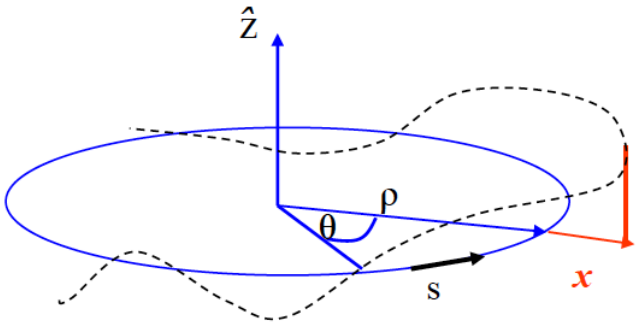
PART THREE- TRANSFER MATRICES, STABILITY AND TUNES

Recall - Hill's Equation and Solutions



$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$
$$z'' - kz = 0$$

Piecewise Solution

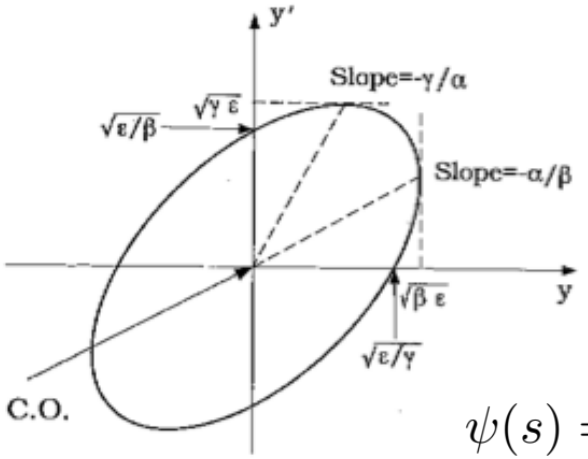
$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Courant-Snyder Solution

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s) + \psi_0)$$



$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

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The beta function

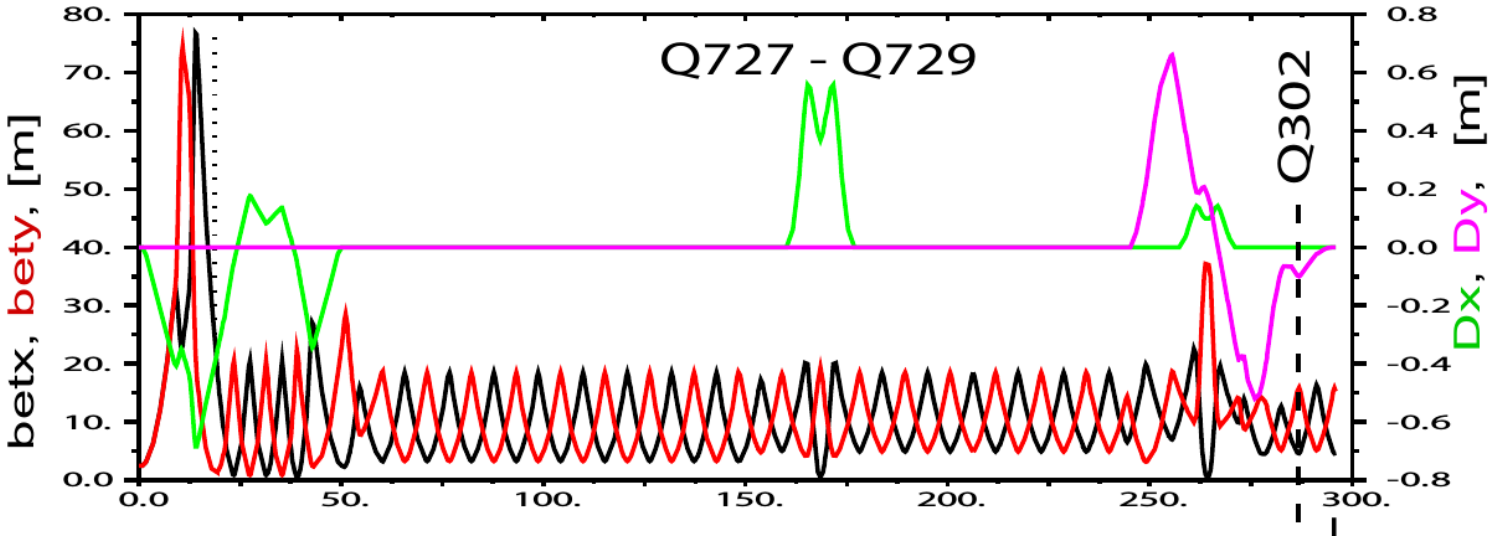
The beta function is a central quantity in the Courant-Synder formalism

It is a positive function of position in the machine, and has the same periodicity as the lattice itself.

It is determined only by the focusing properties of the lattice.

It is maximised in a focusing quadrupole and minimized in a defocusing quadrupole.

Below is a typical example from a transfer line at the g-2 experiment at Fermilab showing “betatron oscillations”.



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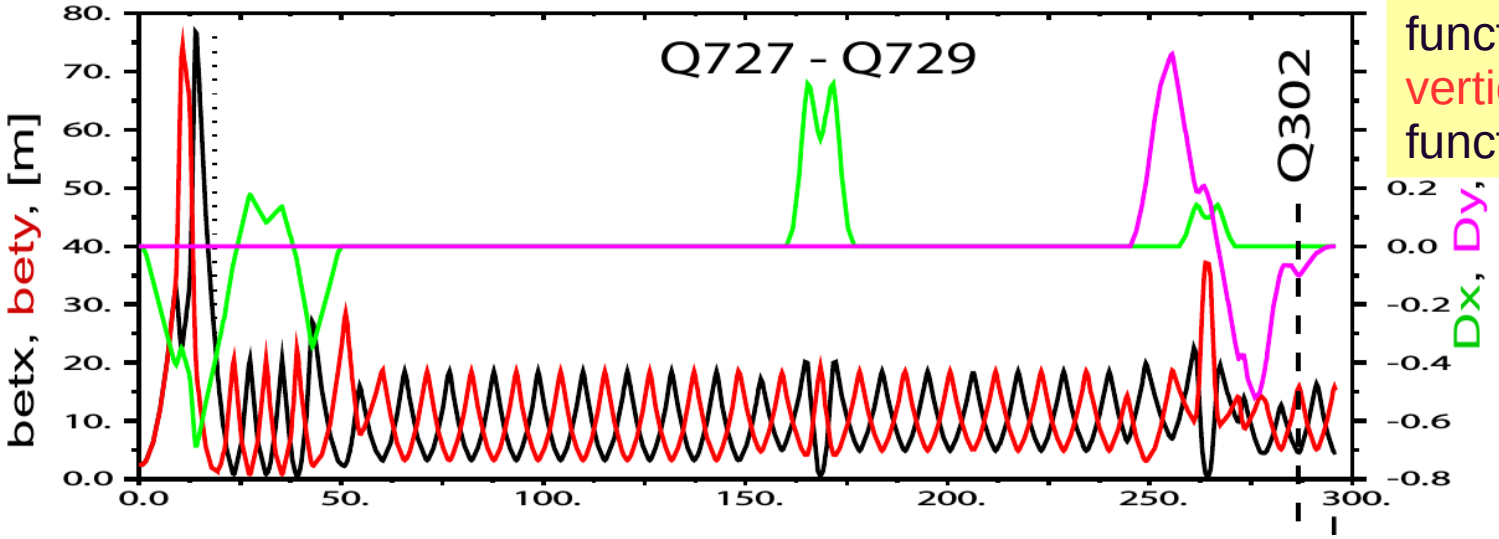

Note that there is a horizontal beta function and a vertical beta function.

