Anapole Moment Calculated within Nuclear DFT

Herlik Wibowo

Collaborators: B. C. Backes, M. Kanerva, J. Dobaczewski, M. Kortelainen, W. Haxton

IOP Nuclear Physics Conference
13 - 15 April 2026
University of Brighton, Brighton, UK

UNIVERSITY of York

UNIVERSITY OF JYVÄSKYLÄ

Anapole Moment Calculated within Nuclear DFT

Herlik Wibowo

Collaborators: B. C. Backes, M. Kanerva, J. Dobaczewski, M. Kortelainen, W. Haxton

IOP Nuclear Physics Conference
13 - 15 April 2026
University of Brighton, Brighton, UK

UNIVERSITY of York

UNIVERSITY OF JYVÄSKYLÄ

1

Anapole Moment and Fundamental Symmetry

1 Anapole Moment and Fundamental Symmetry

★ In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].

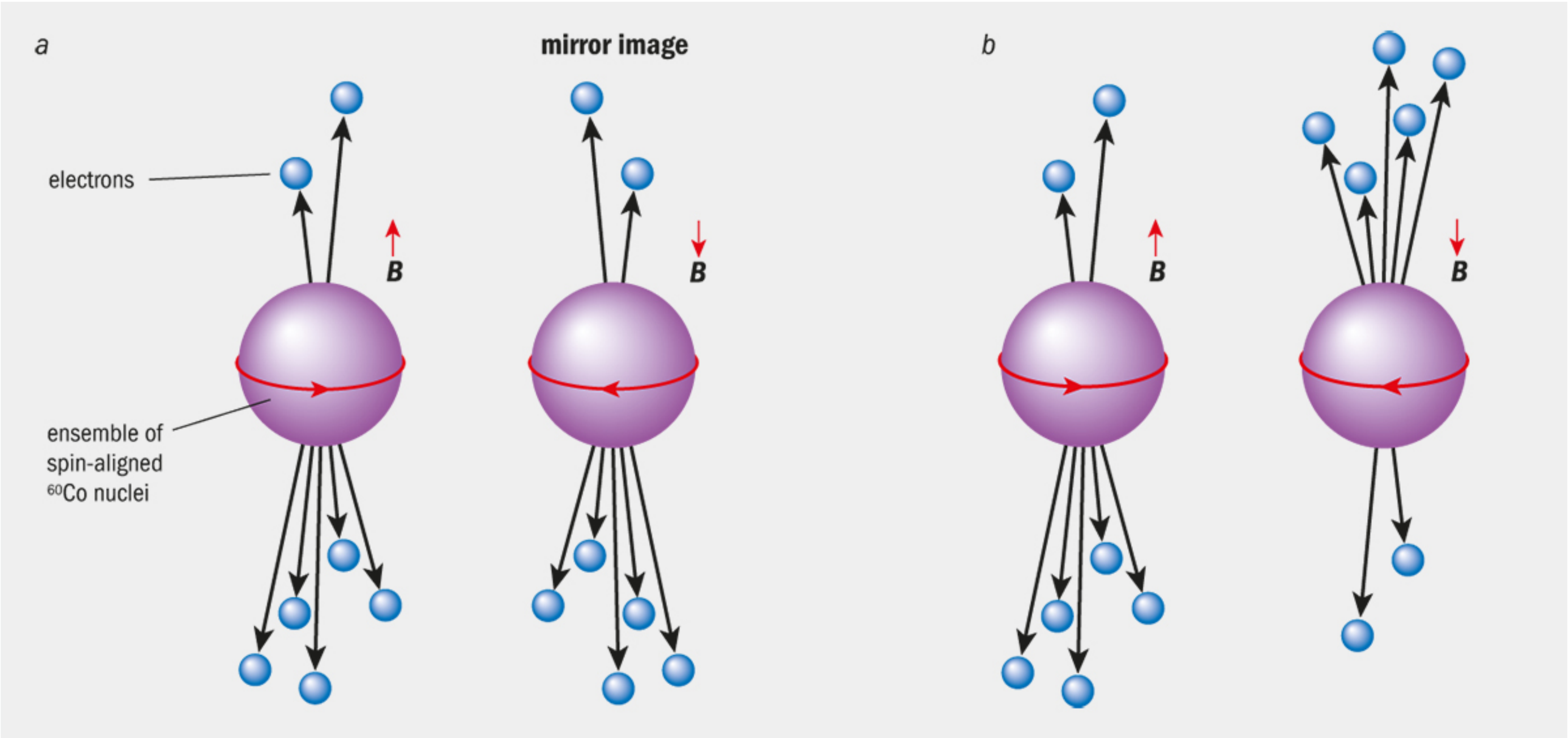
1 Anapole Moment and Fundamental Symmetry

- ★ In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].
- ★ In 1957, C. S. Wu and collaborators tested this hypothesis by studying the beta-decay of cobalt-60 nuclei [C. S. Wu et al., Phys. Rev. **105**, 1413 (1957)].

