Special Relativity and Electromagnetism

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Introduction

	10:30	11:45
Monday 15 October	SR	SR/EM
November 5	EM	EM

- Four lectures is clearly not enough to teach special relativity and electromagnetism.
- This course is therefore just a quick run though of the topics. With emphasis on areas of special relevance for accelerator science.
- Online there are Chris Prior's notes. He used to give the talk and has prepared some impressive handouts.
- For Physicists both SR and EM will be revision; relativistic EM may be new.
- For Engineers EM will be revision and SR may be new.
- For the new cases I can only introduce the subject and hope you will learn the relevant areas when you need it.
- For the students where this is revision I hope I can introduce a slightly different, more pictorial approach.
- There are multiple books on the both subjects as well as the introduction sections of books in accelerator science.

Application of topics in accelerator physics

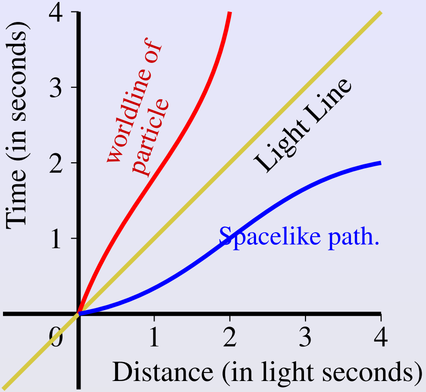
	EM(Maxwell)	EM(Lorentz Force)	Relativistic EM
Beam dynamics	–	$\surd$	$\surd$
Magnet design	$\surd$	–	–
Cavity design	$\surd$	$\surd$	–
Wakefield	$\surd$	$\surd$	$\surd$
Plasma Accel	$\surd$	$\surd$	$\surd$

Special Relativity

Special Relativity

- This is crucial for the understanding of beam dynamics.
- Even at moderate energies, (a few MeVs) electrons have strong relativistic effects.
- At high energies relativistic effects actually make our life easier. Essentially the particles stop talking to each other. So we do not need to consider, to the first approximation, particle-particle interactions.
- I shall try to introduce a more pictorial view of special relativity. This complements the standard undergraduate treatment.

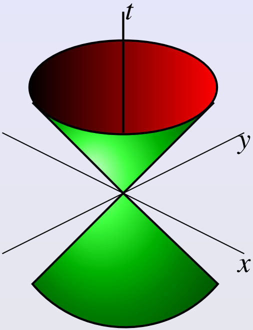
Spacetime diagrams



- Spacetime diagram are brilliant for understanding many problems in SR.
- The coordinates are chosen so that lightline are represented at 45° degrees.
- Particles cannot travel faster than light so much travel more vertical than 45°.
- Their loci are called worldlines.

Spacetime diagrams

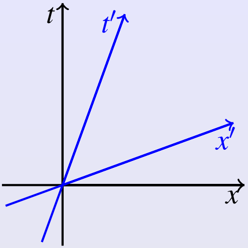
- We sometimes need 3 dimensions so that we can represent 2 dimensions of space and 1 dimension of time.
- In this case we can refer to the lightcone.
- Causality divides space into three regions. The inside of the upper cone (and the cone itself) are all the points which can be influenced by the event (point) at the origin. The lower cone are all events which can influence the origin. All massive particles which path the origin must have worldlines inside this double cone.



- Outside the double cone are not causally connected to the origin.
- Sometimes we need to represent all 4 dimensions. (1 time and 3 space)
- Sometimes we need to represent 7 dimensions: ????

Lorentz transformation

- This are used to compare event according to two different observers, moving relative to each other.
- Each observer has with them a **frame**. You may think of this as arrows representing the directions of x, y and z as well as a clock representing t.
- In accelerator physics this is usually the **lab frame** and the **rest frame of the particle**.



- Using spacetime diagrams we can understand why notions of simultaneity is not meaningful as well as time dilation and Lorentz contractions.(See Worksheet.)
- If the two observers are not moving relative to each other then the only difference in their frames is a rotation.
- If they are moving relative to each other, then there is a “rotation in spacetime”. This is usually referred to as a **boost**.
- In usual rotations we preserve length. In Lorentz boosts we must preserve the speed of light.