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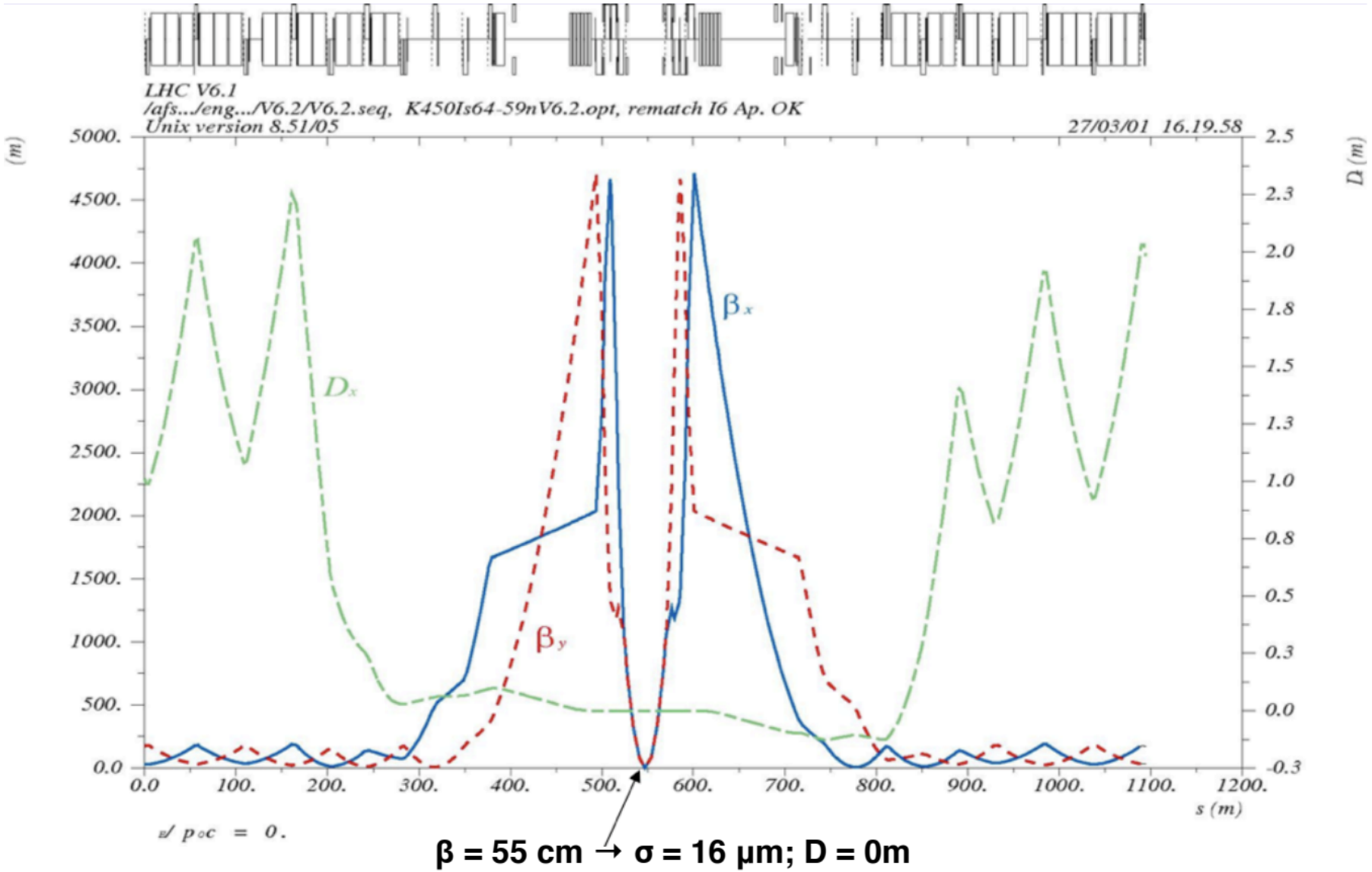
The beta functions of the LHC



β = 55 cm → σ = 16 μm; D = 0m

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The transfer matrix in terms of Courant-Snyder parameters / functions / lattice functions

Can we write a general transfer matrix between any two points in terms of the ‘lattice functions’?

The transfer matrix in terms of Courant-Snyder parameters / functions / lattice functions

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$$x(s) = c_1\sqrt{\beta(s)}\cos\psi(s) + c_2\sqrt{\beta(s)}\sin\psi(s)$$

where c_1 and c_2 are constants yet to be determined. If we define the initial conditions at the point '0' to be

$$\beta(0) = \beta_0 \quad \alpha(0) = \alpha_0 \quad \psi(0) = 0$$

and write the initial particle coordinates to be x_0 and x_0' then we can fix the unknown constants.

We also need to recall

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)} \qquad \alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds}$$

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Determines $\psi(0)$

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$$c_1 = \frac{x_0}{\sqrt{\beta_0}} \qquad c_2 = \sqrt{\beta_0}x_0' + \frac{\alpha_0}{\sqrt{\beta_0}}x_0$$

And so we can write x(s) in the form

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0\sin\psi(s)\right]x_0 + \sqrt{\beta_0\beta(s)}x_0'\sin\psi(s)$$

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The transfer matrix in terms of the Courant-Snyder parameters / functions / lattice

As in the piecewise solutions, we see the expression for x(s) is linear in x_0 and x_0'.

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[\cos\psi(s) + \alpha_0 \sin\psi(s)\right] x_0 + \sqrt{\beta_0\beta(s)}\, x_0' \sin\psi(s)$$

Taking the derivative of this expression, we can cast this equation into a convenient matrix form (as it's linear)

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M(s_1|s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

where

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

$$\psi = \psi(s_1) - \psi(s_0)$$

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$$\psi = \psi(s_1) - \psi(s_0)$$

The subscripts 0 and 1 refer to the beginning and end of the transfer map.

This means the transfer matrix between two points is purely determined by the lattice functions at each point and the phase advance between the points!

The one-turn map

The one turn (one period) map is a very useful quantity (we mentioned it previously for piece-wise solutions to Hill's equation).

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

The map for one turn of the ring means we come back to the same s position, and so

$$\beta_1 = \beta_0 = \beta \quad \alpha_1 = \alpha_0 = \alpha \quad \gamma_1 = \gamma_0 = \gamma$$

And so the one turn map is

$$M(s+C|s) = \begin{pmatrix} \cos(\Psi) + \alpha\sin(\Psi) & \beta\sin(\Psi) \\ -\gamma\sin(\Psi) & \cos(\Psi) - \alpha\sin(\Psi) \end{pmatrix}$$

where we have used $\gamma(s) = \dfrac{1+\alpha^2(s)}{\beta(s)}$

and the phase advance for one turn is:

$$\psi_1 - \psi_0 = \Psi$$

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The one-turn map

If we know this map we can determine the lattice functions.