Anapole Moment and Fundamental Symmetry

1

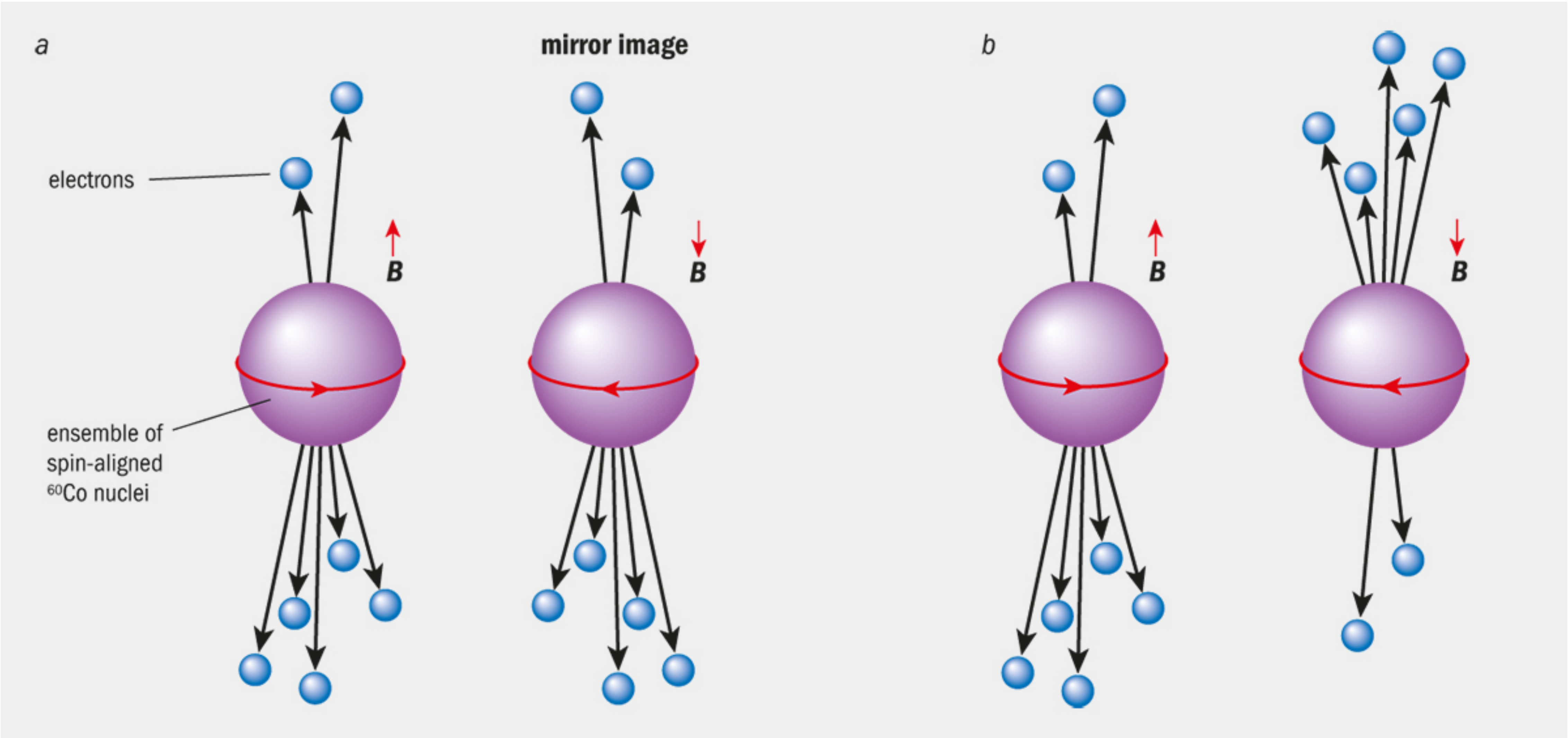
- In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].
- In 1957, C. S. Wu and collaborators tested this hypothesis by studying the beta-decay of cobalt-60 nuclei [C. S. Wu et al., Phys. Rev. **105**, 1413 (1957)].



