



# Useful things to know about accelerators – part I

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C. T. Rogers

ISIS

Rutherford Appleton Laboratory



**Science and  
Technology  
Facilities Council**



# Accelerators - A Window on Nature

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- Particle accelerators provide the source for most high energy physics experiments
  - Provide high luminosity, high energy beams for colliders
  - Provide high brightness beams for secondary particle production
  - Also key technology for life sciences, engineering, chemistry
- How do they work?
  - How can we get to high energy?
  - How can we keep the beam in the accelerator?
  - How can we get to high luminosity?
- What are the main HEP facilities in the world today?
- What might HEP facilities look like in the future?





# Accelerator Components

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- Most accelerators share similar components
- Main components of an accelerator
  - Bending – dipoles
  - Focussing – quadrupoles
  - Acceleration - RF cavities
- Also
  - Vacuum
  - Diagnostics
  - Targets for secondary particle production
- First Lecture: Derive basic theory of accelerator physics
- Second Lecture: Discuss accelerator equipment and techniques

# Lorentz force law

- Fundamental equation for particles moving through fields

The diagram illustrates the Lorentz force law equation,  $\vec{F} = q \vec{v} \times \vec{B} + q \vec{E}$ , with arrows pointing from labeled boxes to the corresponding terms in the equation:

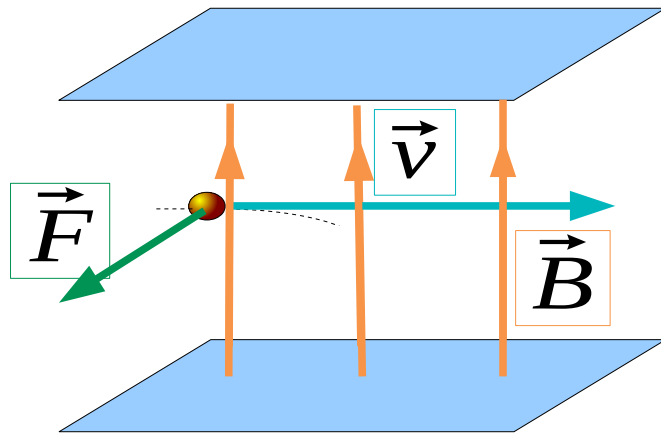
- Force (green box) points to  $\vec{F}$
- Charge (cyan box) points to  $q$
- Velocity (red box) points to  $\vec{v}$
- Magnetic Field (purple box) points to  $\vec{B}$
- Electric Field (orange box) points to  $\vec{E}$

(eq. 1)

- **Magnetic force** is perpendicular to velocity
  - Magnetic field conserves energy
- **Electric force** is weaker by factor velocity
  - Magnets are better for bending and focussing

# Magnetic Rigidity and Bending

- Simplest magnet - “dipole”
  - Uniform magnetic field perpendicular to beam direction



Lorentz force (eq. 1) + centripetal motion:

$$qvB = \frac{pv}{\rho}$$

Radius

Rearranging:

$$B\rho = \frac{p}{q}$$

Magnetic Rigidity

- Constant force  $\rightarrow$  constant curvature  $\rightarrow$  circular motion
- Magnetic rigidity parameterises momentum
- Charge-to-mass ratio important when accelerating multiple particle species

# Worked example – LHC

- If we wanted to accelerate, say, 7 TeV/c particles, what bending radius is required?
- Maximum dipole field around 8.3 T

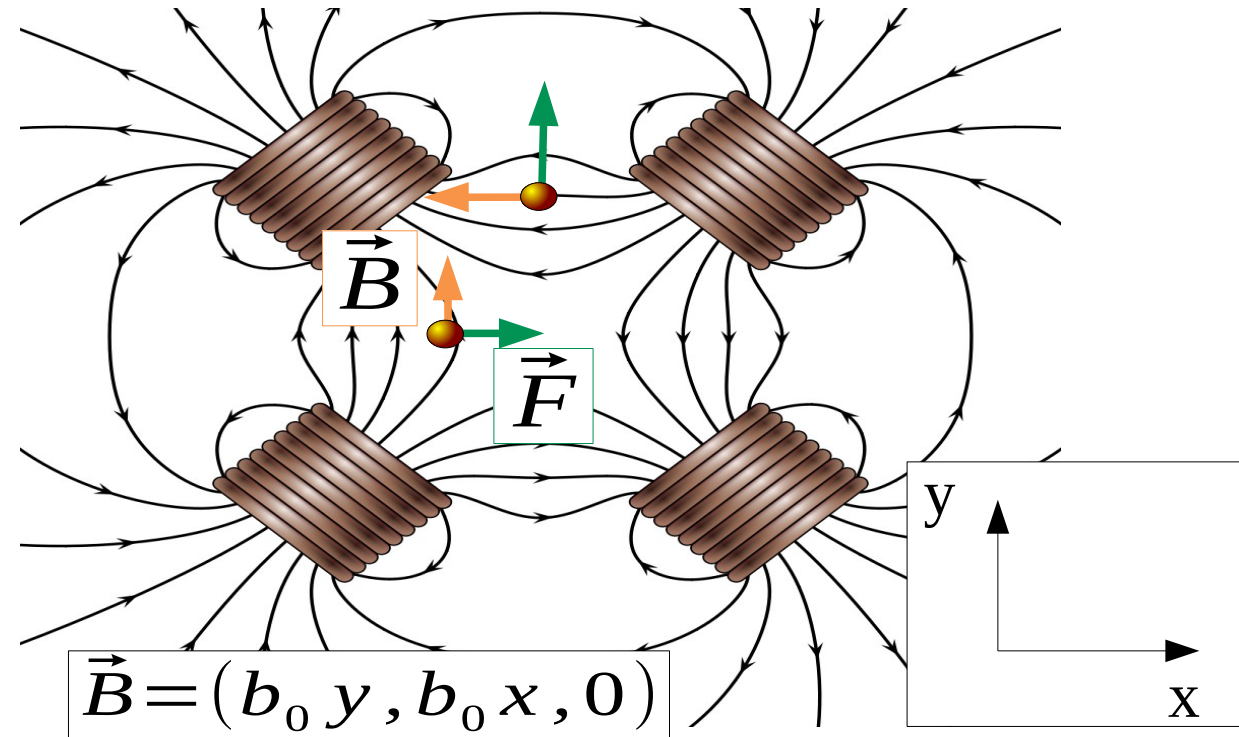
$$B \rho = \frac{p}{q}$$

$$\rho = \frac{p}{q B} = \frac{7e12}{8.3 \times 3e8} = 2800 \text{ m}$$

- Nb: LHC radius ~ 4.1 km
  - Need space for detectors, etc

# Quadrupole magnets

- If we only had bending magnets, particles would soon be lost from the accelerator
- Need to keep the particles in the accelerator using focussing elements
  - Usually use quadrupoles
- Field stronger away from beam centre
  - Like a spring or pendulum
  - Simple harmonic motion
- “F” quad focuses in x and defocuses in y
- “D” quad focuses in y and defocuses in x
- Overall focussing by alternating “F” and “D”
  - Just reverse the field