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We can multiply all the piece-wise matrices for all the elements in the ring together to obtain the total matrix for one turn of the machine

$$M\left(s+C|s\right)=\prod_{i}M\left(s_{i+1}|s_{i}\right)$$

$$M(s+C|s)=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

which we can compare with

$$M(s+C|s)=\begin{pmatrix} \cos(\Psi)+\alpha\sin(\Psi) & \beta\sin(\Psi) \\ -\gamma\sin(\Psi) & \cos(\Psi)-\alpha\sin(\Psi) \end{pmatrix}$$

The one-turn map

We now have

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We can get the one-turn phase from the trace of this matrix!

$$\Psi = \arccos\left(\frac{m_{11} + m_{22}}{2}\right)$$

We can get the lattice function from the other matrix elements.

$$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi} \qquad \gamma = -\frac{m_{21}}{\sin\Psi}$$

Note for the phase advance to be real-valued and hence **stable**, we need

$$|\mathrm{Tr}M| \leq 2$$

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The one-turn map at a different location

If we know the one-turn map at one location, s, is there a way to calculate it at another location, s', provided we know the transfer matrix M for s to s'?

The answer is yes. They are related to each other by the similarity transform

$$M(s'+C|s') = M(s'|s) \cdot M(s+C|s) \cdot M^{-1}(s'|s)$$

Similarity transforms come from matrix theory. They preserve eigenvalues, traces, etc.

We'll now denote the matrix M(s'|s) (i.e. the map from s to s') by

$$M(s'|s) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

(redefining m_{11}, etc). Let's use this to calculate how the lattice functions transform from place to place if we know the transfer matrix.

The transformation of the lattice functions

Starting with the similarity transform,

$$M(s'+C|s') = M(s'|s)\cdot M(s+C|s)\cdot M^{-1}(s'|s)$$

We can express the one-turn maps in terms of the lattice functions at the locations s and s'

$$M\,(s+C|s) = \begin{pmatrix} \cos(\Psi_0)+\alpha_0\sin(\Psi_0) & \beta_0\sin(\Psi_0) \\ -\gamma_0\sin(\Psi_0) & \cos(\Psi_0)-\alpha_0\sin(\Psi_0) \end{pmatrix}$$

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and after a page of algebra we obtain the lattice functions at point s' (or 1) in terms of the lattice functions at point s (or 0) and the elements of the matrix M. The answer is

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} m_{11}m_{22}+m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

Knowing M(s'|s), we can transform the lattice functions to any point in the beam line.

Courant-Snyder parameter evolution in a drift

E.g. In a drift space of length L we have

$$M_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

And so

$$m_{11} = 1 \;\; m_{12} = L \;\; m_{21} = 0 \;\; m_{22} = 1$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -L \\ -2L & 1 & L^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

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The lattice functions evolve

$$\beta_1 = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha_1 = \alpha_0 - \gamma_0 L$$

$$\gamma_1 = \gamma_0$$

A particle evolves

$$x(L) = x_0 + L x'_0$$

$$x'(L) = x'_0$$

The phase advance and tune

Several times we have used the phase advance for one turn of the closed orbit (or any period of a periodic structure). It is

$$\Psi = \int_s^{s+C} \frac{ds}{\beta(s)}$$

We call the phase advance for one complete turn of a ring the **tune**, and express it in units of 2π

$$\nu = \frac{\Psi}{2\pi} = \frac{1}{2\pi} \int_s^{s+C} \frac{ds}{\beta(s)}$$

(or Q)

There is one tune for each plane, including the longitudinal plane.
It's an important function for beam stability.
Note we can evaluate the tune at any point in the ring and always get the same answer (a property not shared by α, β and γ)

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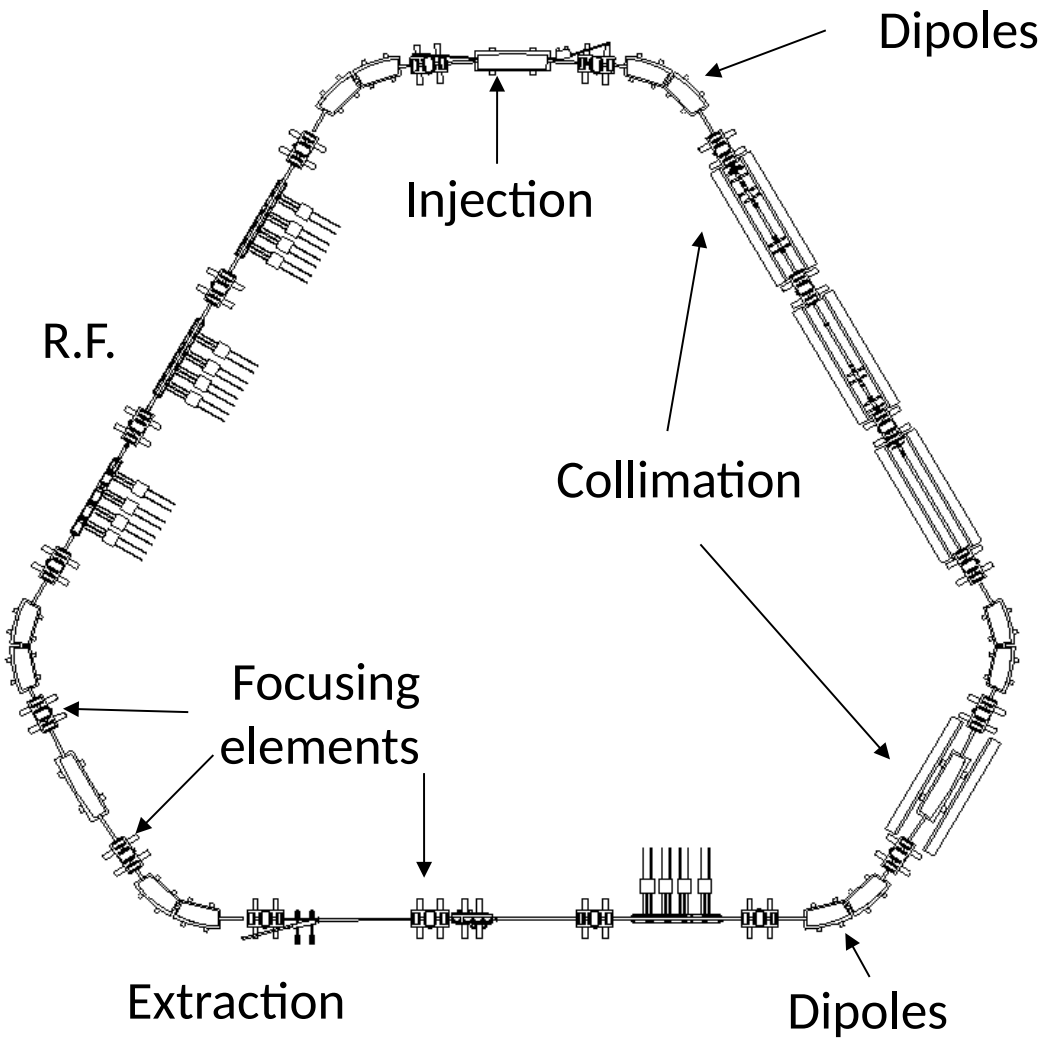
A simple approximation to the tune can come from the average value of the beta function $\bar{\beta}$ and average radius R.

$$\nu \approx \frac{1}{2\pi}\frac{C}{\bar{\beta}} \approx \frac{R}{\bar{\beta}}$$

Applying beam dynamics tools to a lattice

Let's apply the tools we've developed to a storage ring.



