DISPERSION, CHROMATICITY, EMITTANCE

The Cockcroft Institute
of Accelerator Science and Technology

Off-momentum particles

So far we have considered beam motion where all particles have the design momentum p.
We refer to these particles as **on-momentum** particles.

In general, a particle's momentum will be p + Δp

$$p + \Delta p = p(1+\delta) \qquad \delta = \Delta p / p$$

Recall in our solutions to Hill's equation we had

$$\left(\frac{\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho+x} = -\frac{eB_y}{m\gamma v_s}$$

Which we can write as

$$x'' - \frac{\rho+x}{\rho^2} = -\frac{(\rho+x)^2}{\rho^2}\frac{eB_y}{m\gamma v_s}$$

Expanding the right hand side and dropping small terms we obtained

$$x'' - \frac{\rho+x}{\rho^2} = -\frac{eB_y}{m\gamma v_s}\left(1+\frac{2x}{\rho}\right)$$

$$(\rho+x)^2 = \rho^2 + x^2 + 2x\rho$$

The Cockcroft Institute
of Accelerator Science and Technology

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the fractional deviation 'delta'

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the fractional deviation ‘delta’

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

The Cockcroft Institute
of Accelerator Science and Technology

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the fractional deviation 'delta'

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the fractional deviation 'delta'

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

The Cockcroft Institute
of Accelerator Science and Technology

The inhomogeneous equation of motion

Replace the momentum of an on-momentum particle with that of the off-momentum particle expressed in terms of the fractional deviation 'delta'

$$m\gamma v_s = p(1+\delta)$$

Expand the vertical magnetic field and binomially-approximate the momentum

$$B_y \approx B_{y0} + gx \qquad \frac{q}{p(1+\delta)} \approx \frac{q}{p}(1-\delta)$$

Plug these results back into the equation of motion

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

Expand all of the brackets, keeping only terms linear in x and δ, and using

$$\frac{qB_{y0}}{p} = \frac{1}{\rho}$$

we obtain the inhomogeneous equation of motion:

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} \qquad K_x(s) = \frac{g}{B\rho} + \frac{1}{\rho^2}$$

The Cockcroft Institute of Accelerator Science and Technology

Homogeneous and inhomogeneous Hill's equations

Now we see a little more structure to Hill's equation

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

It looks like the homogeneous version (below) apart from a term linear in δ

$$x'' + K_x(s)x = 0$$

The extra term on the RHS will drive the x motion of an off-momentum particle, which we shall call 'horizontal dispersion', or simply ***dispersion***.

Homogeneous and inhomogeneous Hill's equations

Now we see a little more structure to Hill's equation

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

It looks like the homogeneous version (below) apart from a term linear in δ

$$x'' + K_x(s)x = 0$$

The extra term on the RHS will drive the x motion of an off-momentum particle, which we shall call 'horizontal dispersion', or simply ***dispersion***.

The general solution for the horizontal motion of a particle is given by the sum of two terms : the betatron motion term (soln to homogeneous eqn) and an off-momentum dispersion term (soln to inhomogeneous eqn):

$$x(s) = x_h(s) + x_i(s)$$

We can think of $x_i(s)$ as a closed orbit term, around which $x_h(s)$ oscillates.

Let's define a special orbit, D(s), which is followed by a particle with δ=1

$$D(s) = \frac{x_i(s)}{\Delta p/p}$$

The Cockcroft Institute
of Accelerator Science and Technology

Dispersion

Our newly-defined dispersion function D(s):

- Is the orbit of a particle with $\delta=\Delta p/p=1$.
- Obeys Hill's equation.
- Determines the orbit of any (slightly) off-momentum particle

$$x(s) = x_h(s) + D(s)\delta$$

This is similar to a dipole error closed-orbit distortion.

Typical values:

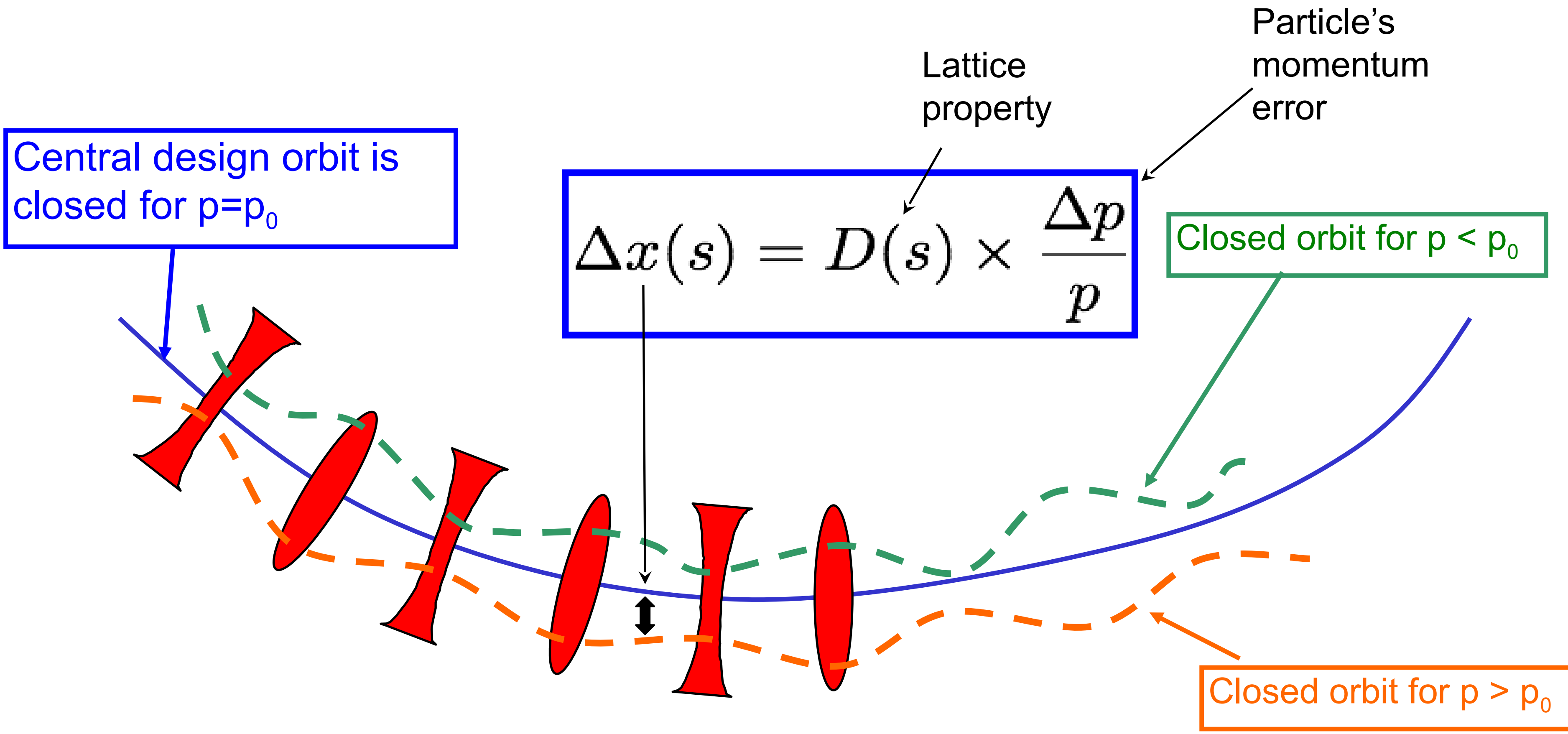
$$x_h \sim 2 - 5\mathrm{mm}$$

$$D(s) \sim 1 - 2\mathrm{m}$$

$$\delta \sim 10^{-3}$$

The Cockcroft Institute
of Accelerator Science and Technology

Dispersion



$$\Delta x(s) = D(s) \times \frac{\Delta p}{p}$$

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

Calculating D(s)

We need to find a solution to the inhomogeneous Hill's equation and add it to the general solution of the homogeneous equation.