Courtesy: IOP Publishing, Physics World, March 2026

1 Anapole Moment and Fundamental Symmetry

- ★ In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].
- ★ In 1957, C. S. Wu and collaborators tested this hypothesis by studying the beta-decay of cobalt-60 nuclei [C. S. Wu et al., Phys. Rev. **105**, 1413 (1957)].

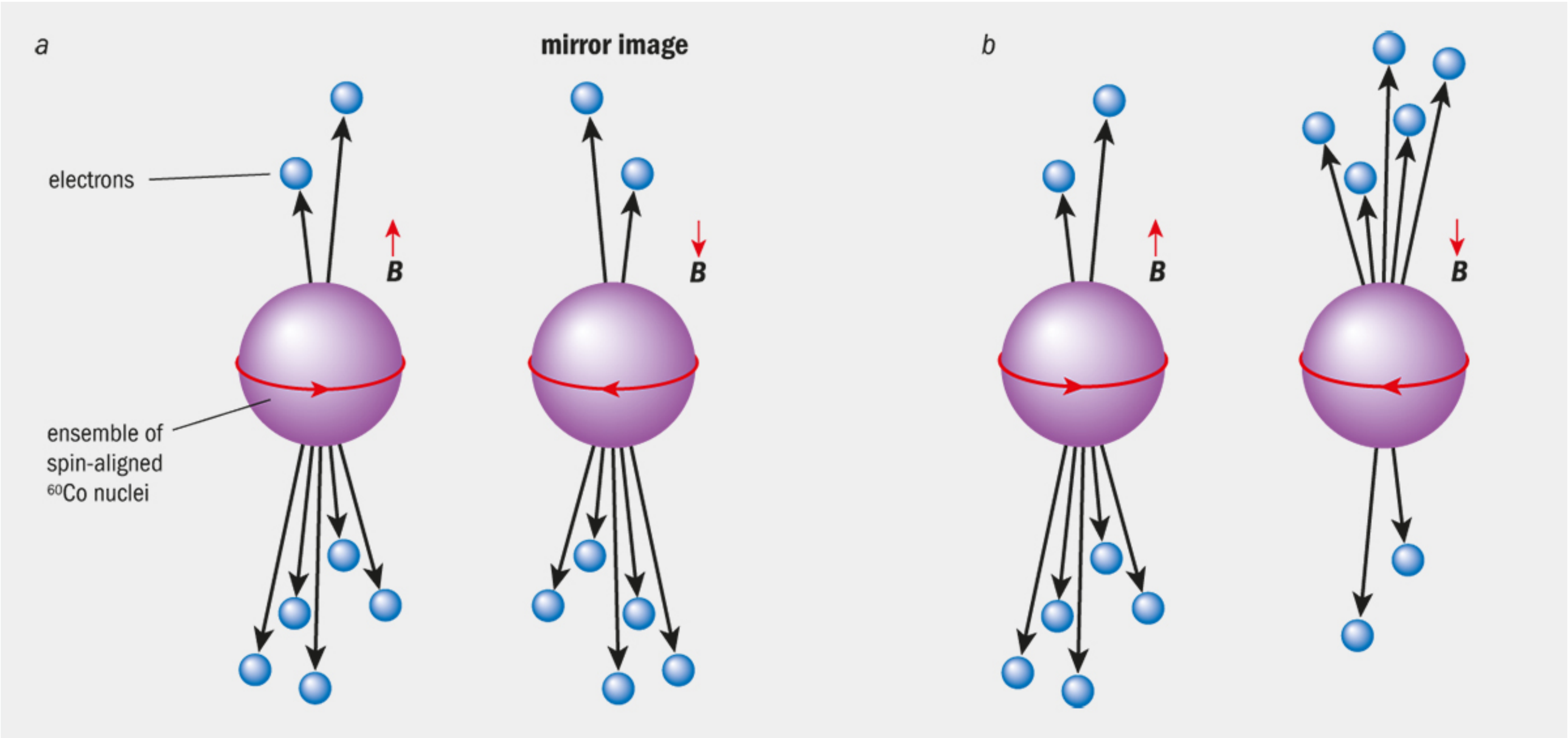


Courtesy: IOP Publishing,
Physics World, March 2026

- ★ In 1958, V. G. Vaks and Ya. B. Zel'dovich introduced the parity-violating moment, which was called the anapole moment by A. S. Kompaneets [Ya. B. Zel'dovich, Sov. Phys. JETP **6**, 1681 (1957)].