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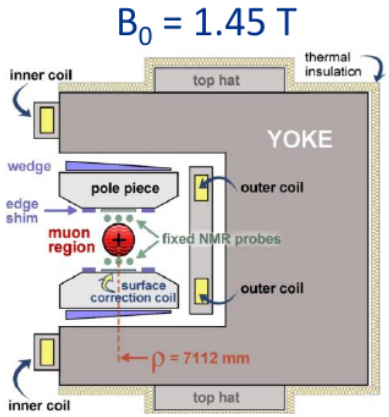
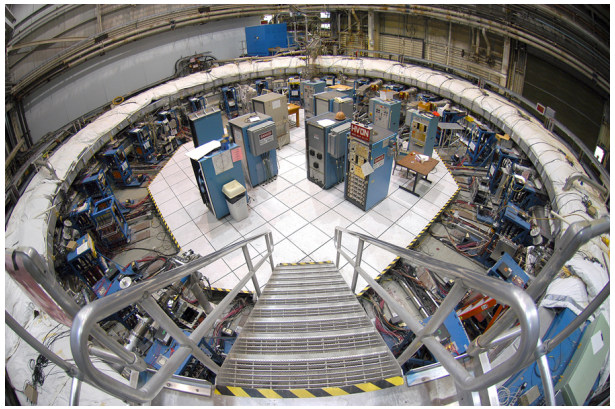
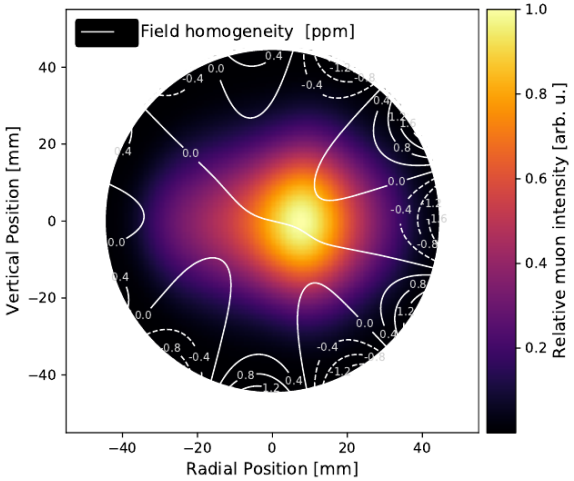
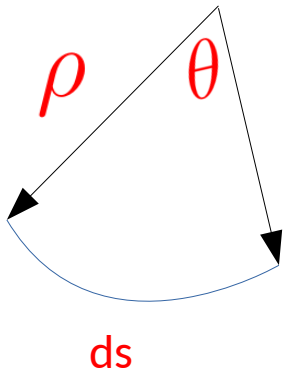
Bending

The first task is to define the curved reference orbit using a layout of dipole magnets. This forms the fundamental footprint of the machine and defines our coordinate system for future analysis. E.g. Fermilab g-2 storage ring









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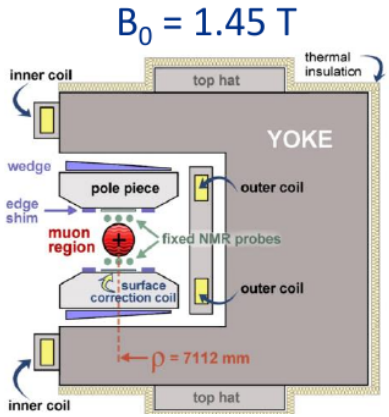
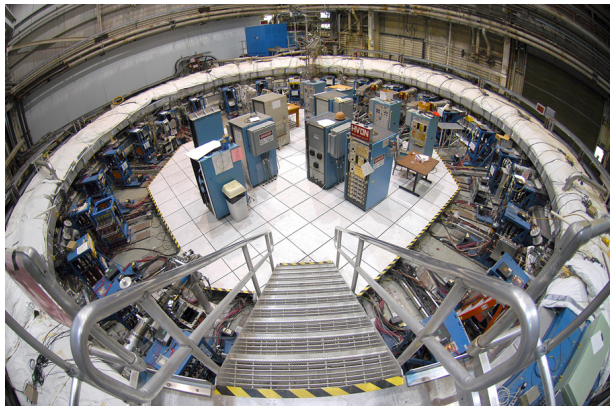
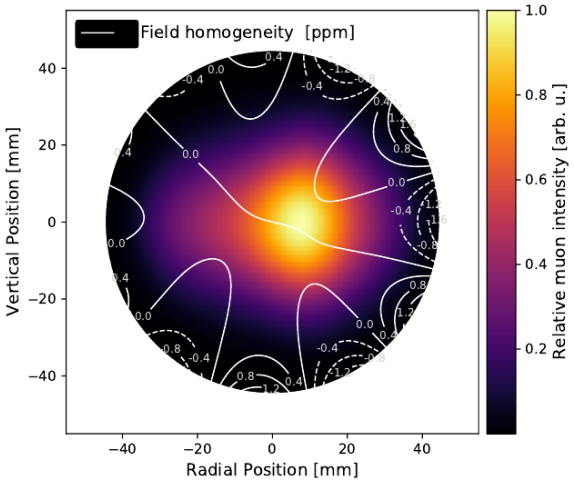
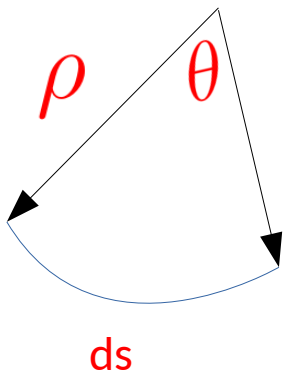




$$d\theta = \frac{ds}{\rho(s)} \approx \frac{dl}{\rho(l)} \approx \frac{B(l)q}{p} dl$$

$$\theta \approx \frac{\int B(l)dl}{(p/q)} = \frac{\int B(l)dl}{B_0\rho_0} = 2\pi$$





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Example – the LHC

$$N_{\text{dipoles}} = 1232$$

$$l_{\text{dipole}} = 15\text{m}$$

$$[B] = Vsm^{-2}$$

$$\int Bdl = NlB = 2\pi\frac{p}{q}$$

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$$B = \frac{2\pi \cdot 7000 \cdot 10^9[eV]}{1232 \cdot 15\text{m} \cdot 3 \times 10^8[m/s]q} = 8.3\text{T}$$

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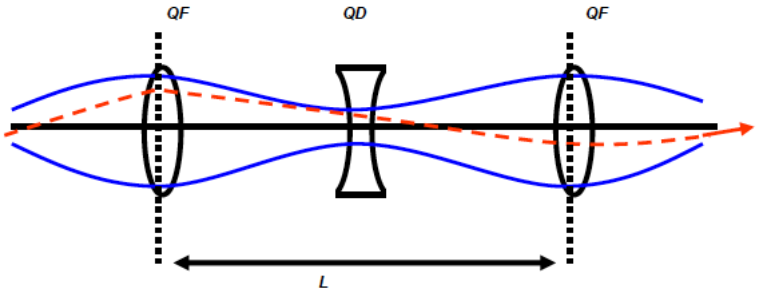
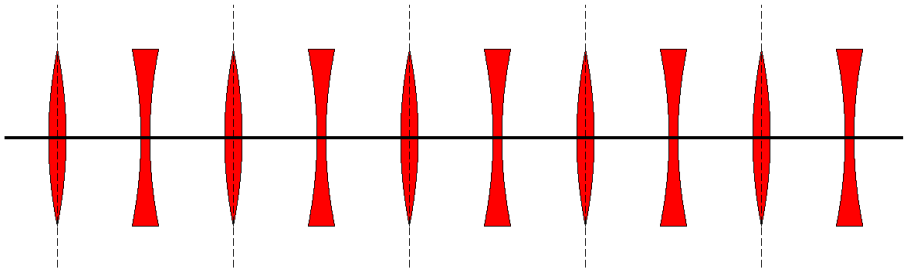
The FODO cell

Recall that two quadrupoles of opposite polarity could provide focusing in both planes at the same time. This is the fundamental building block of the FODO lattice.