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)}$$

For D(s) on the closed orbit we assume the only field is the dipole field. This means that D(s) is a solution of

$$D(s)'' + D(s)\frac{1}{\rho^2} = \frac{1}{\rho}$$

We have already solved the homogeneous equation (below).

$$D''(s) + D(s)\frac{1}{\rho^2} = 0$$

$$M_{\mathrm{dipole,x}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

We only need to find a particular solution of the inhomogeneous equation and add this solution to the solution of the homogeneous equation. If the RHS is just a constant, then a valid choice of a particular solution is also a constant

$$D_p = C = \mathrm{const}$$

Calculating D(s)

$$D(s)'' + D(s)\frac{1}{\rho^2} = \frac{1}{\rho}$$
$$D_p = C = \text{const}$$

Inserting this solution for D into the inhomogeneous equation above immediately gives

$$\frac{C}{\rho^2} = \frac{1}{\rho} \Rightarrow C = \rho$$

And so our general solution for D(s) is

$$D(s) = A\cos(s/\rho) + B\sin(s/\rho) + \rho$$
$$D'(s) = -\frac{A}{\rho}\sin(s/\rho) + \frac{B}{\rho}\cos(s/\rho)$$

The matrix equation for D(s)

As before we determine A and B using the initial conditions at s=0

$$D(0) = D_0 \qquad D'(0) = D'_0$$

Inserting these into our general solution yields

$$A = D_0 - \rho \quad B = \rho D'_0$$

Hence we can write the dispersion function as

$$D(s) = D_0 \cos(s/\rho) + D'_0 \sin(s/\rho) + \rho(1 - \cos(s/\rho))$$
$$D'(s) = -\frac{D_0}{\rho} \sin(s/\rho) + D'_0 \cos(s/\rho) + \sin(s/\rho)$$

Which we can write as a matrix equation

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(s/\rho) & \rho \sin(s/\rho) & \rho(1 - \cos(s/\rho)) \\ -\frac{1}{\rho} \sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

The matrix equation for D(s)

As before we determine A and B using the initial conditions at s=0

$$D(0) = D_0 \qquad D'(0) = D'_0$$

Inserting these into our general solution yields

$$A = D_0 - \rho \quad B = \rho D'_0$$

Hence we can write the dispersion function as

$$D(s) = D_0 \cos(s/\rho) + D'_0 \sin(s/\rho) + \rho(1 - \cos(s/\rho))$$
$$D'(s) = -\frac{D_0}{\rho} \sin(s/\rho) + D'_0 \cos(s/\rho) + \sin(s/\rho)$$

Which we can write as a matrix equation

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(s/\rho) & \rho \sin(s/\rho) & \rho(1 - \cos(s/\rho)) \\ -\frac{1}{\rho} \sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

The Cockcroft Institute of Accelerator Science and Technology

The dispersion

Note the upper-left 2x2 matrix is just the transfer matrix for a dipole we have already derived.

The additional terms in the dipole transfer matrix produce or ‘drive’ the dispersion.

As the motion is given as the sum of the betatron motion and the dispersion

$$x(s) = x_\beta(s) + D(s)\delta$$

We can write the general motion as a matrix equation

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & D \\ m_{21} & m_{22} & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

The Cockcroft Institute of Accelerator Science and Technology

Dispersion in a short 'sector' dipole and a quadrupole

For a short sector dipole with a small bending angle θ

$$\theta = \frac{l}{\rho} \ll 1$$

we can find a simplified matrix from its entrance to its exit

$$\begin{pmatrix} \cos(s/\rho) & \rho\sin(s/\rho) & \rho\,(1-\cos(s/\rho)) \\ -\frac{1}{\rho}\sin(s/\rho) & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & l & l\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

This is useful for quick calculations and corresponds to a thin-lens kick for an off-momentum particle.

In a quadrupole the dispersion function is focussed/defocussed, but there is no driving term for the dispersion and so the 3x3 map is given by

$$\begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion in a FODO cell

Consider a FODO cell with thin lens quadrupoles and dipoles in the drift sections. We can calculate dispersion in the same way we computed the beta functions in a FODO cell previously.

Start at the middle of the F quad, so we have a magnetic arrangement

$$\frac{\mathrm{QF}}{2} \quad \mathrm{B} \quad \mathrm{QD} \quad \mathrm{B} \quad \frac{\mathrm{QF}}{2}$$

Looking at only the x motion in the thin-lens and small angle approximations we find

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which evaluates to

$$M = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) & 2L\theta(1 + \frac{L}{4f}) \\ -\frac{L}{2f}(1 - \frac{L^2}{2f^2}) & 1 - \frac{L^2}{2f^2} & 2\theta(1 - \frac{L}{4f} - \frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix}$$

Here L is the length of each dipole, θ is the bend angle and f is the quadrupole focal length. The upper 2x2 is the same as previously calculated, but now we've added the dispersion.

Dispersion in a FODO cell

By symmetry, the dispersion in the middle of QF must satisfy the closed orbit condition

$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = M \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

and if we solve the resulting equation, noting that in a FODO cell the phase advance is given by

$$\sin\left(\frac{\psi}{2}\right) = \frac{L}{2|f|}$$

we get the dispersion in the middle of the QF

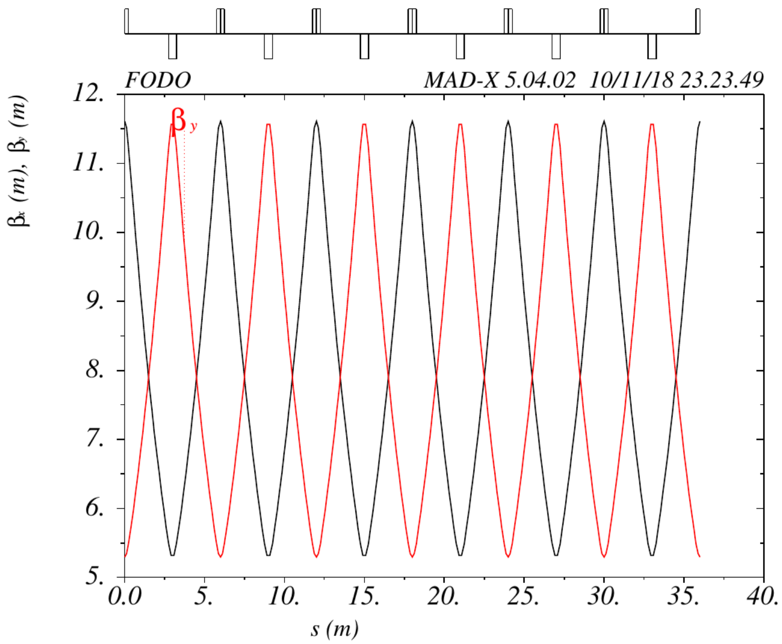
$$D_F = \frac{L\theta\left(1 + \frac{1}{2}\sin\left(\frac{\psi}{2}\right)\right)}{\sin^2(\frac{\psi}{2})} \qquad D'_F = 0$$