1 Anapole Moment and Fundamental Symmetry

- ★ In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].
- ★ In 1957, C. S. Wu and collaborators tested this hypothesis by studying the beta-decay of cobalt-60 nuclei [C. S. Wu et al., Phys. Rev. **105**, 1413 (1957)].

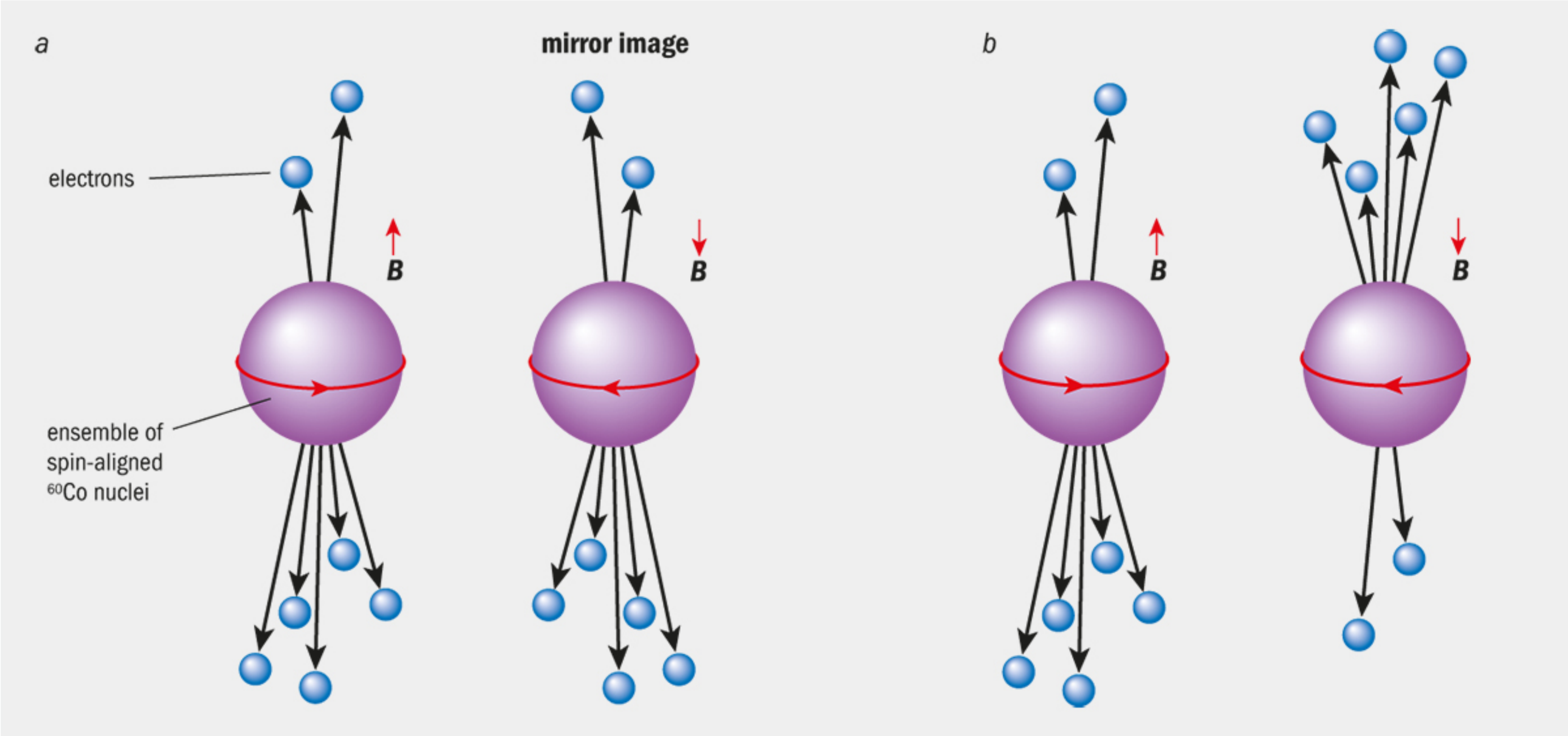


Courtesy: IOP Publishing, Physics World, March 2026

- ★ In 1958, V. G. Vaks and Ya. B. Zel'dovich introduced the parity-violating moment, which was called the anapole moment by A. S. Kompaneets [Ya. B. Zel'dovich, Sov. Phys. JETP **6**, 1681 (1957)].
- ★ In 1980s, Flambaum and collaborators observed that the anapole moment of a heavy nucleus grows as $A^{2/3}$ and is measurable in heavy atoms [Flambaum et al., Phys. Lett. **146B**, 367 (1984)].