The basic building block of this periodic structure is the FODO cell, consisting of a horizontally focusing quadrupole (F), a space (O), a defocusing quadrupole (D) and a space (O).



We can repeat the FODO cell to make a FODO channel. Note the drift space (O) can contain nothing, a bend, some diagnostics, an RF cavity or even a whole particle physics experiment!



The dynamics in a FODO cell

To understand the beam dynamics in a FODO cell we need to compute the one-period map.

To do this we simply multiply the matrices of the components of the cell together, **conventionally starting in the middle of one of the quadrupoles,** which means we start and end with a quadrupole matrix of half strength (length)

Recall

$$M_{\text{foc quad}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

$$M_{\text{defoc quad}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ +\sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

First element!

and we multiply these matrices in sequence

$$M_{\text{FODO}} = M_{\text{Fh}} \cdot M_{\text{drift}} \cdot M_{\text{D}} \cdot M_{\text{drift}} \cdot M_{\text{Fh}}$$

The FODO cell

Let's be concrete and take some real numbers

K = +/- 0.541244 m^{-2}

l_q = 0.5 m

L = 2.5 m

Multiplying out the matrices

$$\begin{pmatrix} 0.983134 & 0.248593 \\ -0.134549 & 0.983134 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2.5 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1.06842 & 0.511352 \\ 0.276767 & 1.06842 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2.5 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.983134 & 0.248593 \\ -0.134549 & 0.983134 \end{pmatrix}$$

We obtain

$$M_{\mathrm{FODO}} = \begin{pmatrix} 0.707107 & 8.20716 \\ -0.0609224 & 0.707107 \end{pmatrix}$$

The FODO cell

Let's be concrete and take some real numbers

K = +/- 0.541244 m^{-2}

l_q = 0.5 m

L = 2.5 m

$$M_{\mathrm{FODO}} = \begin{pmatrix} 0.707107 & 8.20716 \\ -0.0609224 & 0.707107 \end{pmatrix}$$

This is the one period map of the FODO cell, and so has the form

$$M(s+L|s) = \begin{pmatrix} \cos\Psi + \alpha\sin\Psi & \beta\sin\Psi \\ -\gamma\sin\Psi & \cos\Psi - \alpha\sin\Psi \end{pmatrix}$$

Recall

$$x(s) = \sqrt{\epsilon\,\beta(s)}\cos(\psi(s)+\psi_0) \qquad \psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds} \qquad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

Properties of our FODO cell

Is the FODO cell stable? For this we need the trace of the one-turn map to be less than or equal to 2. Here it is 1.414. So this FODO cell will give stable dynamics in this plane.

What is the phase advance per cell? Recall

$$\Psi = \arccos\left(\frac{m_{11} + m_{22}}{2}\right)$$

The phase advance per cell is 45 degrees. This is a “45 degree cell”.

What are the lattice functions at the middle of the focusing quadrupole? We use

$$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi}$$

And find that β=9.645 m and α=0.
i.e. The beam size is at a maximum.

A thin lens FODO cell

We can also make our life easier and compute the matrix for our FODO cell using the thin lens matrices. Again, starting from the middle of QF we have

$$M_{\mathrm{FODO}} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{bmatrix}$$

We end up with the matrix in terms of L and f

$$M_{\mathrm{FODO}} = \begin{bmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{bmatrix}$$

We can ask for what parameters the FODO cell gives stable motion. This means

$$\mathrm{Tr}(M) \leq 2 \rightarrow |f| > \frac{L}{2}$$

We can also write the cell phase advance in terms of the parameters:

$$\cos(\Psi) = \frac{1}{2} Tr(M) = 1 - \frac{L^2}{2f^2} \qquad \sin\left(\frac{\Psi}{2}\right) = \frac{L}{2|f|}$$

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Transfer line stability

Our stability equation from the previous slide seems slightly odd at first

$$\mathrm{Tr}(M) \leq 2 \rightarrow |f| > \frac{L}{2}$$

It seems to say motion is stable when focusing is weak…

This makes sense though. If the focussing is too strong then the periodicity of beta can't match the periodicity of the lattice.

Beta in a FODO cell

Finally…

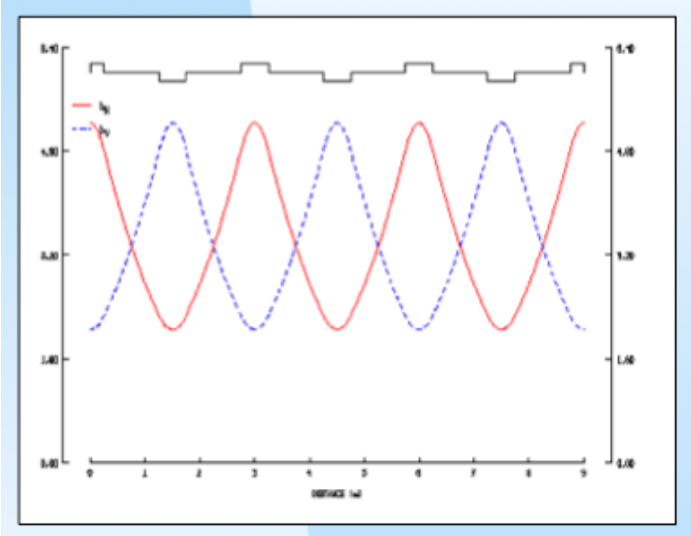
We can now compute all the lattice functions for a FODO cell.

Note that β_x is maximised in the middle of the focusing quadrupoles, and this maximum depends solely on the cell length and phase advance.

Using $$\beta = \frac{m_{12}}{\sin\Psi} \qquad \alpha = \frac{m_{11} - m_{22}}{2\sin\Psi}$$

We get $$\beta_F = \frac{2L(1+\frac{L}{2f})}{\sin(\Psi)} \qquad \alpha_F = 0 \qquad \gamma_F = \frac{1}{\beta_F}$$

In the D quad $$f \rightarrow -f \qquad \beta_D = \frac{2L(1-\frac{L}{2f})}{\sin(\Psi)}$$



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ERROR AND RESONANCES

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A reminder of some of the terms we ignored...