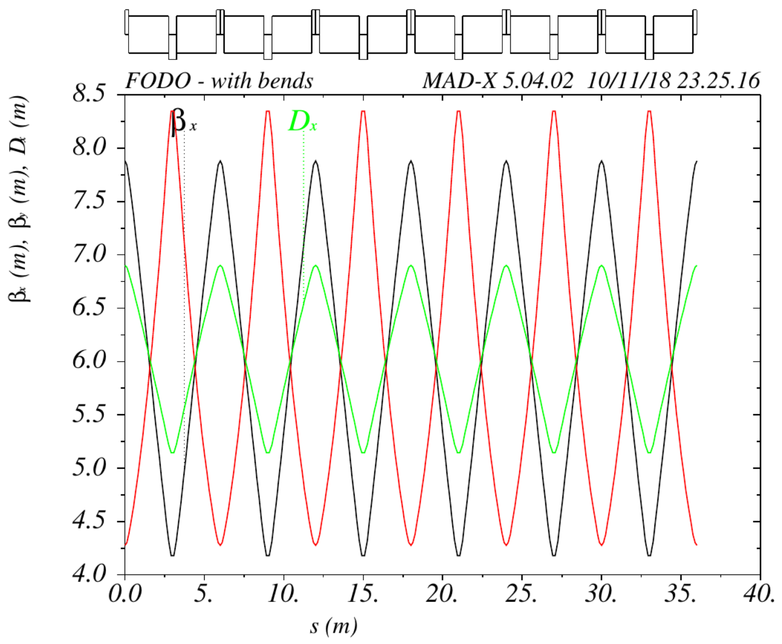
we can get the dispersion elsewhere by transforming this vector using our 3x3 maps. For example in the middle of QD we get

$$D_D = \frac{L\theta\left(1 - \frac{1}{2}\sin\left(\frac{\psi}{2}\right)\right)}{\sin^2(\frac{\psi}{2})} \qquad D'_D = 0$$

Simulating dispersion

D(s) is created (or driven) by dipoles, focused by quadrupoles and will grow in a drift if the angular dispersion D' is non-zero





Controlling dispersion

Dispersion-free lattices are important in many applications. These allow bending of the beam without generating spatial spread (dispersion).

Examples are: Chasman-green, triple-bend achromat,…

We also can build a dispersion suppressor, which matches the periodic dispersion in an arc (perhaps made of FODO cells) into a dispersion-free straight section.

We can also displace the beam transversely without generating dispersion using a sequence of bends, sometimes called a geometrical achromat.

These will be covered in later courses….
But let's look briefly at some examples

The double bend achromat (DBA)

If the dispersion function is non-zero the orbits of particles depend on the particles' momenta. An "achromatic system" means the beam positions at each end do not depend on momentum.

i.e. we require an arrangement of magnets, including bends, which does not generate any dispersion through the structure.

A single bend is not achromatic. In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet.

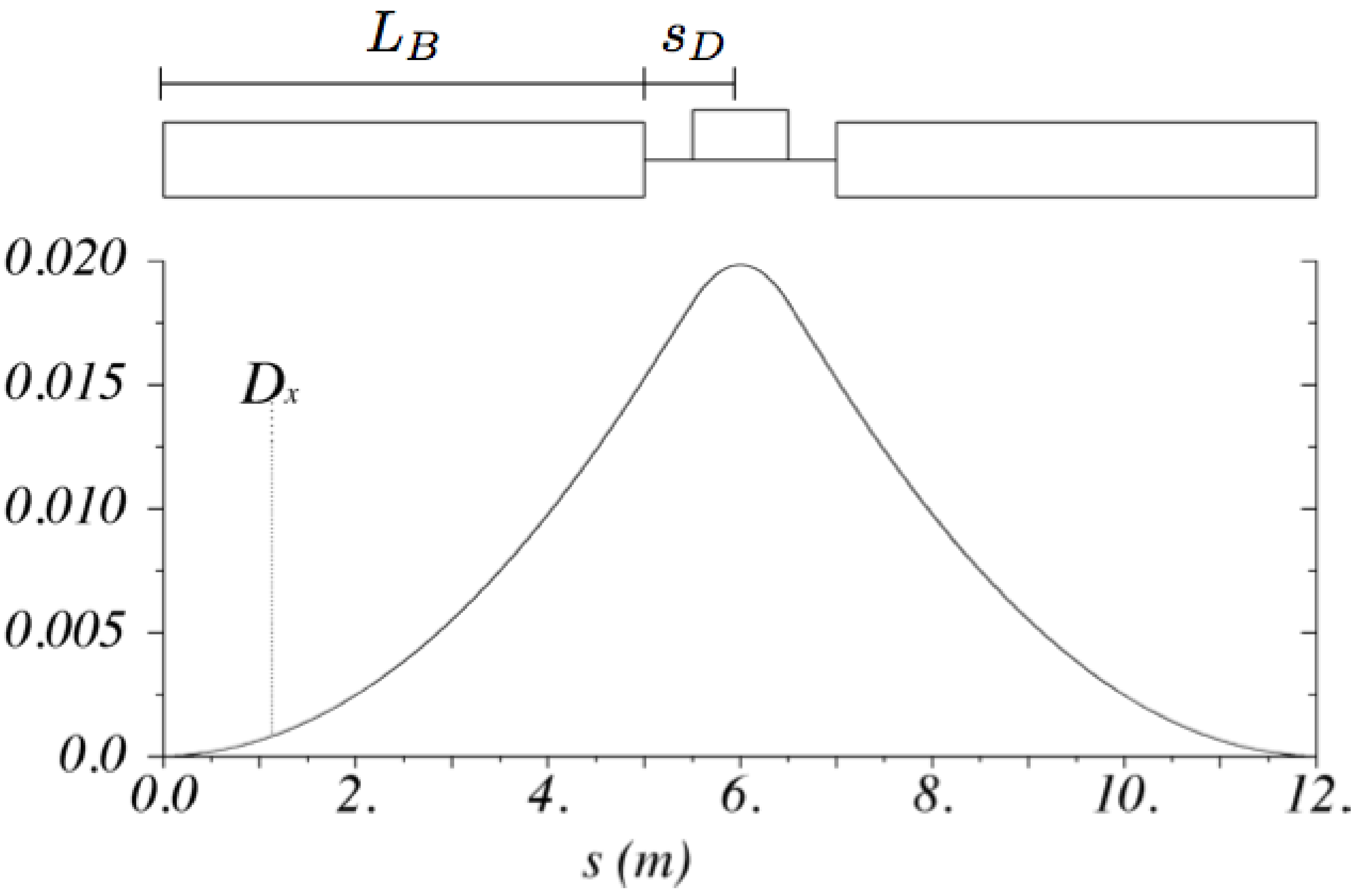
With one focusing quad in between two dipoles, one can get the achromat condition, which means no additional dispersion is generated by the structure.

Due to the mirror symmetry of the lattice w.r.t. to the middle quadrupole D' should be zero in the centre of the lattice. This is the so-called double bend achromat (DBA) structure.