Lorentz transformation

- We will use usual units of furlongs and fortnights so that we have to re-introduce the constant c. What are better units?
- Consider a matrix representing a rotation or a boost or both.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} & & & \\ & \text{a } 4 \times 4 \text{ Matrix} & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- This encodes the fact that the coordinates (ct', x', y', z') are linearly related to (ct, x, y, z)
- Show that $\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$ preserves length, i.e.

$$\sqrt{x^2 + y^2} = \sqrt{(x')^2 + (y')^2}$$

if and only if there exists an angle θ such that

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Lorentz transformation, γ-factor

- Show that $\begin{pmatrix} ct' \\ x' \end{pmatrix} = B \begin{pmatrix} ct \\ x \end{pmatrix}$ preserves the quantity,

$$(ct)^2 - x^2 = (ct')^2 - (x')^2$$

if and only if there exists a hyperbolic angle χ such that

$$B = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}$$

- This may also be represented as

$$B = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

- What is the relationship between γ, v, β and χ.
- Express γ in terms of β and visa versa. What is the matrix equation satisfied by B and R.

4 dimensions

- For a boost in all four directions we can write

$$ct' = \gamma(ct - \underline{\beta} \cdot \underline{x})$$
$$\underline{x}' = \underline{x} - \underline{\beta}\Big(\gamma ct - (\gamma - 1)\frac{\underline{\beta} \cdot \underline{x}}{\underline{\beta} \cdot \underline{\beta}}\Big)$$

- The maximal transformation which obeys the invariant

$$(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - (x')^2 - (y')^2 - (z')^2$$

One can use matrix notation. A general transformation is given by 6 parameters, three rotations and three boosts.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \chi_x & \sinh \chi_x & 0 & 0 \\ \sinh \chi_x & \cosh \chi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \chi_y & 0 & \sinh \chi_y & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \chi_y & 0 & \cosh \chi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \chi_z & 0 & 0 & \sinh \chi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \chi_z & 0 & 0 & \cosh \chi_z \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & 0 & \sin \theta_3 & \cos \theta_3 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

4-vectors and invasions

- Let

$$\boldsymbol{x} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{x}' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  Then

$$(\boldsymbol{x}')^T G \boldsymbol{x}' = \boldsymbol{x}^T G \boldsymbol{x}$$

  where T is the transpose.

- Setting Λ equal to the product of the six matrices on the previous slide then

$$\Lambda^T G \Lambda = G$$

- Given two 4-vectors $\boldsymbol{x}$ and $\boldsymbol{a}$ then the combination $\boldsymbol{a}^T G \boldsymbol{x}$ is invariant under Lorentz transformation. Why?
- We write $\boldsymbol{a} : \boldsymbol{x} = \boldsymbol{a}^T G \boldsymbol{x}$

4-vectors in SR Mechanics

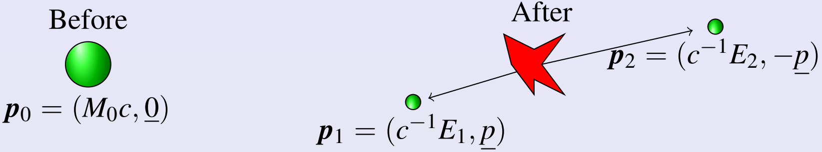
- Let $\boldsymbol{x}(\tau)$ be the position of the particle at its proper time τ.
- It's proper time is defined as the time as measured by a clock moving with the particle.
- The particles 4-velocity is given by

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{d\tau} = \gamma(c, \underline{v}) = c\gamma(1, \underline{\beta})$$

- Calculate $\boldsymbol{v} : \boldsymbol{v}$
- Draw the spacetime diagram for a particle and explain why $\dfrac{d}{d\tau} = \gamma \dfrac{d}{dt}$.
- 4-Momentum $\boldsymbol{p} = m_0 \boldsymbol{v}$, where m_0 is the rest mass. (Mass as measured by a comoving observer).
  Some people set $m = m_0\gamma$ and call it the relativistic mass. Energy.
- 4-Force $\boldsymbol{f} = \dfrac{d\boldsymbol{p}}{d\tau}$ show that $\boldsymbol{f} = m_0\gamma\left(c\dfrac{d\gamma}{dt}, \dfrac{d(\gamma\underline{v})}{dt}\right)$.

Use of invariants: Particle interactions

- We can use the 4-momentum and invariants to calculate particle interaction.