# Quadrupole field – horizontal (1)

- For a particle moving near to the z-axis

$$\vec{F} = q \vec{v} \times \vec{B} + q \vec{E}$$

- Considering only  $p_x$  for now

$$\frac{dp_x}{dt} = -q \frac{dz}{dt} B_y$$

- Use the chain rule

$$\frac{dp_x}{dt} = \frac{dp_x}{dz} \frac{dz}{dt}$$

- Combining these equations:

$$\frac{dp_x}{dz} = -q b_0 x$$

$$\vec{B} = (b_0 y, b_0 x, 0)$$

$$\vec{v} \times \vec{B} = \begin{pmatrix} -v_z B_y \\ v_z B_x \\ 0 \end{pmatrix}$$

## Quadrupole field – horizontal (2)

$$\frac{dp_x}{dz} = -q b_0 x \quad \text{😊}$$

- Definition of x-component of momentum

$$p_x = m \gamma v_x = m \gamma \frac{dz}{dt} \frac{dx}{dz} = p_z \frac{dx}{dz}$$

- Substitute this definition into 😊 gives

$$p_z \frac{d^2 x}{dz^2} = -q b_0 x$$

- Rearrange and wrap up constant terms in focussing strength  $k$

$$\frac{d^2 x}{dz^2} + k x = 0$$

# Quadrupole field – vertical

- Lorentz force law with quadrupole field definition

$$\frac{dp_y}{dt} = q b_0 v_z y$$

- Use chain rule and eliminate  $v_z$

$$p_z \frac{d^2 y}{dz^2} = q b_0 y$$

- Rearrange and wrap up constant terms in defocussing strength  $k$

$$\frac{d^2 y}{dz^2} - k y = 0$$

# Solutions

- Motion is governed by

$$\frac{d^2 x}{dz^2} + k x = 0 \qquad \frac{d^2 y}{dz^2} - k y = 0$$

- This is simple harmonic motion - solutions are of form

$$x = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \frac{1}{\sqrt{k}} \sin(\sqrt{k} z)$$

- Taking derivative

$$\frac{dx}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \cos(\sqrt{k} z)$$

For y

$$y = y_0 \cosh(\sqrt{k} z) + \frac{dy_0}{dz} \frac{1}{\sqrt{k}} \sinh(\sqrt{k} z)$$

$$\frac{dy}{dz} = y_0 \sqrt{k} \sinh(\sqrt{k} z) + \frac{dy_0}{dz} \cosh(\sqrt{k} z)$$

# Transfer Matrix

- Just thinking about  $x$ , the particles move according to

$$x_1 = x_0 \cos(\sqrt{k} z) + \frac{dx_0}{dz} \sin(\sqrt{k} z)$$
$$\frac{dx_1}{dz} = -x_0 \sqrt{k} \sin(\sqrt{k} z) + \frac{dx_0}{dz} \sqrt{k} \cos(\sqrt{k} z)$$

- We can rewrite this as a matrix

$$\begin{pmatrix} x_1 \\ \frac{dx_1}{dz} \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k} z) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} z) \\ -\sqrt{k} \sin(\sqrt{k} z) & \cos(\sqrt{k} z) \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$

- This matrix is known as the quadrupole's **transfer matrix**

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$



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# Questions



# Questions

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- Exercise – what is the transfer matrix for a drift space, that is a region with no fields at all?
  - What is the force acting on the particle?
  - What is  $x(z)$  in terms of  $dx_0/dz$  and  $x_0$
  - What is  $dx/dz$  in terms of  $dx_0/dz$
  - Now write that as a matrix

# Questions

- Exercise – what is the transfer matrix for a drift space?

- What is the force acting on the particle?

- No force

- What is  $x(z)$  in terms of  $dx_0/dz$  and  $x_0$

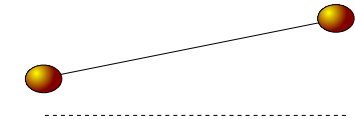
$$x = x_0 + \frac{dx_0}{dz} z$$

- What is  $dx/dz$  in terms of  $dx_0/dz$

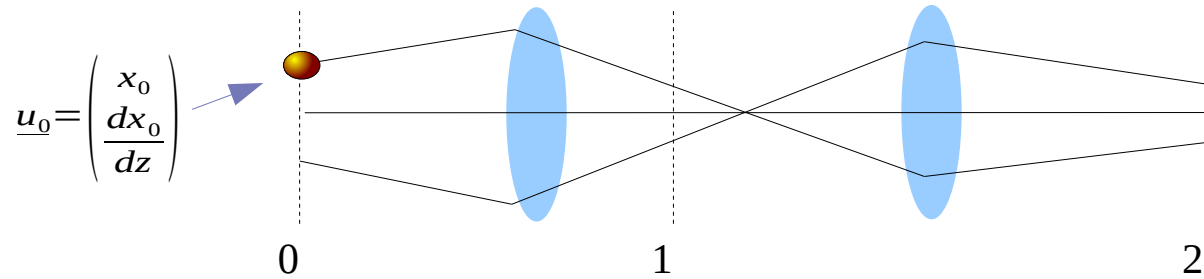
$$\frac{dx}{dz} = \frac{dx_0}{dz}$$

- Now write that as a matrix

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$



# Transfer Lines



- Transfer matrix defines transport through a region
- Transfer matrices can be combined by multiplication
- Say we have transfer matrices like:

$$\underline{u}_1 = \mathbf{M}_{01} \underline{u}_0$$

$$\underline{u}_2 = \mathbf{M}_{12} \underline{u}_1$$

$$\underline{u}_2 = \mathbf{M}_{12} \mathbf{M}_{01} \underline{u}_0$$

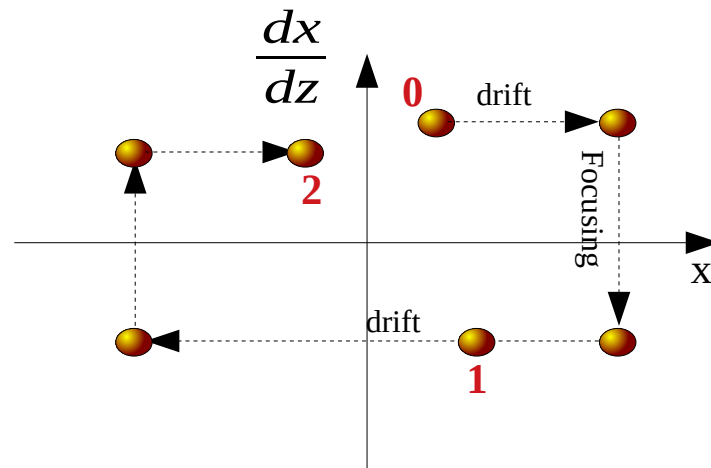
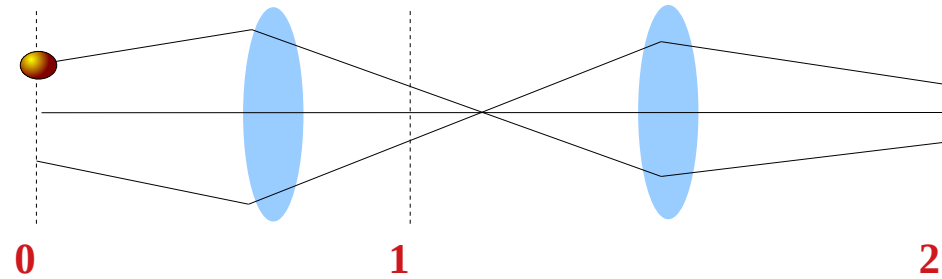
- Then

$$\mathbf{M}_{02} = \mathbf{M}_{12} \mathbf{M}_{01}$$

- i.e. we can define a combined transfer matrix like

# Phase space

- Another instructive way to look at beam optics is by considering the phase space



- There is a general rule for what transfer matrices are allowed by equations of motion
  - “Symplectic condition”
- Formally a matrix  $\mathbf{M}$  is *symplectic* if it satisfies

$$\mathbf{M}^{\text{T}} \mathbf{S} \mathbf{M} = \mathbf{S}$$

- Where

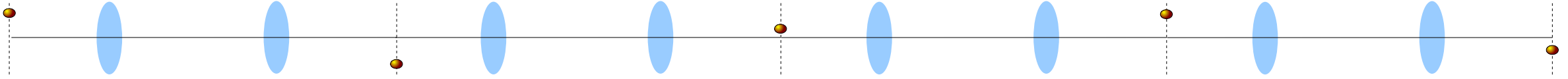
$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- It can be shown that any *symplectic* matrix  $\mathbf{M}$  can be written as

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & \alpha \end{pmatrix} \text{ with } \gamma\beta - \alpha^2 = 1 \text{ and } \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Periodic Lattices



- Following  $n$  identical cells or turns in a ring with one-turn matrix  $\mathbf{M}$

$$\underline{u}_n = \mathbf{M}^n \underline{u}_0$$

- Rewrite

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

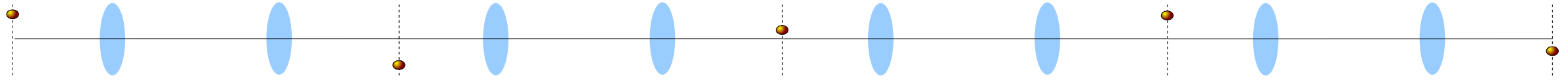
- So

$$\mathbf{J}^2 = -\mathbf{I}$$

- And

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

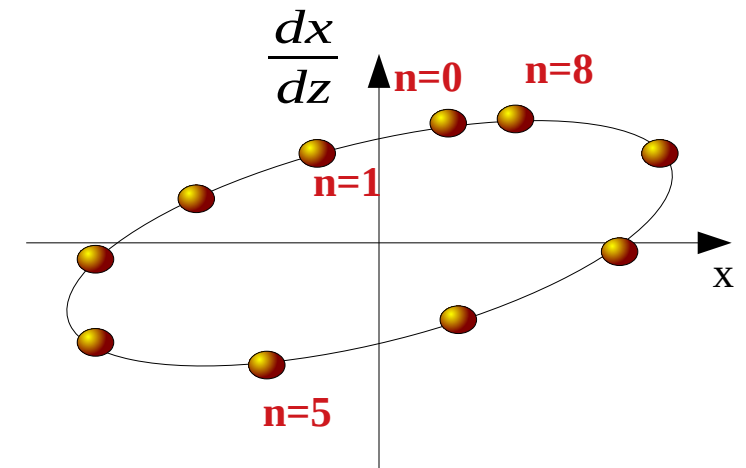
# Periodic Lattices



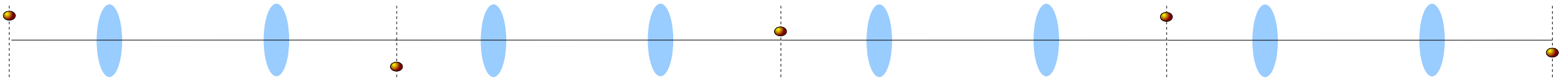
- What does this mean?