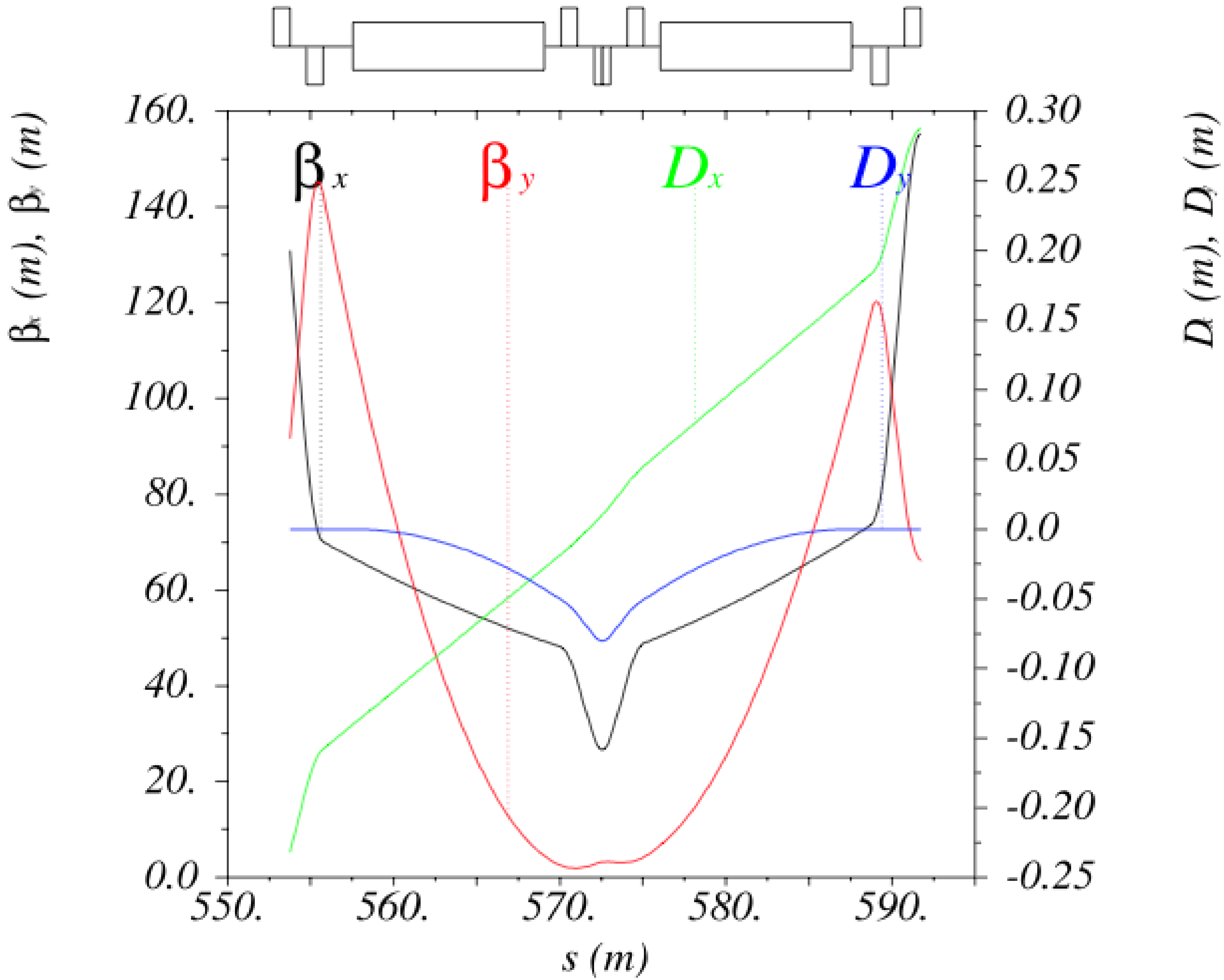
We generally need further quads outside the DBA section to match the betatron functions, tunes, etc.

Similarly, one can design triple bend achromat (TBA), quadrupole bend achromat (QBA), and multi-bend achromat (MBA or nBA) structures.

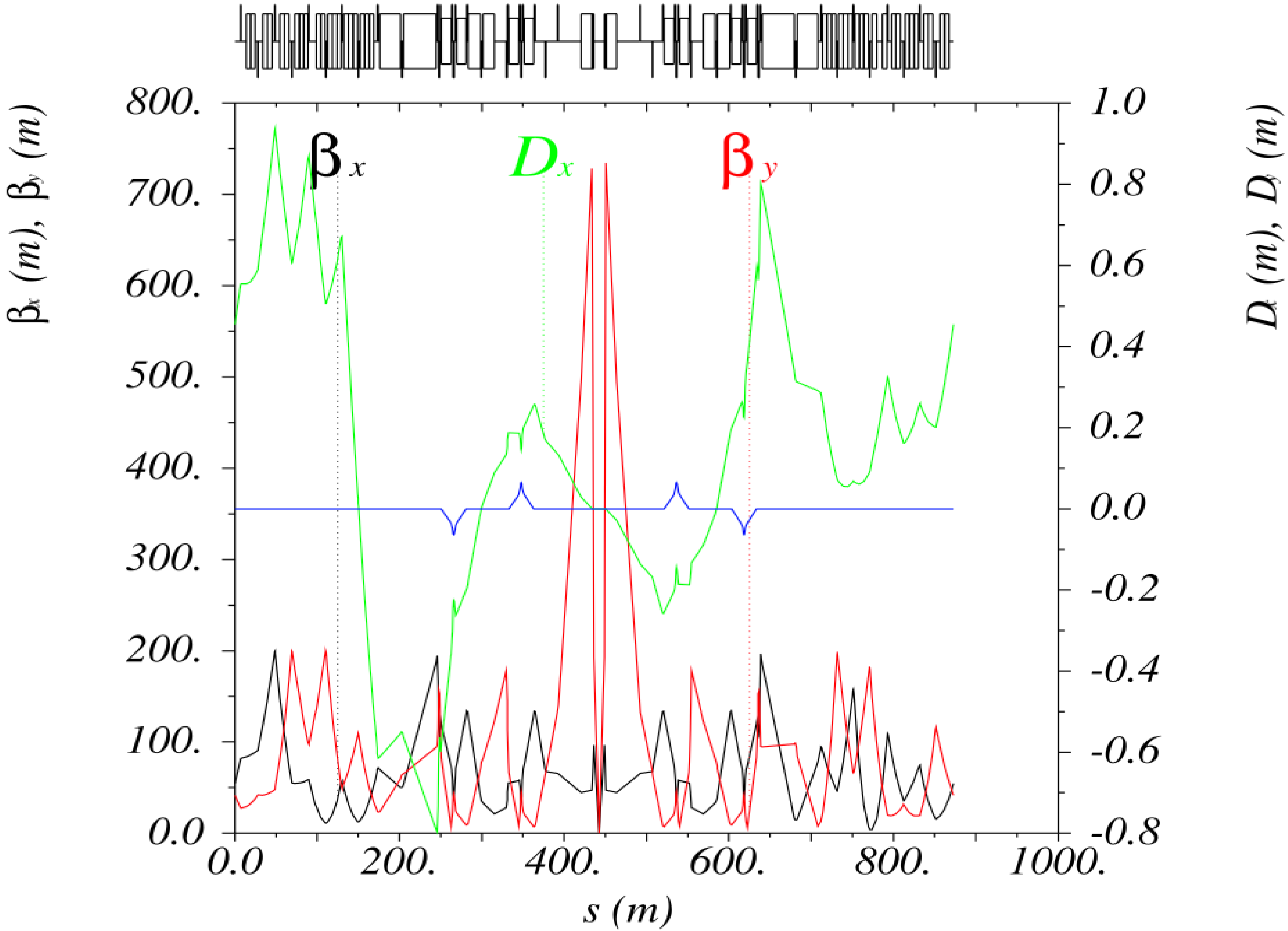
DBA structure with a single quadrupole (sometimes called Chasman-Green)



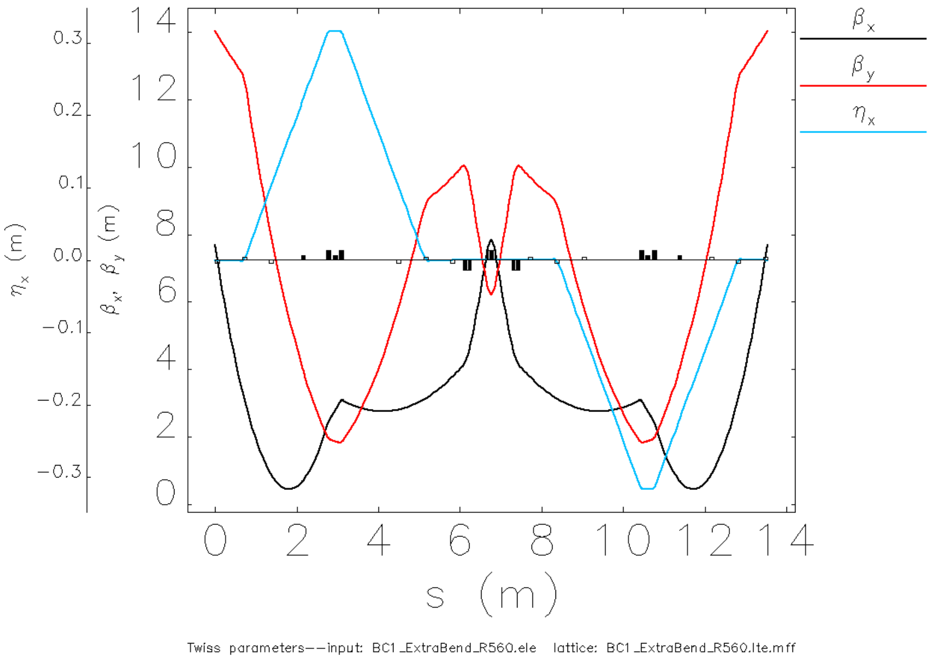
A DBA structure with a quadrupole triplet (vertical)