- Consider a single particle of rest mass M_0 which disintegrates into two particle of masses M_1 and M_2.
  Show that the energies are given by

$$E_1 = c^2 \frac{M_0^2 + M_1^2 - M_2^2}{2M_0}, \qquad E_2 = c^2 \frac{M_0^2 - M_1^2 + M_2^2}{2M_0}$$

Hint: consider $\boldsymbol{p}_1 : \boldsymbol{p}_2 = \boldsymbol{p}_1 : (\boldsymbol{p}_0 - \boldsymbol{p}_1) = (\boldsymbol{p}_0 - \boldsymbol{p}_2) : \boldsymbol{p}_2$ and $\boldsymbol{p}_1 : \boldsymbol{p}_1 = c^2 M_1^2$.

Use of invariants: Photon Emission

- We can show that is it impossible for a particle to emit a single photon.
- The 4-momentum of a photon is any 4-vector $\boldsymbol{p}_\gamma$ such that $\boldsymbol{p}_\gamma : \boldsymbol{p}_\gamma = 0$. I.e. it has zero rest mass.
- Setting $\boldsymbol{p}_\gamma = (E_\gamma/c, \underline{p}_\gamma)$ then $E_\gamma^2 = c^2 \left\| \underline{p}_\gamma \right\|^2$
  Show that for any timelike 4-vector $\boldsymbol{v}$ one has the identity

$$\boldsymbol{p}_\gamma : \boldsymbol{v} > 0$$

- Let $\boldsymbol{p}_0$ be the 4-momentum of a particle. Use the above identity to show that the particle cannot emit a photon and retain its rest mass.
- Observe that this means that an electron cannot simply emit a photon, for example the quantum version of CSR (coherent syncrotron emission).
- To emit a photon we must work around the above constraint. Examples include
  - The particle is a composite particle and looses mass when it emits a photon.
  - The particle being bent by a magnetic field, receives a photon from the magnetic field and emits a photon of radiation.
  - The particle emits a virtual photon (I.e. with $\boldsymbol{p}_\gamma : \boldsymbol{p}_\gamma \neq 0$). This only becomes a real photon when interacting with matter.
  - Some fudge in numerics.

Index notation

- Once we are used to 4-vectors it is usual to introduce index notation to 4-vectors.
- Given a 4-vector, say momentum we introduce

$$p^\mu = m_0\, c\gamma(1, \underline{\beta}) \qquad \text{and} \qquad p_\mu = m_0\, c\gamma(1, -\underline{\beta})$$

  Note the index runs over $\mu = 0, 1, 2, 3$ so that

$$p^0 = m_0\gamma c\,, \quad p^1 = m_0\gamma c\beta^1\,, \quad p^2 = m_0\gamma c\beta^2\,, \quad p^3 = m_0\gamma c\beta^3$$

- We then write

$$c^2 m_0^2 = \boldsymbol{p} : \boldsymbol{p} = \sum_{\mu=0}^{3} p_\mu\, p^\mu = p_\mu\, p^\mu$$

  The last of these is the "Einstein's summation convention". Which is an implicit summation over the index $\mu = 0, 1, 2, 3$. I will not use it here.
- The Lorentz transformation of a 4-vector is

$$p'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu p^\nu$$

Electromagnetism

Electromagnetism

- Electromagnetism can be summarised as four equations Maxwell

$$\nabla \cdot \underline{B} = 0\,, \qquad \nabla \times \underline{E} + \partial_t \underline{B} = 0$$
$$\nabla \cdot \underline{D} = \rho\,, \qquad \nabla \times \underline{H} - \partial_t \underline{D} = \underline{J}$$

and the Lorentz force equation.

$$\boldsymbol{f} = c\, q\, \gamma (c^{-1} \underline{E} \cdot \underline{\beta}\,,\; c^{-1} \underline{E} + \underline{\beta} \times \underline{B})$$

where $\partial_t = \frac{\partial}{\partial t}$.

- Here $\underline{E}$ is the electric field, $\underline{B}$ is the magnetic (flux) field, $\underline{D}$ is the electric displacement field and $\underline{H}$ is the (auxiliary) magnetic field.
  ρ is the charge density and $\underline{J}$ is the current density.
  $\boldsymbol{f}$ is the 4-force, q is the charge on a point particle.
- However there is very little one can do with these equations.
  Show conservation of charge

$$\partial_t \rho + \nabla \cdot \underline{J} = 0$$

- Write down the Lorentz force equations. Show that the Energy equation can be deduced from the 3–momentum equation.