$$\mathbf{M}^n = \mathbf{I} \cos(n\mu) + \mathbf{J} \sin(n\mu)$$

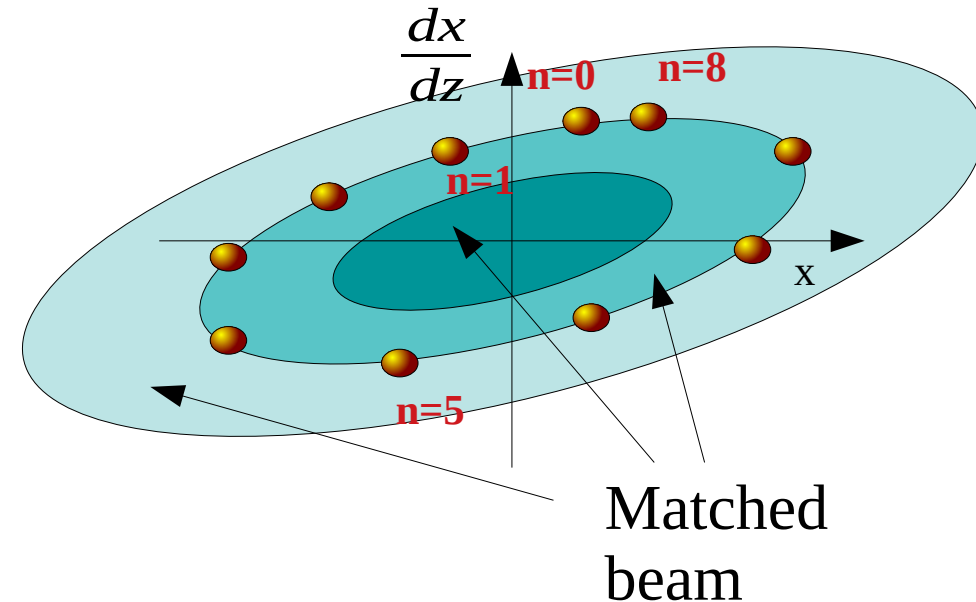
- Particles move around an ellipse in phase space if  $\text{Trace}(\mathbf{M}) < 2$
- $\mu$  is the “phase advance”
  - Sometimes use “tune” ...  $2\pi\nu = \mu$
- $\alpha$ ,  $\beta$  and  $\gamma$  are “Twiss parameters”
  - Tell us the alignment of the ellipse
- Each particle sits on ellipse area  $\varepsilon$  - the particle’s amplitude



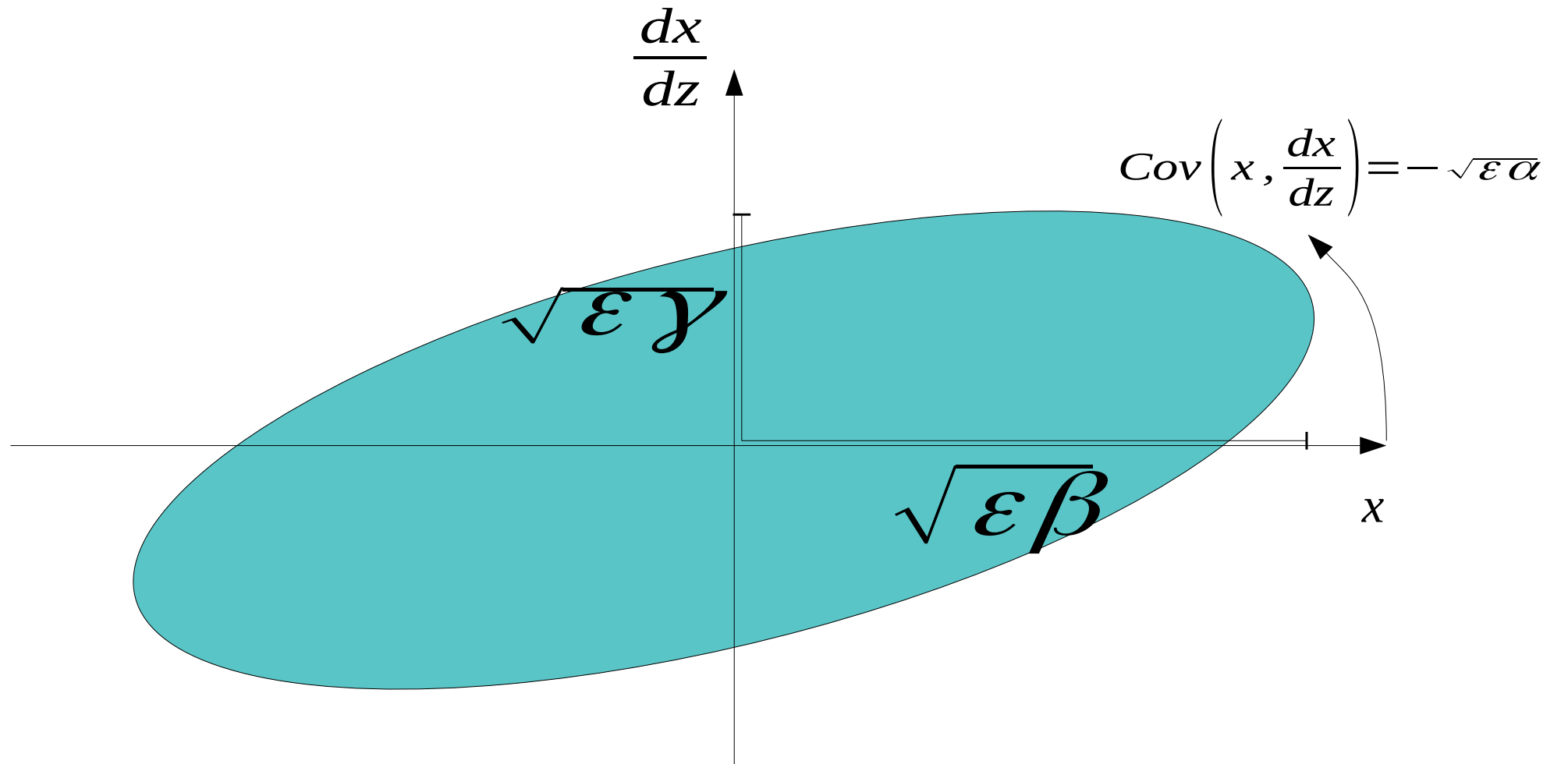
# Periodic Lattices and beams



- Beam is composed of many particles
  - Particles occupy a region in phase space
- “Emittance” is area occupied by the entire beam
- Sometimes classify “RMS emittance”
  - Area occupied by ellipse 1 RMS distance from beam centre
- Low emittance is crucial for
  - High luminosity
  - Low losses



# Beam ellipse





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# Questions



# Questions

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- What is behaviour of particles in phase space if
  - $\text{Trace}(M) < 2$
  - $\text{Trace}(M) = 2$
  - $\text{Trace}(M) > 2$



# Questions

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- What is behaviour of particles in phase space if
  - $\text{Trace}(M) < 2$ 
    - Motion is an ellipse
  - $\text{Trace}(M) = 2$ 
    - $x \rightarrow +/- x$
  - $\text{Trace}(M) > 2$ 
    - Motion is a hyperbola