The long straight section of the LHeC collider (optical work done by CI)

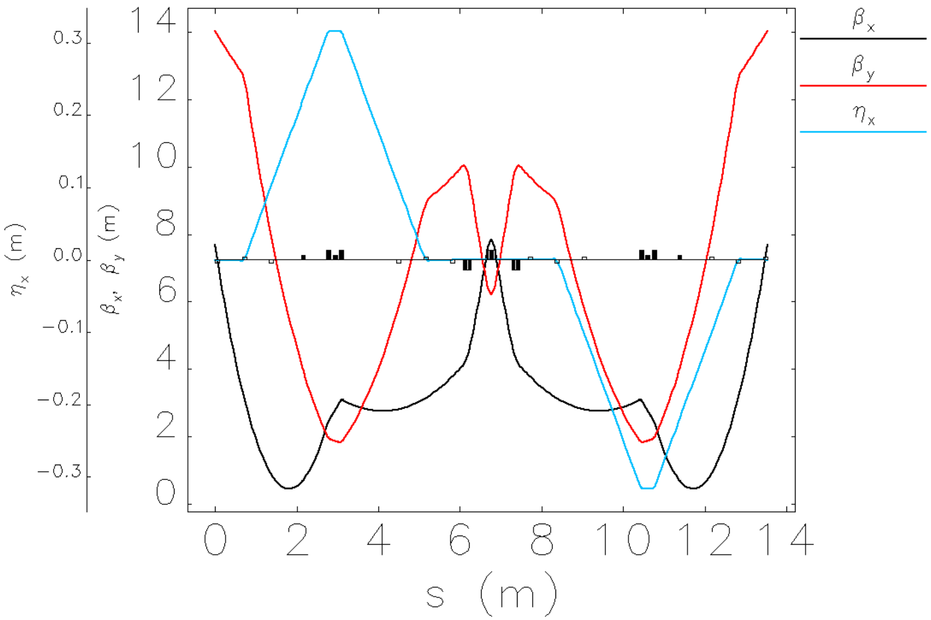


MAX-IV (Gus Perez-Segurana CI)



In addition to affecting the transverse motion, dispersion also has longitudinal effects...

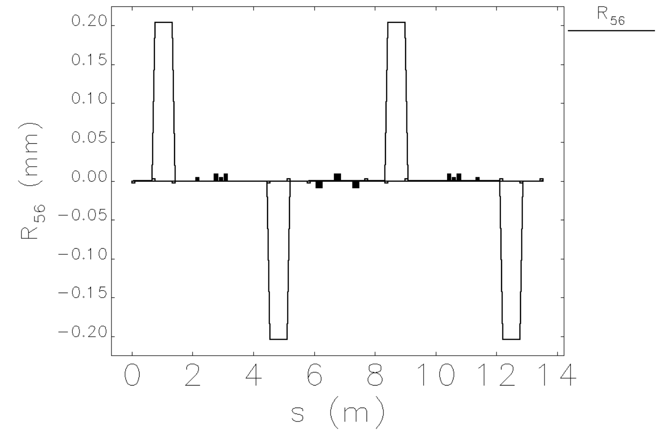
MAX-IV (Gus Perez-Segurana CI)



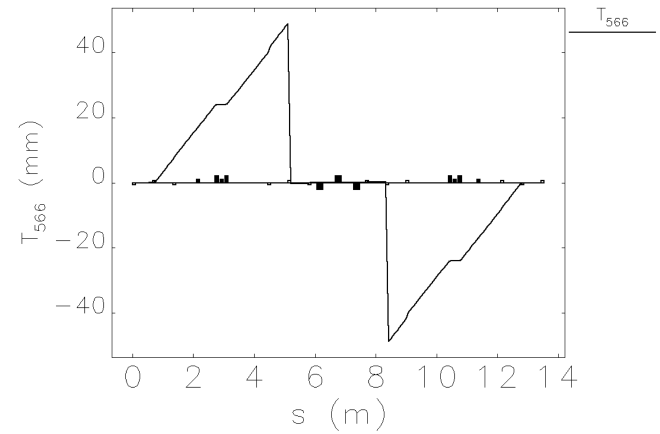
Twiss parameters--input: BC1_ExtraBend_R560.ele lattice: BC1_ExtraBend_R560.lte.mff

In addition to affecting the transverse motion, dispersion also has longitudinal effects...

Longitudinal effects



matrix--input: BC1_ExtraBend_R560.ele lattice: BC1_ExtraBend_R560.lte.mff



matrix--input: BC1_ExtraBend_R560.ele lattice: BC1_ExtraBend_R560.lte.mff

The Cockcroft Institute of Accelerator Science and Technology

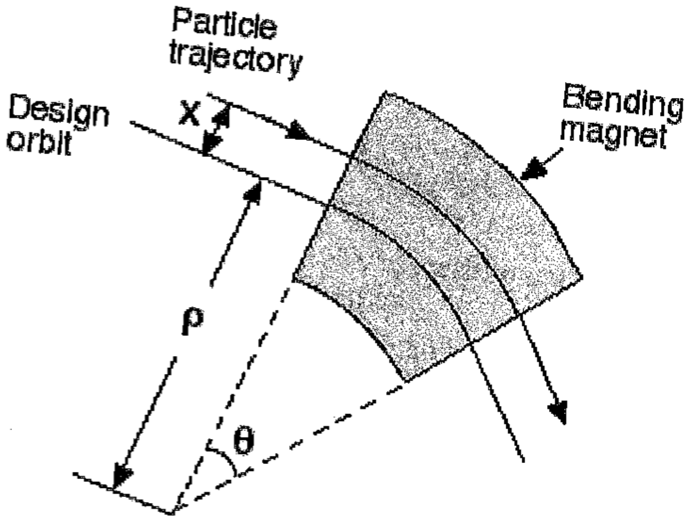
Momentum compaction

A momentum offset changes the horizontal orbit of a particle through dispersion.

Ideally, a machine with only horizontal bends does not generate any vertical dispersion

However, dispersion does generate a longitudinal effect, as the total circumference of an off-momentum particle's trip around the machine will be different to the reference particle.