1 Anapole Moment and Fundamental Symmetry

- ★ In 1956, T. D. Lee and C. N. Yang hypothesized that the parity is not conserved in weak interactions [T. D. Lee and C.N.Yang, Phys. Rev. **104**, 1 (1956)].
- ★ In 1957, C. S. Wu and collaborators tested this hypothesis by studying the beta-decay of cobalt-60 nuclei [C. S. Wu et al., Phys. Rev. **105**, 1413 (1957)].



Courtesy: IOP Publishing, Physics World, March 2026

- ★ In 1958, V. G. Vaks and Ya. B. Zel'dovich introduced the parity-violating moment, which was called the anapole moment by A. S. Kompaneets [Ya. B. Zel'dovich, Sov. Phys. JETP **6**, 1681 (1957)].
- ★ In 1980s, Flambaum and collaborators observed that the anapole moment of a heavy nucleus grows as $A^{2/3}$ and is measurable in heavy atoms [Flambaum et al., Phys. Lett. **146B**, 367 (1984)].
- ★ In 1997, C. S. Wood and collaborators experimentally discovered the anapole moment in cesium-133 [C. S. Wood, et al., Science **275**, 1759 (1997)].

2

Anapole Moment in Classical Electromagnetism

A. Zangwill, *Modern Electrodynamics*

2

Anapole Moment in Classical Electromagnetism

A. Zangwill, *Modern Electrodynamics*

The multipole expansion of the vector potential $\mathbf{A}(\mathbf{r})$ in the Cartesian coordinates reads

$$A_k(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\left[\int d^3s\; j_k(\mathbf{s})\right]\frac{1}{r} - \left[\frac{1}{1!}\int d^3s\; j_k(\mathbf{s})s_\ell\right]\nabla_\ell\frac{1}{r} + \left[\frac{1}{2!}\int d^3s\; j_k(\mathbf{s})s_\ell s_m\right]\nabla_\ell\nabla_m\frac{1}{r} - \cdots\right\}$$

$$\equiv \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}(-1)^n T^{(n)}_{k\ell\ldots m}\underbrace{\nabla_\ell\cdots\nabla_m}_{n\text{ terms}}\frac{1}{r}.$$

2 Anapole Moment in Classical Electromagnetism

A. Zangwill, *Modern Electrodynamics*

The multipole expansion of the vector potential $\mathbf{A}(\mathbf{r})$ in the Cartesian coordinates reads

$$A_k(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\left[\int d^3s\; j_k(\mathbf{s})\right]\frac{1}{r} - \left[\frac{1}{1!}\int d^3s\; j_k(\mathbf{s})s_\ell\right]\nabla_\ell\frac{1}{r} + \left[\frac{1}{2!}\int d^3s\; j_k(\mathbf{s})s_\ell s_m\right]\nabla_\ell\nabla_m\frac{1}{r} - \cdots\right\}$$

$$\equiv \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}(-1)^n T^{(n)}_{k\ell\ldots m}\underbrace{\nabla_\ell\cdots\nabla_m}_{n\text{ terms}}\frac{1}{r}.$$

Here tensor $T^{(n)}_{k\ell\ldots m}$ is defined as

$$T^{(n)}_{k\ell\ldots m} = \frac{1}{n!}\int d^3s\; j_k(\mathbf{s})\underbrace{s_\ell\cdots s_m}_{n\text{ terms}}.$$

Anapole Moment in Classical Electromagnetism

A. Zangwill, *Modern Electrodynamics*

The multipole expansion of the vector potential $\mathbf{A}(\mathbf{r})$ in the Cartesian coordinates reads

$$A_k(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\left[\int d^3s\; j_k(\mathbf{s})\right]\frac{1}{r} - \left[\frac{1}{1!}\int d^3s\; j_k(\mathbf{s})s_\ell\right]\nabla_\ell\frac{1}{r} + \left[\frac{1}{2!}\int d^3s\; j_k(\mathbf{s})s_\ell s_m\right]\nabla_\ell\nabla_m\frac{1}{r} - \cdots\right\}$$

$$\equiv \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}(-1)^n T^{(n)}_{k\ell\ldots m}\underbrace{\nabla_\ell\cdots\nabla_m}_{n\ \text{terms}}\frac{1}{r}.$$