$$
\begin{aligned}
x'' + (\kappa_x^2 + k)\, x = {} & \kappa_x\delta - \kappa_x\delta^2 + \kappa_x\delta^3 - (\underline{k} + \kappa_x\kappa_y)\, y \qquad (3.75)\\
& - (\underline{m} + 2\kappa_x\underline{k} + 2\kappa_y k + 2\kappa_x^2\kappa_y)\, xy - \tfrac{1}{2}m\,(x^2 - y^2)\\
& - (\kappa_x^3 + 2\kappa_x k)\, x^2 - (\kappa_x\kappa_y^2 - \tfrac{1}{2}\,\kappa_x k + \tfrac{2}{3}\kappa_y\,\underline{k} - \tfrac{1}{2}\,\kappa_x'')\, y^2\\
& + \tfrac{1}{2}\,\kappa_x(x'^2 - y'^2) + \kappa_x'\,(xx' + yy') + \kappa_y'\,(x'y - xy') + \kappa_y x'y'\\
& - \tfrac{1}{6}\,r\,x(x^2 - 3y^2) + \tfrac{1}{6}\,\underline{r}\,y\,(y^2 - 3x^2)\\
& + \tfrac{1}{12}(\kappa_y\underline{m} - 11\kappa_x m + 2\kappa_y^2 k - 10\kappa_x^2 k + k'' - \kappa_y\kappa_y''\\
& \qquad + \kappa_x\kappa_x'' - \kappa_y'^{\,2} + \kappa_x'^{\,2})\, x^3\\
& - (2\kappa_x\underline{m} + \kappa_y m + \kappa_x^2\underline{k} + 2\kappa_x\kappa_y k)\, x^2 y\\
& + \tfrac{1}{4}(5\kappa_x m - 7\kappa_y\underline{m} + 6\kappa_x^2 k + k'' - \kappa_y\kappa_y'' - 2\kappa_y^2 k\\
& \qquad + 5\kappa_x\kappa_x'' + \kappa_x'^{\,2} - \kappa_y'^{\,2} - \kappa_x\kappa_y\underline{k})\, xy^2\\
& + \tfrac{1}{6}(10\kappa_x\kappa_y k + 8\kappa_x''\kappa_y + \kappa_x\underline{m} + 4\kappa_y^2\underline{k} + \underline{k}'' + 2\kappa_x'\kappa_y' + 5\kappa_y m)\, y^3\\
& - (2\kappa_x^2 + \tfrac{3}{2}k)\, xx'^2 - (\kappa_x'\kappa_y + \kappa_x\kappa_y')\, xx'y - \kappa_x\kappa_x'\, x^2x'\\
& - \tfrac{1}{2}\underline{k}'x^2y' - \kappa_y\kappa_y' x'y^2 - \kappa_x\kappa_y xx'y' - \tfrac{1}{2}(\underline{k} + 3\kappa_x\kappa_y)\, x'^2y\\
& + k'xyy' - \tfrac{1}{2}(k + \kappa_x^2)\, xy'^2 - (2\kappa_y^2 - \underline{k})x'yy' + \tfrac{1}{2}\underline{k}'y^2y' - \tfrac{1}{2}\underline{k}\,yy'^2\\
& + (2\kappa_x^2 + k)\, x\,\delta + (2\kappa_x\kappa_y + \underline{k})\, y\,\delta - \kappa_x' yy'\delta + \kappa_y'\, xy'\delta\\
& + \tfrac{1}{2}\kappa_x(x'^2 + y'^2)\delta + (\tfrac{3}{2}\kappa_y\underline{k} + \kappa_x\kappa_y^2 - \tfrac{1}{2}\kappa_x k - \tfrac{1}{2}\kappa_x'' - \tfrac{1}{2}m)\, y^2\delta\\
& + (\tfrac{1}{2}m + 2\kappa_x k + \kappa_x^3)\, x^2\delta + (\underline{m} + 2\kappa_x^2\kappa_y + 2\kappa_y k + 2\kappa_x\underline{k})\, xy\delta\\
& - (k + 2\kappa_x^2)\, x\delta^2 - (\underline{k} + 2\kappa_x\kappa_y)\, y\delta^2 + \mathcal{O}(4)\,.
\end{aligned}
$$

H. Wiedemann, Particle Accelerator Physics

Real life and field errors

Recall that we started with an arbitrary magnetic field and made an expansion

$$\frac{e}{p}B_z(x) = \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}ox^3 + \ldots$$

where the first term is the dipole (steering) field, the second is the quadrupole(focussing) term, the next is the sextupole term, etc.

To create these fields we build magnets with a specified field quality. These magnets will never be perfect. Therefore any magnet will have small contributions of higher-order field components.

In addition, magnets will not be perfectly aligned. E.g. if a quadrupole is displaced it will apply an additional dipole field to the beam.

Finally, the magnet strength may differ from the design value and may vary with time. E.g. a power supply may deliver too much or too little current to an electromagnet.

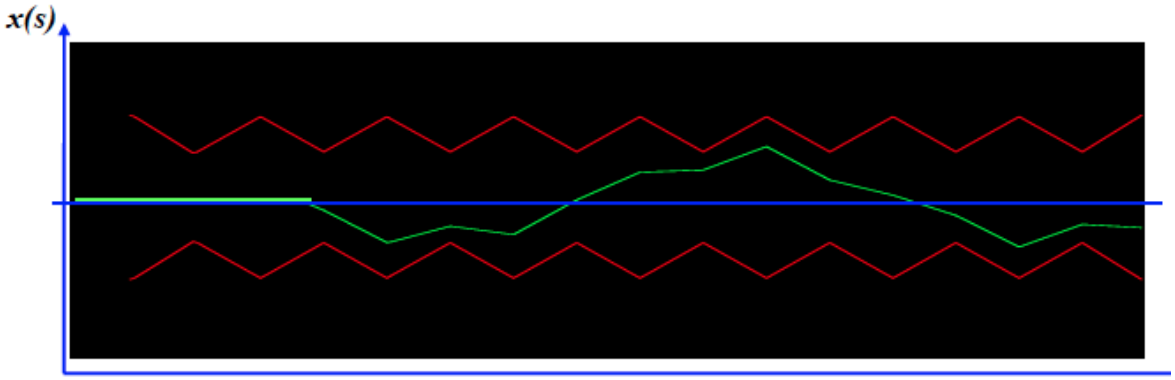
In this lecture we include the effects of some of these 'field errors' into our solutions to Hill's equation.

Closed orbit distortion

The design orbit defined by all of the dipoles in the ring is sometime referred to as the "closed design orbit".

This is the orbit a reference particle would follow in a perfect situation.

If there is a small additional dipole kick – the orbit will distort, and this distortion will affect the orbit around the entire ring. i.e. the effects of a small kick at any location are not localised; they will be seen everywhere in the ring!



This closed orbit distortion defines a position-dependent orbit offset around the ring. In effect the particles no longer undergo betatron oscillations around the design orbit but around a new closed orbit

$$x(s) = x_\beta(s) + x_{co}(s)$$

Closed orbit distortion

Imagine we have a “dipole kick” error of strength ΔB and length l, at some location s_0

$$\theta = \frac{\int B dl}{B\rho} \Rightarrow \Delta\theta = \frac{\Delta B l}{B\rho}$$

Recall the transport map in terms of the Twiss parameters

$$M(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta_1\beta_0}\sin\psi \\ \frac{\alpha_0-\alpha_1}{\sqrt{\beta_1\beta_0}}\cos\psi - \frac{1+\alpha_1\alpha_0}{\sqrt{\beta_1\beta_0}}\sin\psi & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\psi - \alpha_1\sin\psi) \end{pmatrix}$$

- The M_{12} element shows how a horizontal angular kick (Δx’) will translate into a horizontal displacement at another point in the ring.
- On each “turn” we experience another kick.
- We need to apply this map for many turns, summing over the kicks, to see how the displacement accumulates.