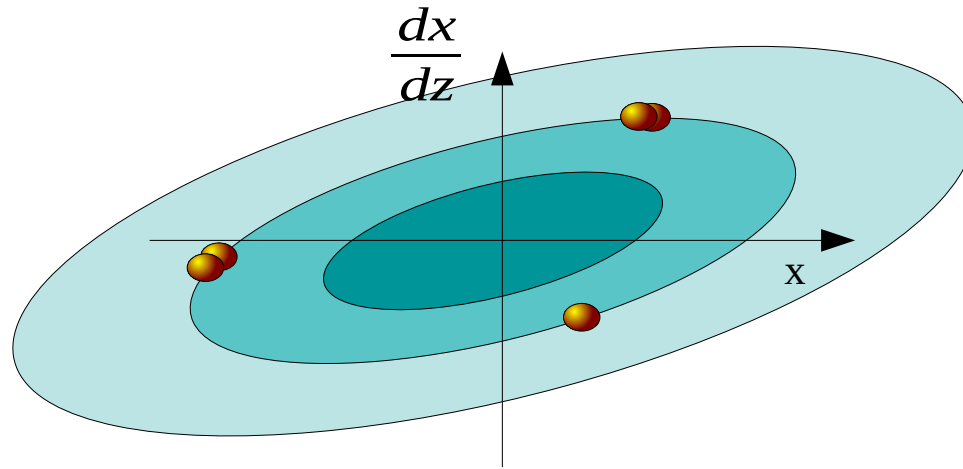


# Emittance Growth

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- Ideally emittance is conserved, but this is not always the case
- Long list of effects that can cause emittance growth
  - Beam mismatch
  - Scattering off residual gas
  - Scattering off particles in the same beam
  - Scattering off particles in other beams (e.g. in collider)
  - Space charge
  - Resonances

# Resonances



- Reminder:-
  - Tune  $\nu$  is number of SHM oscillations per turn
  - Phase advance  $\mu = 2\pi\nu$  is "angle" advanced per turn
- The beam does not behave well when
$$\nu_x = l + \frac{m}{n}$$

*integer*

  - Beam passes through the same field region every  $n^{\text{th}}$  turn
  - Imperfections in the field get amplified
- Resonance



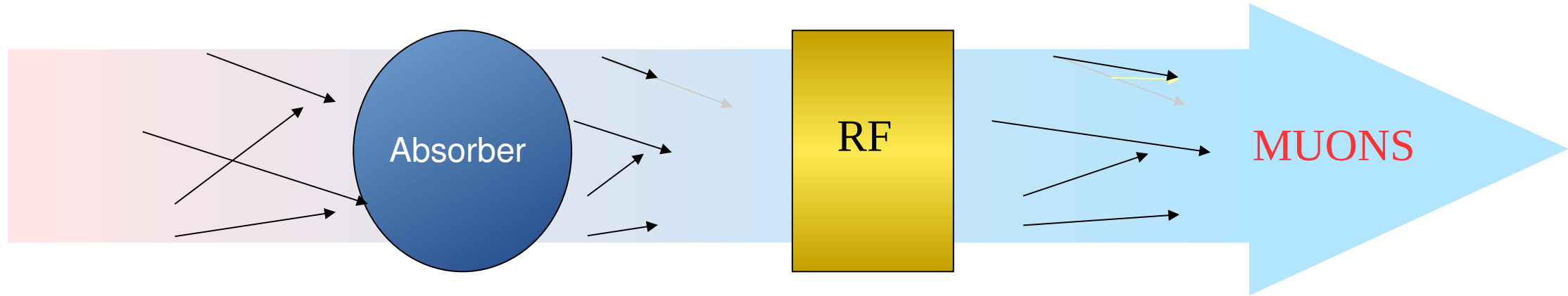


# Emittance Reduction (Cooling)

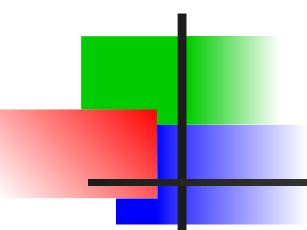
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- Several techniques to reduce emittance
  - Synchrotron radiation cooling
  - Stochastic cooling
  - Laser cooling
  - Electron cooling
  - Ionisation cooling
- Fundamental principle is to remove “heat” from the beam using a neighbouring heat sink
  - Comoving electron beam → electron cooling
  - Comoving laser → laser cooling
  - Emission of synchrotron radiation
    - Photon emission caused by (principally) electrons bending in magnetic field

# E.g. Ionisation Cooling



- Beam loses energy in absorbing material
  - Absorber removes momentum in all directions
  - RF cavity replaces momentum only in longitudinal direction
  - End up with beam that is more straight
- Multiple Coulomb scattering from nucleus ruins the effect
  - Mitigate with tight focussing
  - Mitigate with low-Z materials
  - Equilibrium emittance where MCS completely cancels the cooling