Here tensor $T^{(n)}_{k\ell\ldots m}$ is defined as

$$T^{(n)}_{k\ell\ldots m} = \frac{1}{n!}\int d^3s\; j_k(\mathbf{s})\underbrace{s_\ell\cdots s_m}_{n\ \text{terms}}.$$

Anapole moment is a part of magnetic quadrupole tensor, $T^{(2)}_{k\ell m}$.

$$T^{(2)}_{k\ell m} = \frac{1}{2}\int d^3s\; j_k(\mathbf{s})s_\ell s_m = \frac{1}{3}\epsilon_{ki\ell}\int d^3s\; (\mathbf{s}\times\mathbf{j})_i s_m \equiv \frac{1}{3}\epsilon_{ki\ell}\int d^3s\; N_i s_m.$$

3

Anapole Moment in Classical Electromagnetism

3 Anapole Moment in Classical Electromagnetism

Let us now decompose the product $N_i s_m$ into

$$N_i s_m = \frac{1}{2}(N_i s_m + N_m s_i) + \frac{1}{2}(N_i s_m - N_m s_i).$$

3 Anapole Moment in Classical Electromagnetism

Let us now decompose the product $N_i s_m$ into

$$N_i s_m = \frac{1}{2}(N_i s_m + N_m s_i) + \frac{1}{2}(N_i s_m - N_m s_i).$$

The antisymmetric term gives us the anapole part, that is

$$T^{(2;\text{anapole})}_{k\ell m} = \frac{1}{6}\epsilon_{ki\ell}\int d^3s\ (N_i s_m - N_m s_i) = \frac{1}{4\pi}(\delta_{km}a_\ell - \delta_{\ell m}a_k),$$

where the anapole moment $\mathbf{a}$ is defined as

$$\mathbf{a} = \frac{2}{3}\pi \int d^3s \left[\mathbf{s}(\mathbf{s}\cdot\mathbf{j}) - s^2\mathbf{j}\right].$$

3 Anapole Moment in Classical Electromagnetism

Let us now decompose the product $N_i s_m$ into

$$N_i s_m = \frac{1}{2}(N_i s_m + N_m s_i) + \frac{1}{2}(N_i s_m - N_m s_i).$$

The antisymmetric term gives us the anapole part, that is

$$T^{(2;\text{anapole})}_{k\ell m} = \frac{1}{6}\epsilon_{ki\ell}\int d^3 s\,(N_i s_m - N_m s_i) = \frac{1}{4\pi}(\delta_{km}a_\ell - \delta_{\ell m}a_k),$$

where the anapole moment $\mathbf{a}$ is defined as

$$\mathbf{a} = \frac{2}{3}\pi\int d^3 s\left[\mathbf{s}(\mathbf{s}\cdot\mathbf{j}) - s^2\mathbf{j}\right].$$

Steady-current condition: $\quad \boldsymbol{\nabla}_{\mathrm{s}}\cdot\mathbf{j} + \underbrace{\partial\rho/\partial t}_{=0} = 0 \rightarrow \boldsymbol{\nabla}_{\mathrm{s}}\cdot\mathbf{j} = 0$

4 Anapole Moment in Classical Electromagnetism

4 Anapole Moment in Classical Electromagnetism

The anapole moment $\mathbf{a}$ reduces to

$$\mathbf{a} = -\pi \int d^3s \; s^2 \mathbf{j}.$$

Anapole Moment in Classical Electromagnetism

The anapole moment $\mathbf{a}$ reduces to

$$\mathbf{a} = -\pi \int d^3s\, s^2 \mathbf{j}.$$

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r})$ reads

$$\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \left[\mathbf{a}\, \delta(\mathbf{r}) + \boldsymbol{\nabla}(\mathbf{a} \cdot \boldsymbol{\nabla}) \frac{1}{4\pi r} \right].$$

The magnetic field $\mathbf{B}^{(\text{anapole})} = \boldsymbol{\nabla} \times \mathbf{A}^{(\text{anapole})}$ reads

$$\mathbf{B}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{a} \times \boldsymbol{\nabla} \delta(\mathbf{r}),$$

since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f(\mathbf{r}) = 0$.