Constitutive Relations

- We need constitutive relations which relate $(\underline{D}, \underline{H})$ to $(\underline{E}, \underline{B})$. Therefore the constitutive relations may look something like:

$$\underline{D} = \text{Some function}\big(\underline{E}, \underline{B}, \text{other stuff}\big)$$
$$\underline{H} = \text{Another function}\big(\underline{E}, \underline{B}, \text{other stuff}\big)$$

  Furthermore $(\rho, \underline{J})$ are sometimes also dependent on $(\underline{E}, \underline{B})$, for example in a metal.

Constitutive Relations

- The simplest constitutive relations are those for the vacuum.

$$\underline{D} = \epsilon_0 \underline{E} \qquad \text{and} \qquad \underline{H} = \frac{1}{\mu_0}\underline{B}$$

  In fact ϵ_0 and μ_0 are really just a consequence of the choice of units.
  If we set $c = 1$ then we can set $\underline{D} = \underline{E}$ and $\underline{H} = \underline{B}$.
- For a general media these constitutive relations can be very complicated.
- Simple homogeneous non-dispersive isotropic media:

$$\underline{D} = \epsilon \underline{E} \qquad \text{and} \qquad \underline{H} = \frac{1}{\mu}\underline{B}$$

  We call ϵ the permittivity and μ the permeability.
- Simple anisotropic non-dispersive media:

$$\underline{D} = \underline{\underline{\epsilon}}\,\underline{E} \qquad \text{and} \qquad \underline{H} = \underline{\underline{\mu}}^{-1}\underline{B}$$

- Magneto electric media

$$\underline{D} = \underline{\underline{\epsilon}}\,\underline{E} + \underline{\underline{\alpha}}\,\underline{B} \qquad \text{and} \qquad \underline{H} = \underline{\underline{\mu}}^{-1}\underline{B} + \underline{\underline{\beta}}\,\underline{E}$$

Constitutive Relations

- Dispersive media:

$$\underline{D}(t,\underline{x}) = \int_0^\infty \epsilon(t-t')\,\underline{E}(t',\underline{x})\,dt'$$

In this case we usually write the constitutive relations in frequency space.

$$\widetilde{\underline{D}}(\omega,\underline{x}) = \widetilde{\epsilon}(\omega)\,\widetilde{\underline{E}}(\omega,\underline{x})$$

- Inhomogeneous media

$$\underline{D}(t,\underline{x}) = \epsilon(\underline{x})\,\underline{E}(t,\underline{x})$$

- Spatially dispersive media

$$\widetilde{\underline{D}}(\omega,\underline{k}) = \widetilde{\epsilon}(\omega,\underline{k})\,\widetilde{\underline{E}}(\omega,\underline{k})$$

- Spatially dispersive inhomogeneous media ????
- Non linear media

$$\underline{D} = \epsilon(\underline{E})\,\underline{E}$$

- Non linear spatially dispersive inhomogeneous media ????????

Some simple calculations: EM waves

- Consider a homogeneous non-dispersive medium given by (ϵ, μ), which includes the vacuum, with no charges or currents.
-

$$\nabla \times \underline{E} + \mu\, \partial_t \underline{H} = 0\,, \quad \nabla \times \underline{H} - \epsilon\, \partial_t \underline{E} = 0\,, \quad \nabla \cdot \underline{H} = 0\,, \quad \nabla \cdot \underline{E} = 0$$

- Show this implies

$$\nabla^2 \underline{E} - \epsilon\mu\, \partial_t^2 \underline{E} = 0$$

This is the wave equation. It can be solved by

$$E_x(t,z) = E_{\rm amp}\, e^{-i\omega t + ikz}\,, \qquad H_y(t,z) = H_{\rm amp}\, e^{-i\omega t + ikz}$$

where $\omega^2 = c_{\rm med}^2\, k^2$ and $c_{\rm med} = (\epsilon\mu)^{-1/2}$ is the speed of light in the medium.
- Draw picture.