# Longitudinal Dynamics and Acceleration

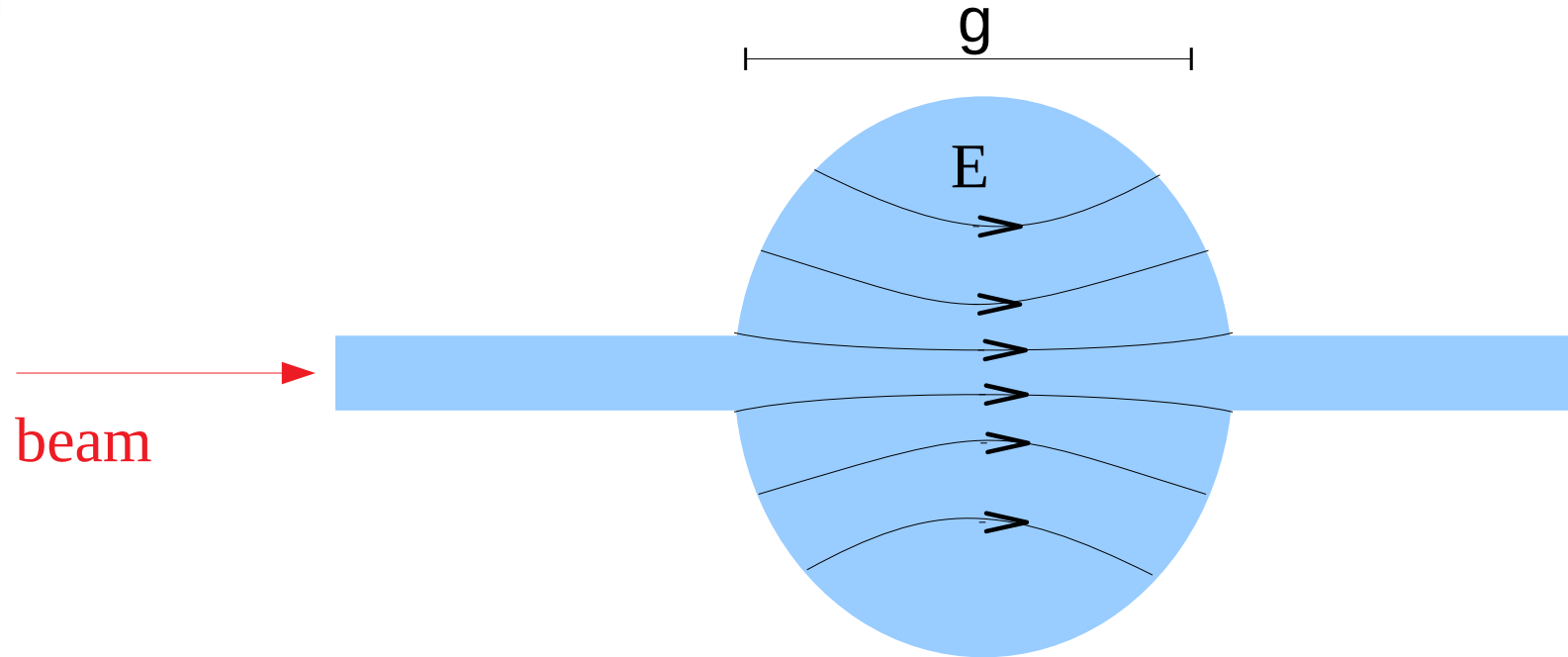


# Longitudinal Dynamics

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- So much for transverse motion (i.e.  $x$  and  $y$  planes)
- What about energy and acceleration?
- Electrostatic acceleration limited by breakdown potential
  - Change in energy is given by voltage differential
  - High voltage differentials cause breakdown (sparks)
  - Practically limits electrostatic acceleration to few MeV
- To accelerate beyond MeV require oscillating electric field
- RF Cavities

# RF cavity field



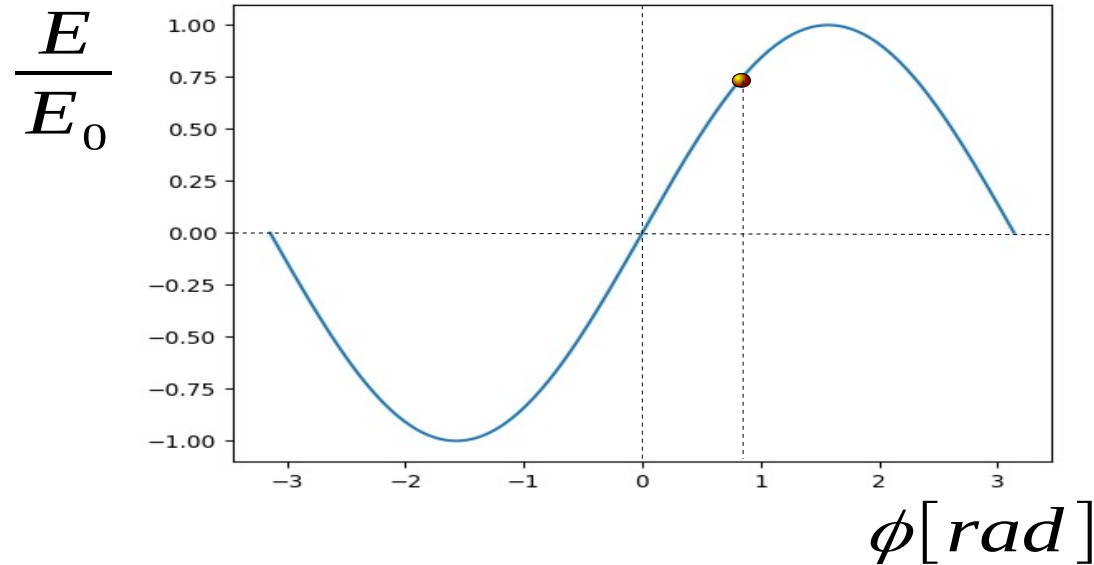
- RF cavity holds a resonating EM wave

- Recall Lorentz force law

$$\vec{F} = q \vec{v} \times \vec{B} + q \vec{E}$$

- Force is in direction of motion - energy changes!

# RF cavity field



- In RF cavity

$$\vec{E} = E_0 \sin(\omega t + \phi) \hat{z}$$

- Energy change of particle crossing at  $\phi$

$$\delta W = \int F dz = q g E_0 \sin(\phi)$$

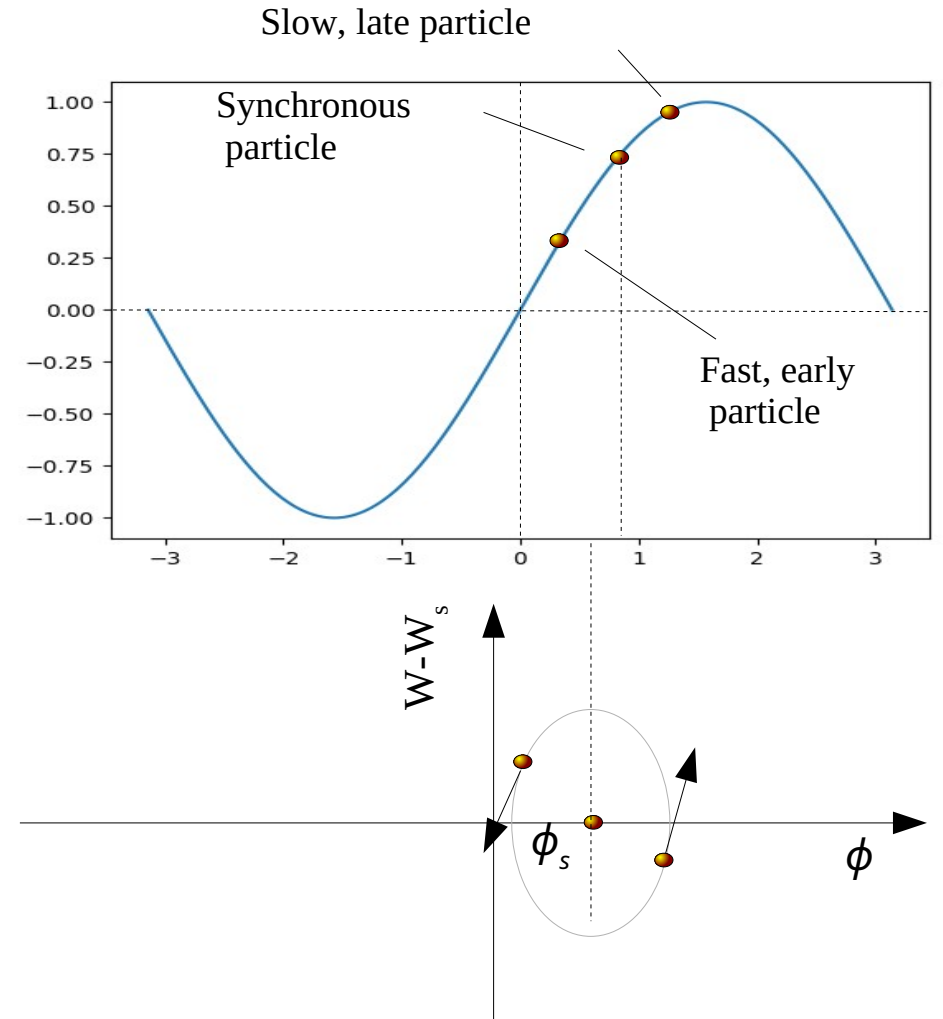
- $g$  is the gap length
- Assumes gap is short so electric field doesn't change much
  - For longer gaps, can introduce an effective gap length  $g T$
  - $T$  is the "transit time factor" → reduces the effective gap length

