Towards an Algebraic Theory of Spin

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Presentation Summary

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- Discuss insights gained by this method.

Deconstructing Non-Relativistic Spin

Familiar representations

s	$\rho(S_x)$	$\rho(S_y)$	$\rho(S_z)$
$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
1	$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$
$\frac{3}{2}$	$\frac{1}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}0 & -i\sqrt{3} & 0 & 0\\ i\sqrt{3} & 0 & -2i & 0\\ 0 & 2i & 0 & -i\sqrt{3}\\ 0 & 0 & i\sqrt{3} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix}$

Deconstructing Non-Relativistic Spin

What lies within

- All spin representations preserve the Lie Bracket:

$$[\rho(S_a), \rho(S_b)] = i \sum_c \varepsilon_{abc} \rho(S_c)$$

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$$S_\rho^2 = \sum_a \rho(S_a) \circ \rho(S_a) = s(s+1) id$$

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- Are these properties enough to describe each spin representation?

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- Are these properties enough to describe each spin representation?
- No.

Deconstructing Non-Relativistic Spin

What we're missing

s	*identity*
$\frac{1}{2}$	$\rho(S_a)\circ\rho(S_b) + \rho(S_b)\circ\rho(S_a) - \frac{1}{2}\delta_{ab} id = 0$
1	$\rho(S_a)\circ\rho(S_b)\circ\rho(S_c) + \rho(S_c)\circ\rho(S_b)\circ\rho(S_a) - \delta_{ab}\rho(S_c) - \delta_{bc}\rho(S_a) = 0$
$\frac{3}{2}$	?

Deconstructing Non-Relativistic Spin

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- If we redefined $S_a \to iS_a$ we could have all of our identities completely real...
- How essential are complex numbers to the story?
- $\mathrm{SU}(2, \mathbb{C})$ is the universal covering group of $\mathrm{SO}(3, \mathbb{R})$, the same geometric symmetry as in classical non-relativistic physics...

Deconstructing Non-Relativistic Spin

While we're at it...

- If we redefined $S_a \rightarrow iS_a$ we could have all of our identities completely real...
- How essential are complex numbers to the story?
- $\mathrm{SU}(2, \mathbb{C})$ is the universal covering group of $\mathrm{SO}(3, \mathbb{R})$, the same geometric symmetry as in classical non-relativistic physics...
- How essential is quantum mechanics to the story?

Towards Algebraic Spin...

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How can we understand spin without spinors?

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How can we understand spin without spinors?

By studying the algebraic structure of the spin generators.

Towards Algebraic Spin...

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Towards Algebraic Spin...

What stands in our way?

- We wish to proceed with as few assumed structures as possible.
- No complex numbers $\rightarrow$ No guarantee of eigenvalues.
- No spinors $\rightarrow$ No matrices.
- No quantum mechanics $\rightarrow$ No calculus.
- Capture all spins $\rightarrow$ Initially dealing with an infinite-dimensional structure.

Towards Algebraic Spin...

Summary of results (to be published!)

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- **All** spin algebras are derivable without complex numbers or quantum mechanics.
- Each spin is really a collection of fundamental non-zero (non-commutative) multipole moments, which emerge naturally and with central importance!
- We expect these multipole moments to be dynamical.
- A closer connection to the real physical symmetry $\mathrm{SO}(3, \mathbb{R})$ is maintained.

...and Beyond

- Relativistic spin from $\mathrm{SO}(3, 1, \mathbb{R})$.

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- Explore conformal symmetries $\mathrm{SO}(p+1, q+1, \mathbb{R})$.
- Symplectic (dynamical) symmetries $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(2p, 2q, \mathbb{R})$.
- Since $\mathrm{U}(n, \mathbb{C}) = \mathrm{O}(2n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R}) \rightarrow$ Towards a completely algebraic theory of quantum mechanics!

Thank you for your attention!

Any questions?

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