Some simple calculations: Charge in Magnetic field

- Consider a particle of mass m_0 and charge q, in a uniform magnetic field $\underline{E} = \underline{0}$ and $\underline{B} = (0, 0, B)$.
- The Lorentz force equation

$$m_0 \gamma c \left(\frac{d\gamma}{dt}, \frac{d(\gamma \underline{\beta})}{dt} \right) = c\, q\, \gamma \left(c^{-1} \underline{E} \cdot \underline{\beta}\,,\, c^{-1} \underline{E} + \underline{\beta} \times \underline{B} \right)$$

Hence

$$m_0 \left(\frac{d\gamma}{dt}, \frac{d(\gamma \underline{\beta})}{dt} \right) = q\, \left(0, \underline{\beta} \times \underline{B}\right)$$

so γ is constant and

$$\frac{d\underline{v}}{dt} = \frac{q}{m_0 \gamma} \underline{v} \times \underline{B}$$

- I.e. the particle undergoes a helical motion in the direction of the magnetic field, z. Let ρ be the radius of the rotation in (x, y) plane and $v_\perp$ be the its speed in the (x, y) plane. Then

$$v_\perp m_0 \gamma = q \rho B$$

- The quantity $B\rho$ is called the magnetic rigidity.

Synchrotrons

- set ω as the angular velocity, $v_\perp = \omega\,\rho$ hence

$$\omega m_0 \gamma = q\,B$$

- Explain why a particle in a synchrotron takes longer to go round when its energy increase.
- Explain briefly how a synchrotron works.

Potentials

- We can automatically solve two of Maxwell's equation by introducing a 4-potential $\boldsymbol{A} = (c^{-1}\phi, \underline{A})$.
  Set

$$\underline{E} = -\nabla\phi - \partial_t \underline{A}\,, \qquad \underline{B} = \nabla \times \underline{A}$$

- Show that two of Maxwell's equation are satisfied.
- Write down the remaining two.
- What is the gauge freedom?
- Lorentz gauge condition:

$$c^{-2}\partial_t\phi + \nabla \cdot \underline{A} = 0$$

Relativistic Maxwell

- It turns out that Maxwell's equations are fully relativistic.
- This is not a coincidence. Einstein was motivated to find relativity by looking at Maxwell's equations (despite what you may have heard about the aether and the Michelson-Morley experiment.)
- The Lorentz transformation of the electromagnet field is given by

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel}\,, \qquad \underline{E}'_{\perp} = \gamma\big(\underline{E}_{\perp} + c\,\underline{\beta} \times \underline{B}\big)\,,$$
$$\underline{B}'_{\parallel} = \underline{B}_{\parallel}\,, \qquad \underline{B}'_{\perp} = \gamma\big(\underline{B}_{\perp} - c^{-1}\underline{\beta} \times \underline{E}\big)$$

Relativistic Transformation of Fields

- Recall

$$t = \gamma(t' + \beta\, x')\,, \quad x = \gamma(x' + \beta\, t')\,, y = y'\,, z = z'$$

- Lorentz transformation of $\boldsymbol{A} = (c^{-1}\phi, \underline{A})$

$$c^{-1}\phi' = \gamma(c^{-1}\phi - \beta\, A_x)\,, \quad A'_x = \gamma(A_x - \beta\, c^{-1}\phi)\,,$$
$$A'_y = A_y \quad \text{and} \quad A'_z = A_z$$

- Then

$$\underline{B}' = \nabla' \times \underline{A}'$$

implies

$$\begin{aligned} B'_z &= \frac{\partial A'_y}{\partial x'} - \frac{\partial A'_x}{\partial y'} = \frac{\partial A_y}{\partial x'} - \frac{\partial}{\partial y'}\big(\gamma(A_x - \beta\, c^{-1}\phi)\big) \\ &= \frac{\partial A_y}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial A_y}{\partial t}\frac{\partial t}{\partial x'} - \frac{\partial}{\partial y}\big(\gamma(A_x - \beta\, c^{-1}\phi)\big) \\ &= \gamma\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial y} + \beta\left(\frac{\partial A_y}{\partial t} + c^{-1}\frac{\partial \phi}{\partial y}\right)\right) = \gamma\Big(B_z + c^{-1}\beta\, E_y\Big) \end{aligned}$$

Relativistic electromagnetism

- What is the 4-vector like quantity which encodes $(\underline{E}, \underline{B})$?
- We introduce the matrix

$$F = \begin{pmatrix} 0 & c^{-1}E_1 & c^{-1}E_2 & c^{-1}E_3 \\ -c^{-1}E_1 & 0 & -B_z & B_y \\ -c^{-1}E_2 & B_z & 0 & -B_x \\ -c^{-1}E_3 & -B_y & B_x & 0 \end{pmatrix}$$