Let's calculate the path length difference. Consider this situation:



$$\Delta C = (\rho + x)\theta - \rho\theta = x\theta$$

Momentum compaction

The path length deviation is given by

$$\Delta C = (\rho + x)\theta - \rho\theta = x\theta$$

The change in circumference of the machine is given by an integral over the whole ring

$$\Delta C = \oint \frac{x_{CO}(s)}{\rho(s)} ds \qquad ds = \rho \, d\theta$$

For the case where the closed orbit distortion is given by a momentum error

$$\Delta C = \delta \oint \frac{D(s)}{\rho(s)} ds \qquad x_{CO}(s) = D(s)\delta$$

We define the **linear** momentum compaction factor

$$\alpha_c = \frac{1}{\delta} \frac{\Delta C}{C} \quad \text{so} \quad \frac{\Delta C}{C} = \alpha_c \delta = \alpha_c \frac{\Delta p}{p}$$

Therefore the linear momentum compaction factor is given by

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds$$

The Cockcroft Institute of Accelerator Science and Technology

Momentum compaction

The path length deviation is given by

$$\Delta C = (\rho + x)\theta - \rho\theta = x\theta$$

The change in circumference of the machine is given by an integral over the whole ring

$$\Delta C = \oint \frac{x_{\mathrm{CO}}(s)}{\rho(s)} ds \qquad ds = \rho \, d\theta$$

For the case where the closed orbit distortion is given by a momentum error

$$\Delta C = \delta \oint \frac{D(s)}{\rho(s)} ds \qquad x_{\mathrm{CO}}(s) = D(s)\delta$$

We define the **linear** momentum compaction factor

$$\alpha_c = \frac{1}{\delta} \frac{\Delta C}{C} \quad \text{so} \quad \frac{\Delta C}{C} = \alpha_c \delta = \alpha_c \frac{\Delta p}{p}$$

Therefore the linear momentum compaction factor is given by

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds$$

The Cockcroft Institute of Accelerator Science and Technology

Typical lattices and momentum compaction

The momentum compaction factor is an important lattice design parameter

If the orbit is exactly circular we get

$$\alpha_C = \frac{1}{2\pi\rho}\frac{1}{\rho}\oint D(s)ds = \frac{\langle D \rangle}{\rho}$$

A large value means the path length varies a lot for off-momentum particles. This means the particles tend to spread out and the bunch length becomes long.

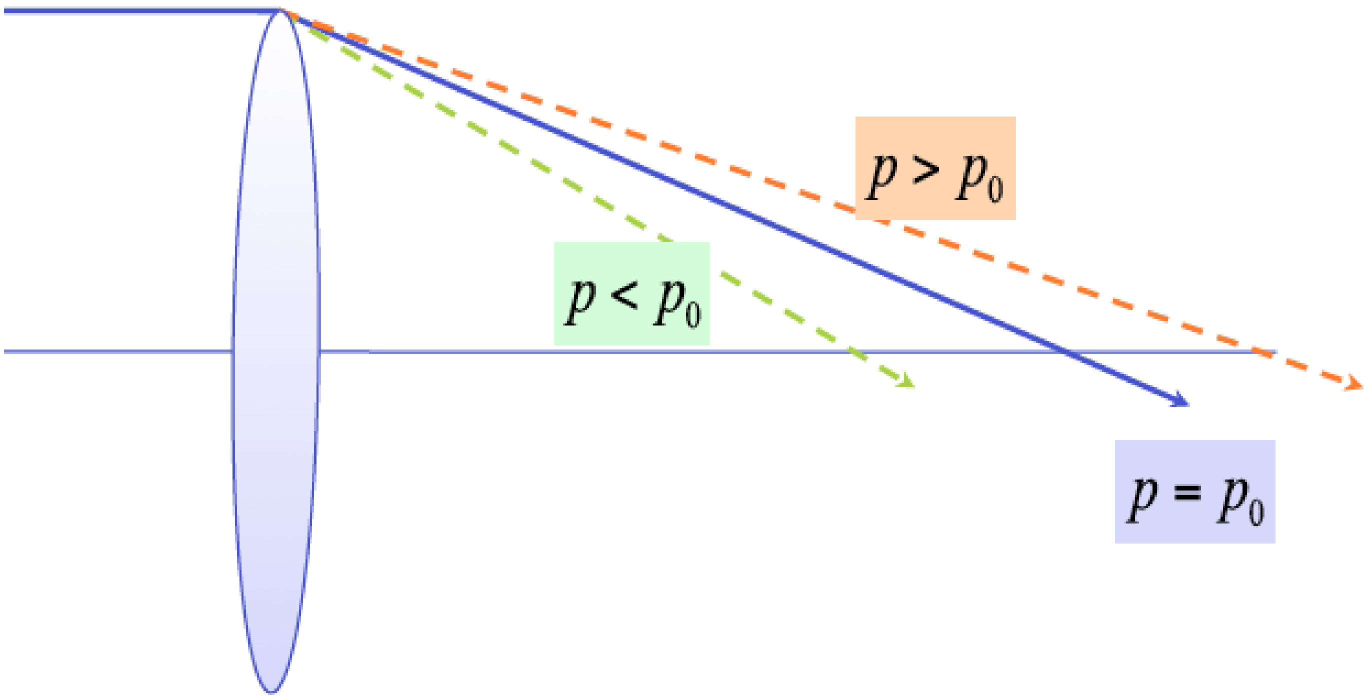
Similarly, a small value means a shorter bunch length.

Typically <D> > 0. In this case the momentum compaction factor is > 0 and the orbit gets longer for positive momentum deviations.

An **isochronous** lattice is designed to counter this natural tendency, i.e. path length doesn’t depend on momentum.

Chromaticity

Consider some particles with slightly different momenta passing through a FODO cell



Higher momentum particles have a higher rigidity, so experience weaker effects when passing through magnetic fields. This means focusing is momentum-dependent and so the machine tune will depend on momentum deviation.

Chromaticity

If a machine's tunes depend linearly on the momentum deviation then

$$\nu_{x,y} = \nu_{x,y}(0) + \xi_{x,y}\delta$$

where the linear chromaticity is ξ. To analyse this we return to the equations of motion, but this time keeping all terms linear in x and δ. Recall

$$\left(\frac{\rho}{\rho + x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho + x} = -\frac{eB_y}{m\gamma v_s}$$

$$x'' - \frac{\rho + x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1 - \delta)(1 + \frac{2x}{\rho})$$

This time, we keep the term (x.δ) we previously dropped. After dropping higher order terms we obtain the equation of motion with both a dispersion term and a 'chromatic' term

$$x'' + K_x(s)x = \frac{\delta}{\rho} + \left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)x\delta$$

As usual

$$K_x(s) = \frac{1}{\rho^2} + \frac{g}{B\rho}$$

Chromaticity

If a machine's tunes depend linearly on the momentum deviation then

$$\nu_{x,y} = \nu_{x,y}(0) + \xi_{x,y}\delta$$

where the linear chromaticity is ξ. To analyse this we return to the equations of motion, but this time keeping all terms linear in x and δ. Recall

$$\left(\frac{\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho+x} = -\frac{eB_y}{m\gamma v_s}$$

$$x'' - \frac{\rho+x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

This time, we keep the term (x.δ) we previously dropped. After dropping higher order terms we obtain the equation of motion with both a dispersion term and a 'chromatic' term

$$x'' + K_x(s)x = \frac{\delta}{\rho} + \left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)x\delta$$

As usual

$$K_x(s) = \frac{1}{\rho^2} + \frac{g}{B\rho}$$

Chromaticity

If a machine's tunes depend linearly on the momentum deviation then

$$\nu_{x,y} = \nu_{x,y}(0) + \xi_{x,y}\delta$$

where the linear chromaticity is ξ. To analyse this we return to the equations of motion, but this time keeping all terms linear in x and δ. Recall

$$\left(\frac{\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho+x} = -\frac{eB_y}{m\gamma v_s}$$

$$x'' - \frac{\rho+x}{\rho^2} \approx -\frac{q}{p}(B_{y0} + gx)(1-\delta)(1+\frac{2x}{\rho})$$

This time, we keep the term (x.δ) we previously dropped. After dropping higher order terms we obtain the equation of motion with both a dispersion term and a 'chromatic' term

$$x'' + K_x(s)x = \frac{\delta}{\rho} + \left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)x\delta$$

As usual

$$K_x(s) = \frac{1}{\rho^2} + \frac{g}{B\rho}$$

Chromaticity

We can think of this chromatic term as a quadrupole field error of strength

$$\Delta K_x = -\left(\frac{2}{\rho^2} + \frac{g}{B\rho}\right)\delta$$

A similar analysis in the vertical plane would have a found a chromatic perturbation of

$$\Delta K_y = \frac{g}{B\rho}\delta$$

We already know how to compute the effect of a quadrupole field error.
Recall the tune shift from a quadrupole error k(s) in our lattice

$$\Delta\nu = \frac{1}{4\pi}\oint ds\beta(s)k(s)$$

Which means we can write down the tune-shift arising from the chromatic perturbation term,

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)(-1)\left[\frac{2}{\rho^2(s)} + \frac{g(s)}{B\rho}\right]\delta$$

An expression which is linear in the momentum deviation.

