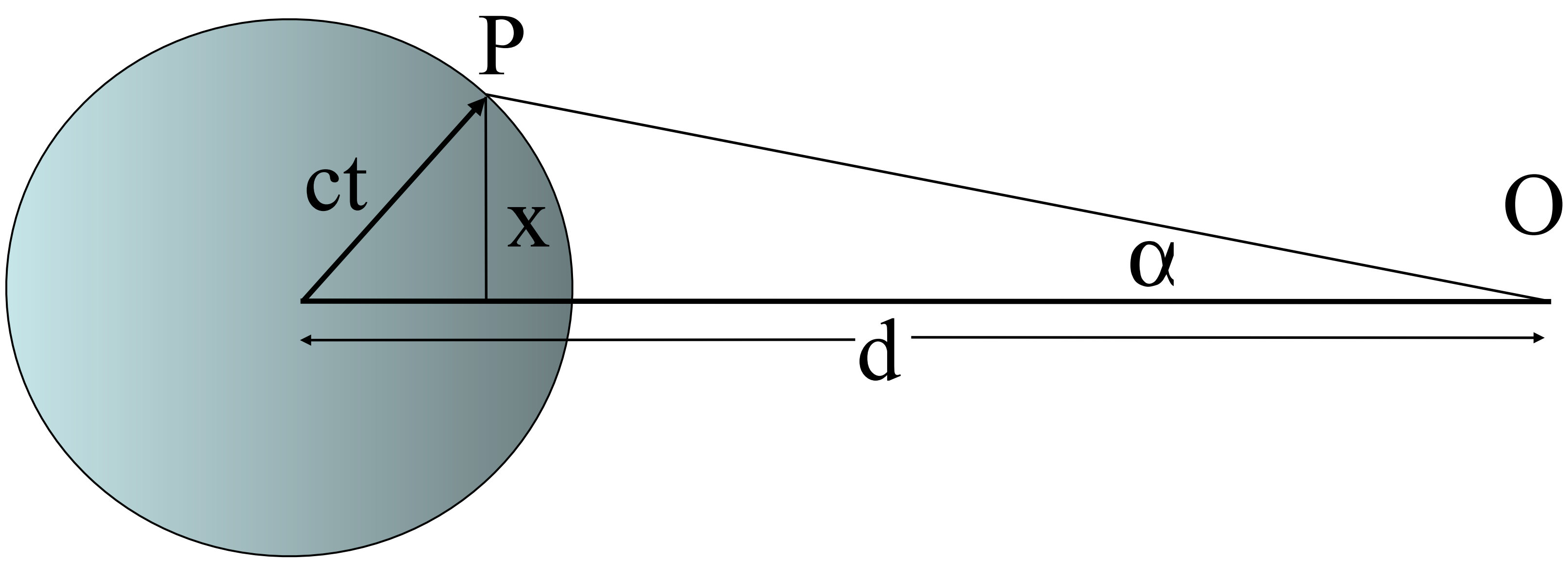
Science & Technology Facilities Council

Motion faster than light

1. Two rods sliding over each other. Speed of intersection point is v/sinα, which can be made greater than c.



2. Explosion of planetary nebula. Observer sees bright spot spreading out. Light from P arrives $t=d\alpha^2/2c$ later.



$$t = \frac{d\alpha^2}{2c} \approx \frac{x}{c}\frac{\alpha}{2} << \frac{x}{c}$$