- The vacuum matrix equations can now be written

$$\frac{\partial F_{\mu\nu}}{\partial x^\rho} + \frac{\partial F_{\nu\rho}}{\partial x^\mu} + \frac{\partial F_{\rho\mu}}{\partial x^\nu} = 0 \qquad \text{and} \qquad \sum_{\nu=0}^{3} \frac{\partial F_{\mu\nu}}{\partial x^\nu} = J_\mu$$

where $J_\mu = (-c\rho, \underline{J})$ is the 4-current. The Lorentz force is

$$f_\mu = \sum_{\nu=0}^{3} \beta^\nu \, F_{\nu\mu}$$

- The Lorentz transformation of F is given by

$$F'_{\mu\nu} = \sum_{\sigma,\rho=0}^{3} \Lambda^\rho{}_\mu \, \Lambda^\sigma{}_\nu \, F_{\rho\sigma}$$

Phase velocity and group velocity in a waveguide.

- Explain why there is a dispersion relation of the form

$$\omega^2 - c^2 k^2 = \omega_0^2$$

  Draw dispersion plot.
- What is the phase velocity and why is it faster than light?
- What is the group velocity?

Radiation from a moving source.

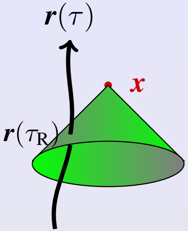
- Consider a moving particle with position on spacetime given by $\boldsymbol{r}(\tau)$.
- For each point $\boldsymbol{x}$ in spacetime there is a unique τ_{R} such that $\boldsymbol{r}(\tau_{\mathrm{R}})$ is connected to $\boldsymbol{x}$ via a future pointing lightlike curve. I.e.

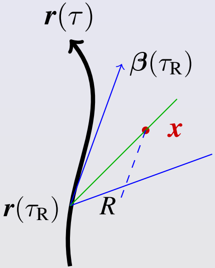
$$\big(\boldsymbol{x} - \boldsymbol{r}(\tau_{\mathrm{R}})\big) : \big(\boldsymbol{x} - \boldsymbol{r}(\tau_{\mathrm{R}})\big) = 0$$

  This is known as the retarded time. It depends on $\boldsymbol{x}$ so we can write $\tau_{\mathrm{R}}(\boldsymbol{x})$.
- The potential due to the curve $\boldsymbol{r}(\tau)$ is given by the Liénard-Wiechart Potential

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{q\,\boldsymbol{\beta}(\tau_{\mathrm{R}})}{\big(\boldsymbol{x} - \boldsymbol{r}(\tau_{\mathrm{R}})\big) : \boldsymbol{\beta}(\tau_{\mathrm{R}})} = \frac{q\,\boldsymbol{\beta}(\tau_{\mathrm{R}})}{R}$$

  where $\boldsymbol{\beta}(\tau) = \frac{d}{d\tau}\boldsymbol{r}(\tau)$ and R is the spatial distance between $\boldsymbol{x}$ and the particle in the rest frame of the particle at the retarded time.





Radiation from a moving source.

- We can directly calculate the corresponding electromagnetic fields.

$$\underline{E} = \left(\frac{q(\underline{n} - \underline{\beta})}{\gamma^2 (1 - \underline{n} \cdot \underline{\beta})^3 R^2} + \frac{q\underline{n} \times \left((\underline{n} - \underline{\beta}) \times \underline{\dot{\beta}} \right)}{c(1 - \underline{n} \cdot \underline{\beta})^3 R} \right) \Bigg|_{\tau_R}$$

$$\underline{B} = \underline{n} \times \underline{E}$$

where

$$R = \|\underline{x} - \underline{r}\| \qquad \text{and} \qquad \underline{n} = \frac{\underline{x} - \underline{r}}{R}$$

- We see there are two terms, the first is called the coulomb term as it falls off as $1/\|\underline{x} - \underline{r}\|^2$. The second is called the radiation term as it falls of as $1/\|\underline{x} - \underline{r}\|$. It depends on the acceleration of the particle.

Pictorial description of Electromagnetism.

- Draw Maxwell's equations in 4-dimensions
- Draw Maxwell's equations in 3+0 dimensions
- Draw Maxwell's equations in 2+1 dimensions