Natural Chromaticity

The tune change per unit delta is the linear chromaticity we defined earlier

$$\xi_{x,nat} = -\frac{1}{4\pi}\oint ds\beta_x(s)\left[\frac{2}{\rho^2(s)}+\frac{g(s)}{B\rho(s)}\right]$$

$$\xi_{x,nat} \approx -\frac{1}{4\pi}\oint ds\beta_x(s)\left[\frac{g(s)}{B\rho(s)}\right] \qquad 1/\rho^2(s)\approx 0$$

We call this chromaticity 'natural' as any lattice with quadrupoles generates this chromaticity. Similarly in the vertical plane

$$\xi_{y,nat} = +\frac{1}{4\pi}\oint ds\beta_y(s)\frac{g(s)}{B\rho}$$

As the beta function is biggest in focusing quadrupoles the natural chromaticity is normally negative in both planes.

The linear chromaticity is sometimes written as Q'

$$\Delta Q = Q'\delta$$

For a FODO cell we can show that

$$\xi_{x,nat} = -\frac{\beta_F-\beta_D}{4\pi f}$$

The Cockcroft Institute
of Accelerator Science and Technology

Is chromaticity bad?

Chromaticity is naturally generated by any focusing lattice. So when we have non-zero k we have chromaticity, and it tends to be negative in both planes.

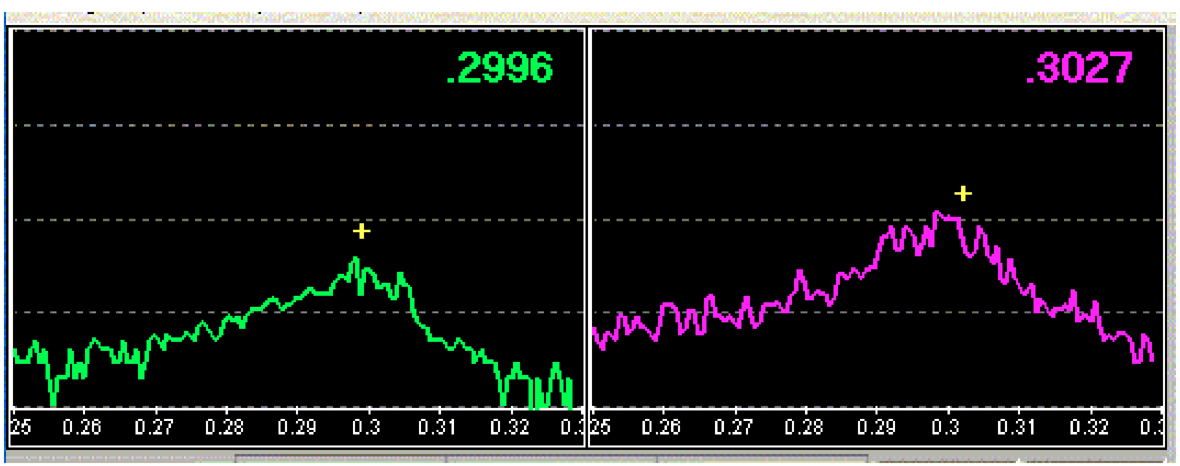
$$\xi_{x,nat} = -\frac{1}{4\pi}\oint ds\beta_x(s)\frac{g(s)}{B\rho}$$

It tells us how much the tune shifts per a unit shift in the momentum deviation.

Since any beam will an energy/momentum spread, chromaticity tells us the spread of the tune of the beam.

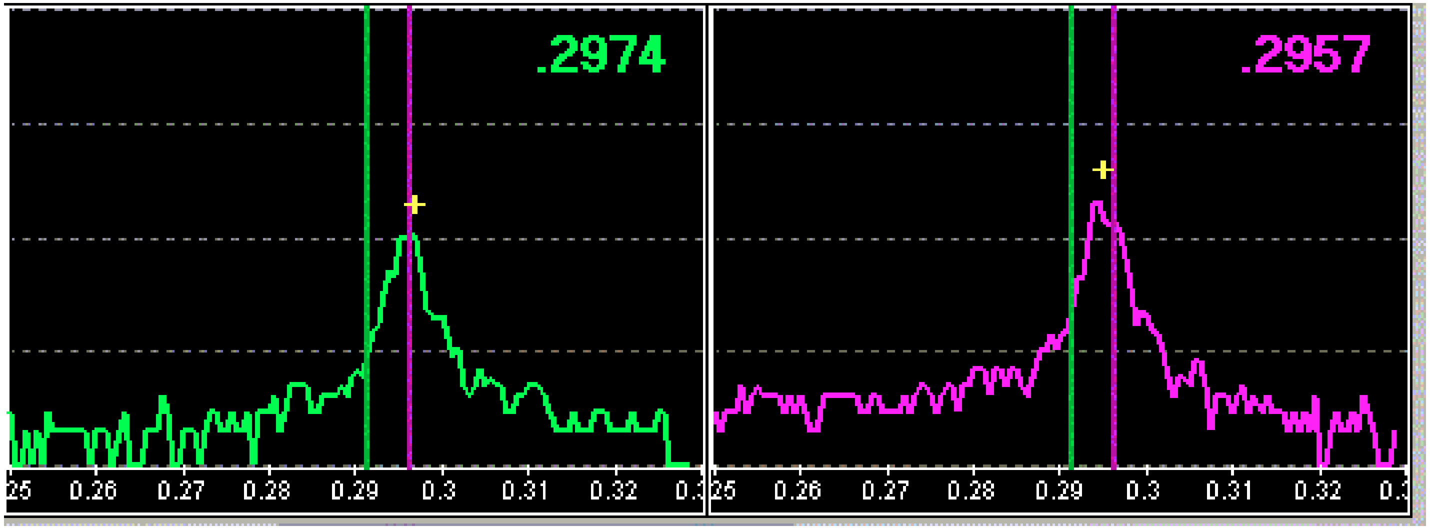
Tune is therefore a blob not a point in tune-space.

This is a plot of tune at HERA, showing the spread of tunes due to uncorrected chromaticity. It crosses many resonances....



The Cockcroft Institute
of Accelerator Science and Technology

Tune after chromaticity correction



We need a mechanism to correct for chromaticity.

Chromaticity originates when off-momentum particles ‘see’ a different quadrupole field than an on-momentum particle.

So we need a correcting device which has some kind of momentum-dependent focusing.....

Sextupoles

A sextupole field has field components given by

$$B_x = Sxy \qquad B_y = \frac{S}{2}(x^2 - y^2)$$

where we define the sextupole strength by

$$S = \frac{d^2 B_y}{dx^2}$$

Note the field is quadratic in x and y, and also (for the first time) we see products of x and y in our equations. A sextupole couples the horizontal and vertical beam motion.