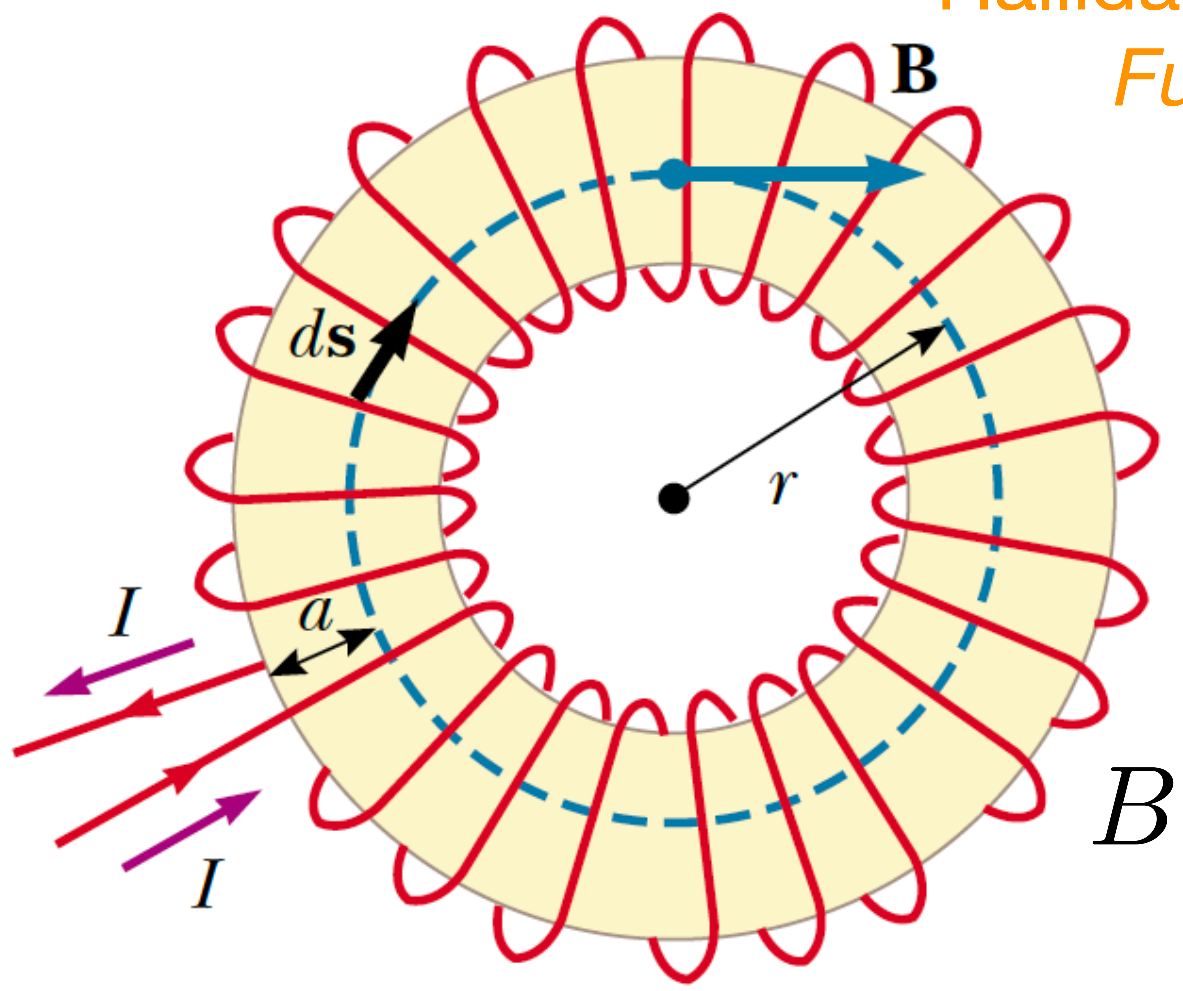
4 Anapole Moment in Classical Electromagnetism

The anapole moment $\mathbf{a}$ reduces to

$$\mathbf{a} = -\pi \int d^3s\, s^2 \mathbf{j}.$$

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r})$ reads

$$\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \left[\mathbf{a}\, \delta(\mathbf{r}) + \boldsymbol{\nabla}(\mathbf{a} \cdot \boldsymbol{\nabla}) \frac{1}{4\pi r} \right].$$



Halliday, Resnick, and Walker, *Fundamentals of Physics*

$$B = \frac{\mu_0 N I}{2\pi r} \text{ (inside)}$$
$$= 0 \text{ (outside)}$$

The magnetic field $\mathbf{B}^{(\text{anapole})} = \boldsymbol{\nabla} \times \mathbf{A}^{(\text{anapole})}$ reads

$$\mathbf{B}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{a} \times \boldsymbol{\nabla} \delta(\mathbf{r}),$$

since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f(\mathbf{r}) = 0$.