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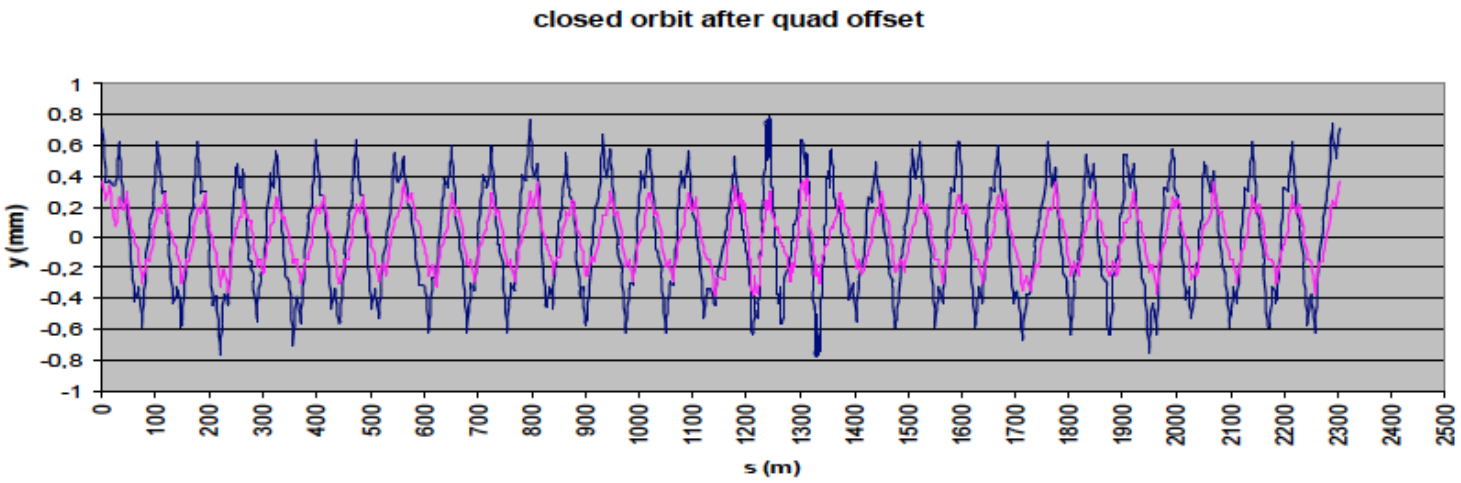
- On each “turn” we experience another kick.
- We need to apply this map for many turns, summing over the kicks, to see how the displacement accumulates.

Closed orbit distortion

A short(ish) analysis adding up the effects from each kick after a large number of turns gives the distorted closed orbit in terms of the beta function at s_0 and the beta function at any other position in the ring s.

$$x_{co}(s) = \Delta\theta \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin(\pi\nu)} \cos(\pi\nu - |\psi(s) - \psi(s_0)|) \qquad \nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$

We can minimise this distortion by monitoring the position of the beam and using orbit correction magnets.



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Resonances

Our expression for the closed orbit distortion has an overall factor of

$$\frac{1}{2\sin\pi\nu}$$

This means that every time the tune becomes an integer, the argument of the sine becomes a multiple of π, and this factor diverges.

This is an example of resonance.

Imagine the tune was 1 in a machine.

- Then the particle would encounter a dipole error at the same point in the machine and at the same phase in its betatron oscillation on every turn.
- This means the effect of the dipole error accumulates constructively on every turn.
- We avoid this by minimising magnet errors and staying away from dangerous values of the tune.

NB Consider what happens to the one turn map in this case.

Resonances

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More generally resonances occur when

$$m\nu_x + n\nu_y = p \qquad m, n, p = \text{integer}$$

Quadrupole errors

Imagine we have an extra quadrupole in our ring (or a quadrupole field error) of strength k and length L at location s_0. Unlike the dipole error, this will change the focussing properties of the lattice causing:

1)A change in the beta function
2)A change in the tune.

Recall that the tune is given by:

$$\nu = \frac{\Psi}{2\pi} = \int_s^{s+C} \frac{ds}{\beta(s)}$$

Quadrupole errors

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2)A change in the tune.

Change in the tune:

$$\Delta\nu = \frac{\beta(s_0)kL}{4\pi} = \frac{\beta(s_0)}{4\pi f}$$

- The perturbed tune increases if k > 0, which corresponds to a focusing quadrupole i.e. focussing more means more oscillations. So we get a positive tune shift for increased particle focusing.

- This means a pure quadrupole field error would shift the tune one way in one plane and the other way in the other plane

- However, we can also get tune shifts from space-charge, beam-beam effects and electron clouds, which can cause same-sign tune shift in both planes

- The effect of the quadrupole error is proportional to the local beta function. This is a common feature - the beta function magnifies local field errors.

A distribution of quadrupole errors

If we have a distribution of quadrupole errors, k(s), around the ring, the approximate tune shift can be calculated from

$$\Delta\nu = \frac{1}{4\pi}\oint ds\beta(s)k(s)$$

This effect can also be used deliberately to measure the beta functions.

- We vary the strength of a single quadrupole in the ring.
- We measure the tune.
- The response is proportional to the beta function at the quadrupole.

In general the beta function tells you how sensitive the beam is to perturbations.

Beta beat

The change in the beta function is itself a function of s, and oscillates twice as fast as the original beta function:

$$\frac{\Delta\beta(s)}{\beta} = -\frac{k\beta(s_0)}{2\sin(2\pi\nu)}\cos(2\pi\nu - 2|\psi(s) - \psi(s_0)|)$$

This is why it's called a 'beta beat'.

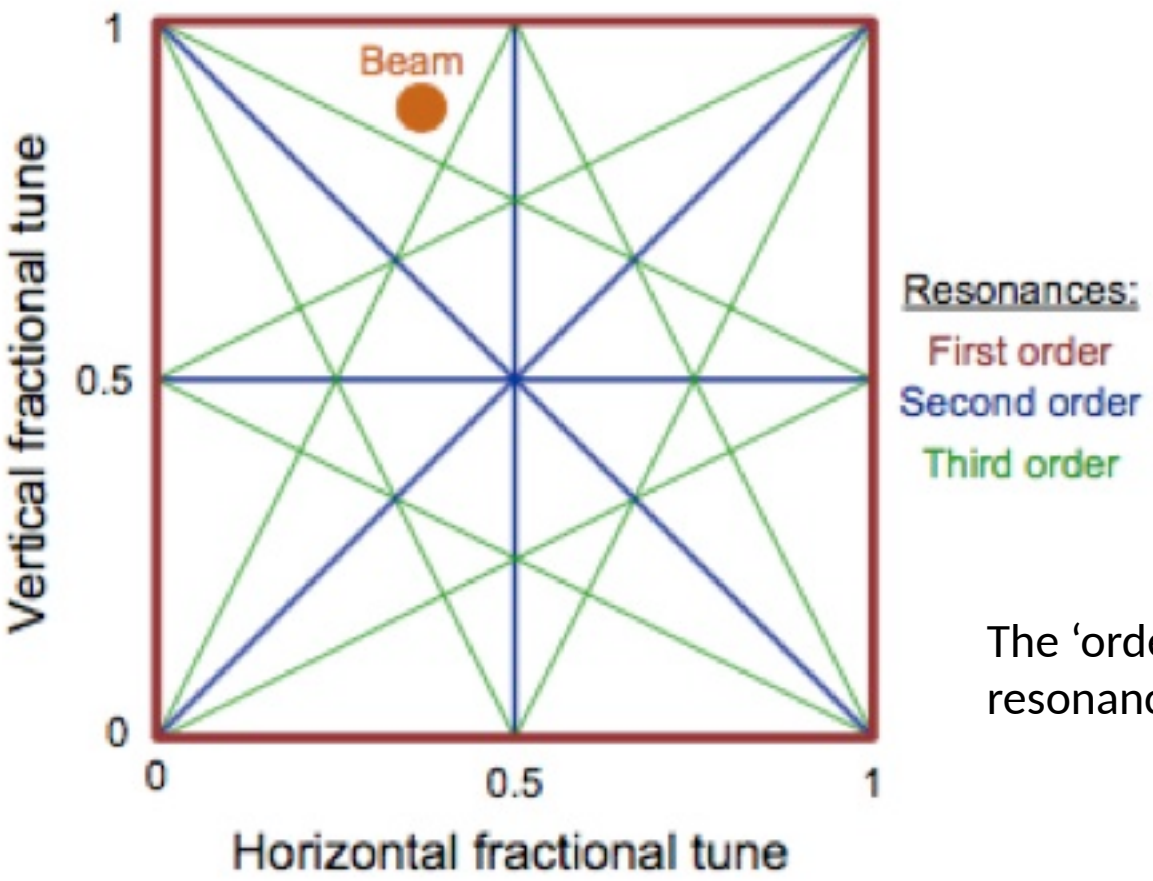
The strength of the distortion is proportional to the quadrupole error (k) and to the beta function at the position of the error s_0.