# Phase stability

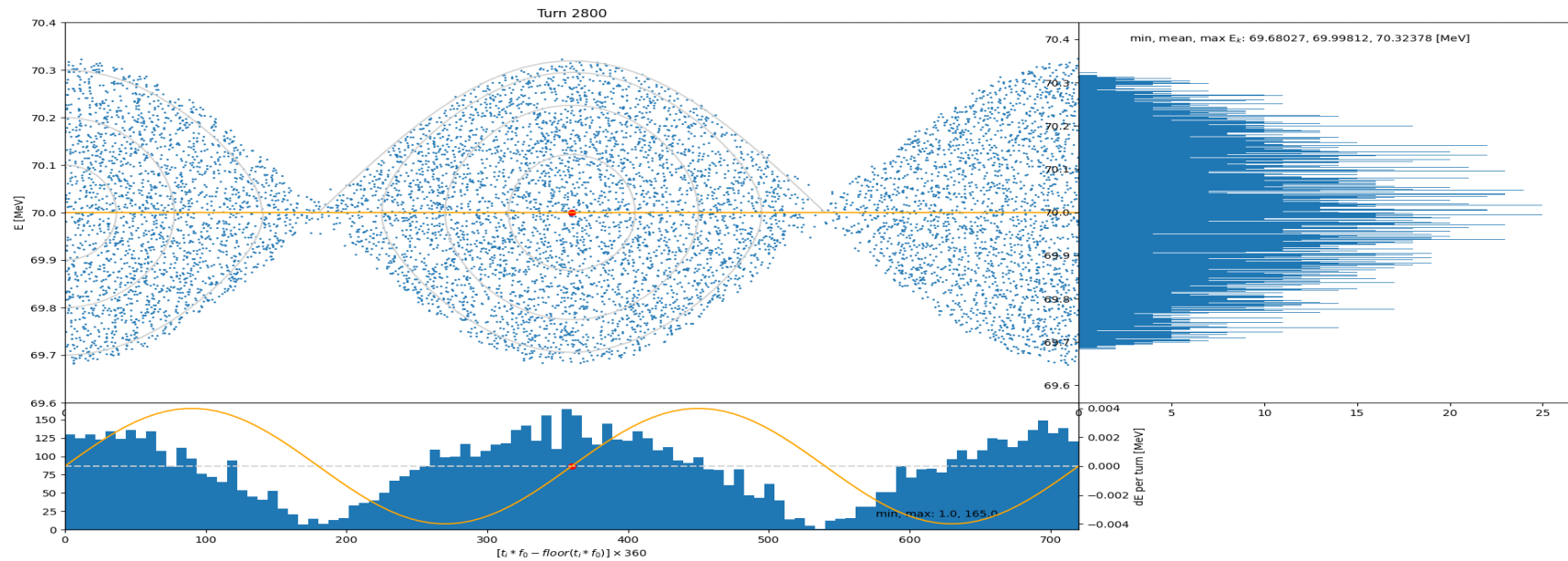
- Phase cavities so that a “synchronous” particle always crosses at phase  $\phi_s$
- Particle crossing at phase  $\phi$  relative to synchronous particle

$$\delta W = q g T E_0 \sin(\phi + \phi_s)$$

- Particle arriving early
  - Fast  $\rightarrow$  t negative
  - Gets smaller energy kick
  - Ends up relatively slower
- Particle arriving late
  - Slow  $\rightarrow$  t positive
  - Gets bigger energy kick
  - Ends up relatively faster
- Phase stability!



# RF Bucket



- RF Bucket – region where phase is stable
- Harmonic number – number of RF oscillations per ring turn

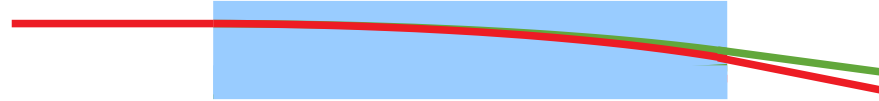


# Dealing with momentum spread

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- Momentum spread introduces a few effects
  - Dispersion
  - Chromaticity
  - Momentum compaction
- Dispersion:
  - Off-momentum particles follow a different trajectory
- Momentum compaction (rings):
  - Different path length yields different time of flight
- Chromaticity:
  - Off-momentum particles get a different focussing strength