Sextupoles

An off-momentum particle passing through the sextupole has displacement

$$x = x_\beta + D\delta \quad y = y_\beta$$

and so the fields seen by the particle are

$$B_x = S(x_\beta + D\delta)y_\beta = Sx_\beta y_\beta + SD\delta y_\beta$$
$$B_y = \frac{S}{2}(x_\beta^2 - y_\beta^2) + Sx_\beta D\delta + \frac{S}{2}D^2\delta^2$$

There are many terms here, some helpful and some harmful. The helpful ones for us are

$$B_x = Sy_\beta D\delta \quad B_y = Sx_\beta D\delta$$

The Cockcroft Institute of Accelerator Science and Technology

Sextupoles

An off-momentum particle passing through the sextupole has displacement

$$x = x_\beta + D\delta \quad y = y_\beta$$

and so the fields seen by the particle are

$$B_x = S(x_\beta + D\delta)y_\beta = Sx_\beta y_\beta + SD\delta y_\beta$$
$$B_y = \frac{S}{2}(x_\beta^2 - y_\beta^2) + Sx_\beta D\delta + \frac{S}{2}D^2\delta^2$$

There are many terms here, some helpful and some harmful. The helpful ones for us are

$$B_x = Sy_\beta D\delta \quad B_y = Sx_\beta D\delta$$

The Cockcroft Institute of Accelerator Science and Technology

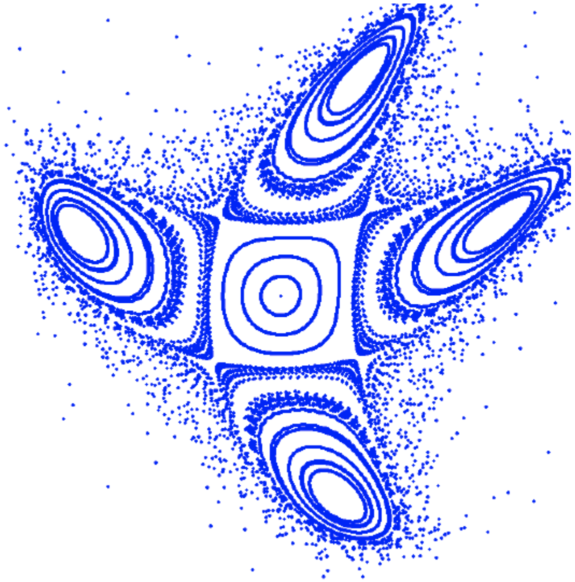
Sextupoles

The dispersion effectively makes the sextupole into a quadrupole with a momentum-dependent focusing gradient

$$B_x = S y_\beta D\delta \quad B_y = S x_\beta D\delta$$

This means we can compensate the chromaticity in the ring, and reduce the tune spread, by adjusting the sextupoles.

But it's not all perfect. Some sextupole terms we ignored introduce non-linearities and coupling into our accelerator ring.



It is difficult to represent a sextupole in our linear formalism, and often the best way to understand the impact of sextupole fields is to track particles with matrices, stopping to be more careful every time a sextupole is encountered. This leads to the study of a machine's dynamic aperture (i.e. how large can a particle's deviation from the closed orbit be if we want the particle to survive for many turns.)

Beams of particles and emittance

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$

So far we've defined 'emittance' as a property of each particle. In the Courant-Snyder analysis we showed the motion of an individual particle is completely specified by its emittance and initial phase.

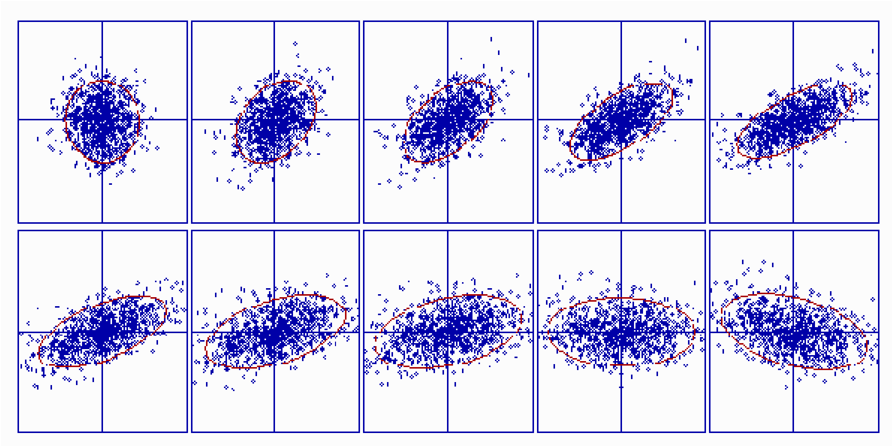
Different particles have different emittances and initial phases but they all have the same Courant-Snyder functions.

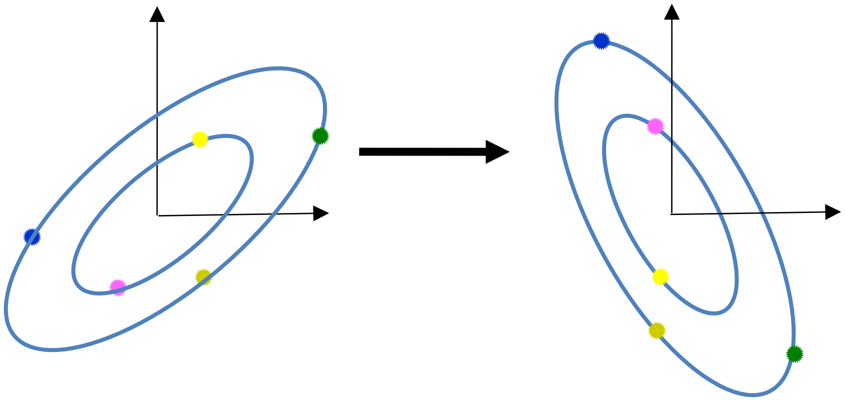
For example, the particle with x=x'=0 will have zero emittance and always stay at x=x'=0. This is the ideal particle.

But, we always have more than one particle in our beam and so need to understand how to characterise a beam of particles, each with their own emittance.

Beams of particles and emittance

We can plot the emittance (also known as the Courant-Snyder invariant) for all the particles in a beam





We choose the emittance of one particle to represent the emittance of the entire beam. For example, we can characterise the beam by the emittance of the particle for which 95% of the beam particles are within the ellipse of this particle.

Another useful definition, when dealing with complicated distributions, is the RMS emittance, which we find by averaging over the beam distribution

$$\epsilon_{\text{rms}} = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$$

The Cockcroft Institute of Accelerator Science and Technology

Beam moments

For a complex and non-linear beam distribution, we often work with the moments of the beam distribution where ρ here is the particle density not the radius of curvature!

$$< x > \;=\; \int dx \int dx' \; x \; \rho(x, x')$$

$$< x' > \;=\; \int dx \int dx' \; x' \; \rho(x, x')$$

$$< x^2 > \;=\; \int dx \int dx' \; (x- < x >)^2 \; \rho(x, x')$$

$$< {x'}^2 > \;=\; \int dx \int dx' \; (x'- < x' >)^2 \rho(x, x')$$

$$< x x' > \;=\; \int dx \int dx' \; (x'- < x' >)(x- < x >) \rho(x, x')$$

Liouville's theorem

J. Liouville
(1809-1882)

Liouville's theorem : the density of points representing particles in 6-D (**x**, **p**) phase space is conserved if all forces are conservative and differentiable.

Radiation and other dissipative forces do not satisfy this requirement, but magnetostatic forces and (Newtonian) gravitational forces do.

There must be no (or very slow) time-dependence in the system.

Note: acceleration keeps (x,p) phase space constant, but reduces (x, x') phase space.

Transfer maps derived from a Hamiltonian have a mathematical property called symplecticity, which is linked to Liouville's theorem. But this is beyond this course…

Symplecitjcty in 2D phase space is equivalent to det(M)=1