As before we have a sinusoidal term in the denominator that depends on the tune.

This gives us a 'half-integer resonance'.

Resonance diagram

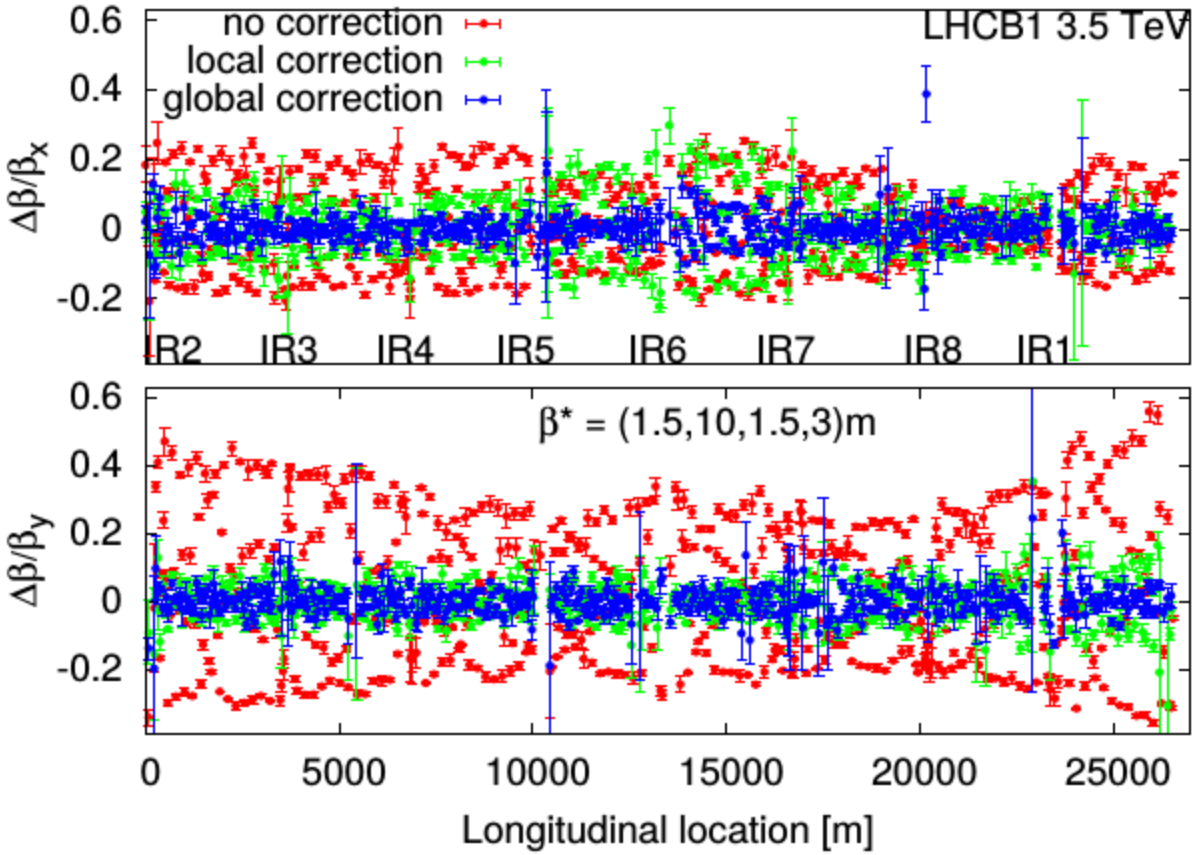
Similarly, higher-order resonances are generated by errors in high-order multipoles.



The 'order' of the resonance is m+n

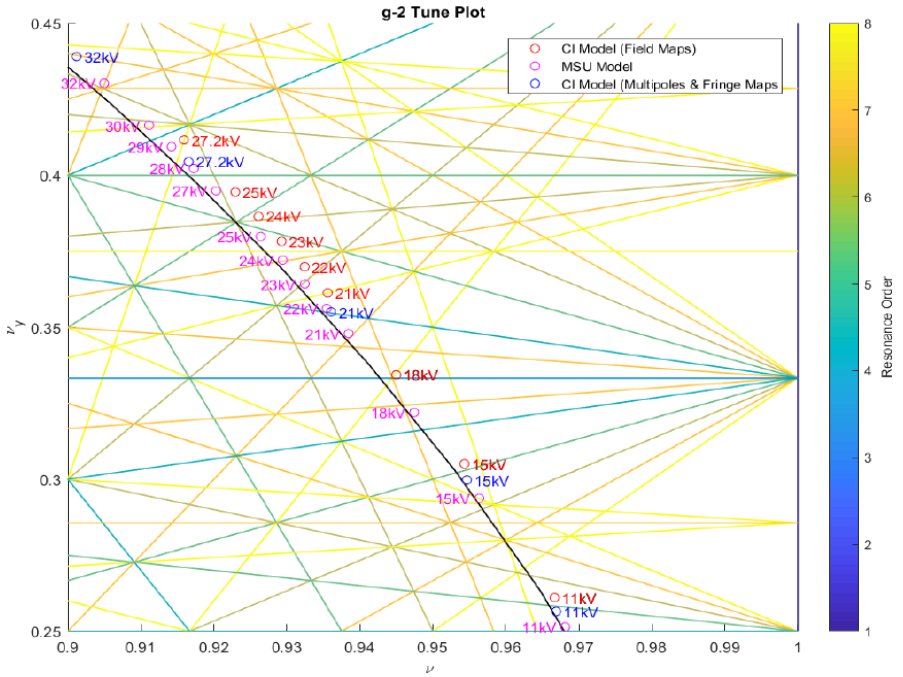
$$m, n, p = \text{integer} \qquad m\nu_x + n\nu_y = p$$

Beta beat at the LHC



Summary

- Dipole Errors
  - Introduce closed orbit distortion
  - Betatron oscillations occur around the new orbit
  - Give resonance on integer values of the tune
- Quadrupole Errors
  - Introduce a ‘beta beat’
  - Introduce a tune shift
  - Give resonances on half-integer values of the tune.
- Detailed simulation needed to calculate tune and avoid all resonances.



Alex Herrod, Liverpool / Cockcroft 2017

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