# Dispersion



- Recall the definition of magnetic rigidity

$$B \rho = \frac{p}{q}$$

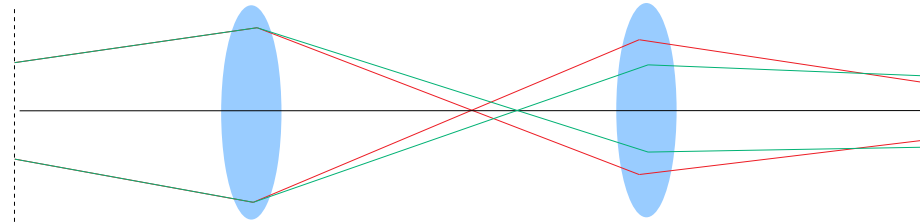
- Particles having different momentum ( $p$ ) get different radius of curvature
  - Introduce dispersion  $D$

$$D = p \frac{dx}{dp}$$

- Which is another optical function that we must make periodic

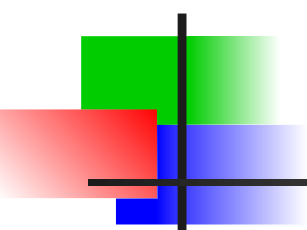
# Chromaticity

- Chromaticity arises because quadrupoles focus differently for different momenta



$$k = q \frac{b_0}{p}$$

- This often limits the degree of focussing at a collision point
  - Limits luminosity
- Can deliberately enhance/reduce chromaticity by
  - Introduce a dispersion
  - Using a magnet with variable focussing strength across the aperture - “sextupole”



# Questions

# Review

- Dipoles are used to bend a beam - rigidity is  $B\rho = \frac{p}{q}$
- Quadrupoles are used to focus a beam:  $k = q \frac{b_0}{p}$
- Beam in each of  $x$  and  $y$  can be characterised by 3 Twiss parameters and an emittance
- Lattices can be characterised by a phase advance
- RF cavities are used to accelerate the beam
- Introducing momentum spread, one can also define a dispersion (and its derivative with respect to  $z$ )

# Finally... luminosity

- Luminosity defines the number of interactions in a collider per unit time for a given cross section
- Luminosity will increase if
  - Beam is narrower
  - Current is higher

Number of particles  
in each bunch

Revolution  
frequency

Number of  
bunches

$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$

Width of  
Each bunch



# What dictates luminosity?

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$$\tilde{L} = \frac{N_1 N_2 f N_b}{4 \pi \sigma_x \sigma_y}$$

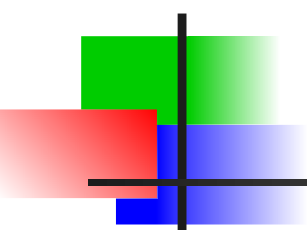
- Typically
  - Number of particles → space charge
  - Revolution frequency → ring circumference
  - Number of bunches → RF frequency
  - Beam width →  $\sqrt{\epsilon \beta}$ 
    - Emittance (cooling?)
    - Twiss beta (final focus and chromaticity)



# Next lecture...

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- Accelerator equipment
- Types of accelerator
- Current facilities
- Future facilities



# Backup

# Transverse Space Charge 1

- Consider a circular beam of radius  $a$  having uniform density

$$\rho(r) = q \frac{I}{\beta_{rel} c \pi a^2} \quad r < a$$

- Quote field around a cylinder of charge/current

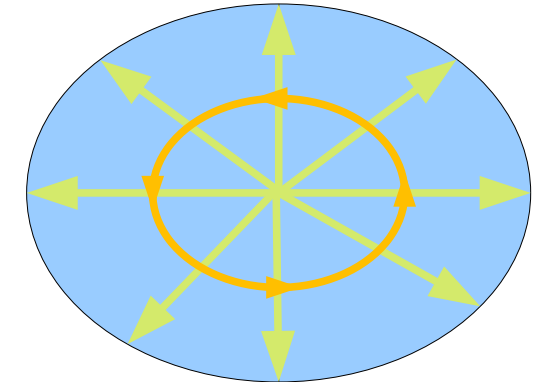
$$E(r) = \frac{1}{2 \pi \epsilon_0} \frac{I}{\beta_{rel} c} \frac{r}{a^2}$$

$$B_\phi(r) = \frac{1}{2 \pi \epsilon_0} \frac{I}{c^2} \frac{r}{a^2}$$

- Apply Lorentz force law

$$\vec{F} = q \vec{v} \times \vec{B} + q \vec{E}$$

$$F_r = q \vec{v} \times \vec{B} + q \vec{E} = \frac{1}{2 \pi \epsilon_0} \frac{r}{a^2} \left( \frac{I}{\beta_{rel} c} - \frac{I}{\beta_{rel} c} \beta_{rel}^2 \right) = \frac{1}{2 \pi \epsilon_0} \frac{r}{a^2} \left( \frac{I}{\gamma^2 \beta_{rel} c} \right)$$



# Transverse Space Charge 2

- Force is defocusing

$$\frac{d^2 x}{dz^2} - (k - K_{sc}) x = 0 \quad \text{with} \quad K_{sc} = \frac{1}{2 \pi \epsilon_0} \frac{1}{a^2} \left( \frac{I}{\gamma_{rel}^3 \beta_{rel}^2 c} \right)$$

- Treat SC as a perturbation

$$\mathbf{M}_p = \mathbf{M} \mathbf{M}_{sc}$$

$$\mathbf{M} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu$$

$$\mathbf{M}_{sc} = \begin{pmatrix} 1 & 0 \\ -K_{sc} & 1 \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

- Change of beam size ( $\beta$ )

- Change of phase advance**

- Drive the beam onto resonances  $\rightarrow$  ruin the acceptance

- Phase advance  $\rightarrow$  look at Trace of  $\mathbf{M}_p$

$$\text{Tr}(\mathbf{M}_p) = 2 \cos(\mu) + \alpha \sin(\mu) - \alpha \sin(\mu) + \beta K \sin(\mu)$$

# Transverse Space Charge 3

- Consider just the  $\text{trace}(\mathbf{M}_p)$

$$\text{Tr}(\mathbf{M}_p) = 2 \cos(\mu) + \beta K \sin(\mu)$$

- Consider compound angle formula

$$\cos(\mu + \delta\mu) = \cos(\mu) \cos(\delta\mu) + \sin(\mu) \sin(\delta\mu)$$

$$\cos(\mu + \delta\mu) \simeq \cos(\mu) + \sin(\mu) \sin(\delta\mu)$$

- Looking at the tune

$$\delta\nu = \frac{\delta\mu}{2\pi} = \frac{\beta K}{4\pi}$$

$$K_{sc} = \frac{1}{2\pi\epsilon_0} \frac{1}{a^2} \left( \frac{I}{\gamma_{rel}^3 \beta_{rel}^2 c} \right)$$

$$\delta\nu = \frac{r_0 N}{2\pi\epsilon\beta_{rel}^2 \gamma_{rel}^3}$$

$$\sigma(x) = \sqrt{\beta\epsilon}$$

# Transverse Space Charge 3