4 Anapole Moment in Classical Electromagnetism

The anapole moment $\mathbf{a}$ reduces to

$$\mathbf{a} = -\pi \int d^3s\, s^2 \mathbf{j}.$$

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r})$ reads

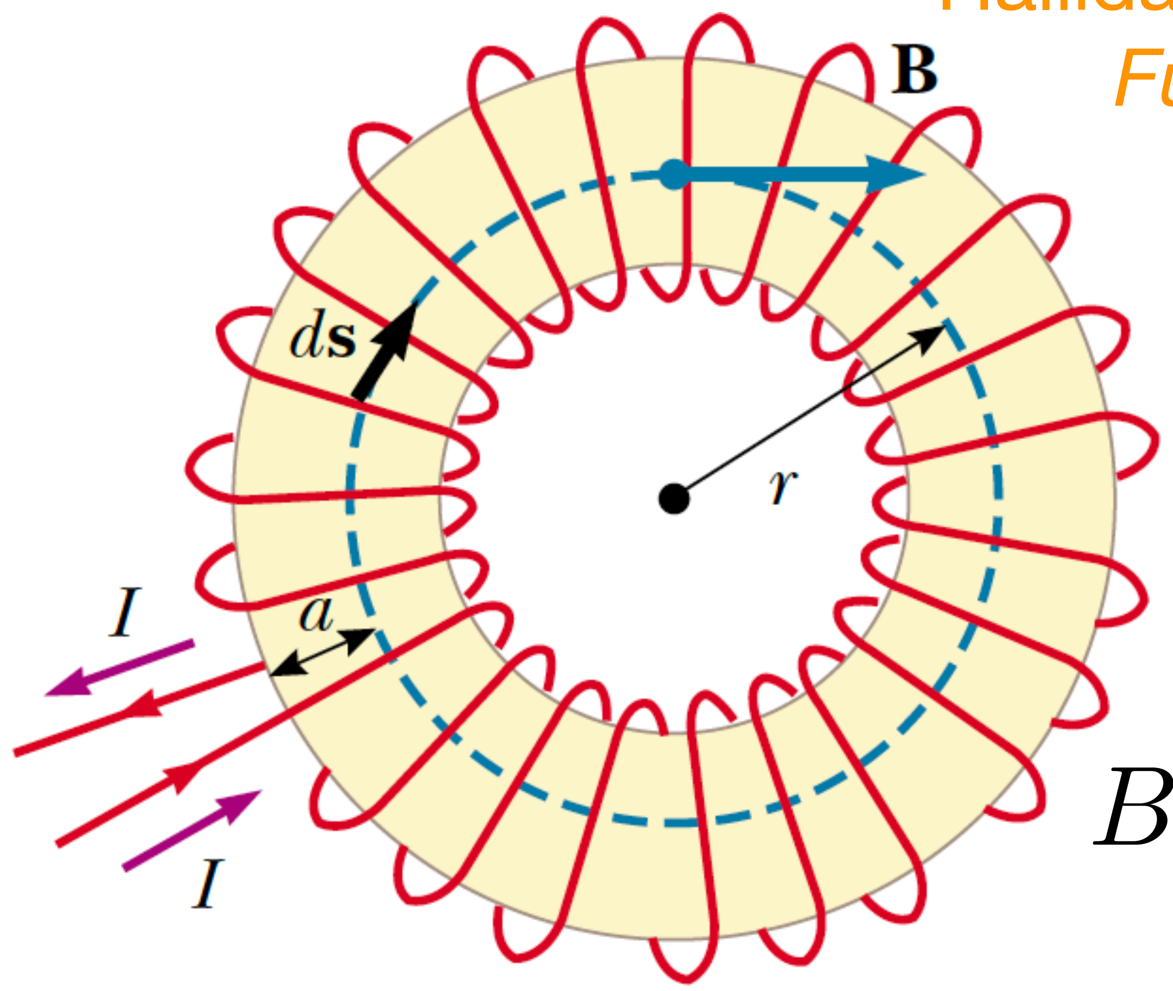
$$\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \left[\mathbf{a}\,\delta(\mathbf{r}) + \boldsymbol{\nabla}(\mathbf{a}\cdot\boldsymbol{\nabla})\frac{1}{4\pi r} \right].$$

The magnetic field $\mathbf{B}^{(\text{anapole})} = \boldsymbol{\nabla} \times \mathbf{A}^{(\text{anapole})}$

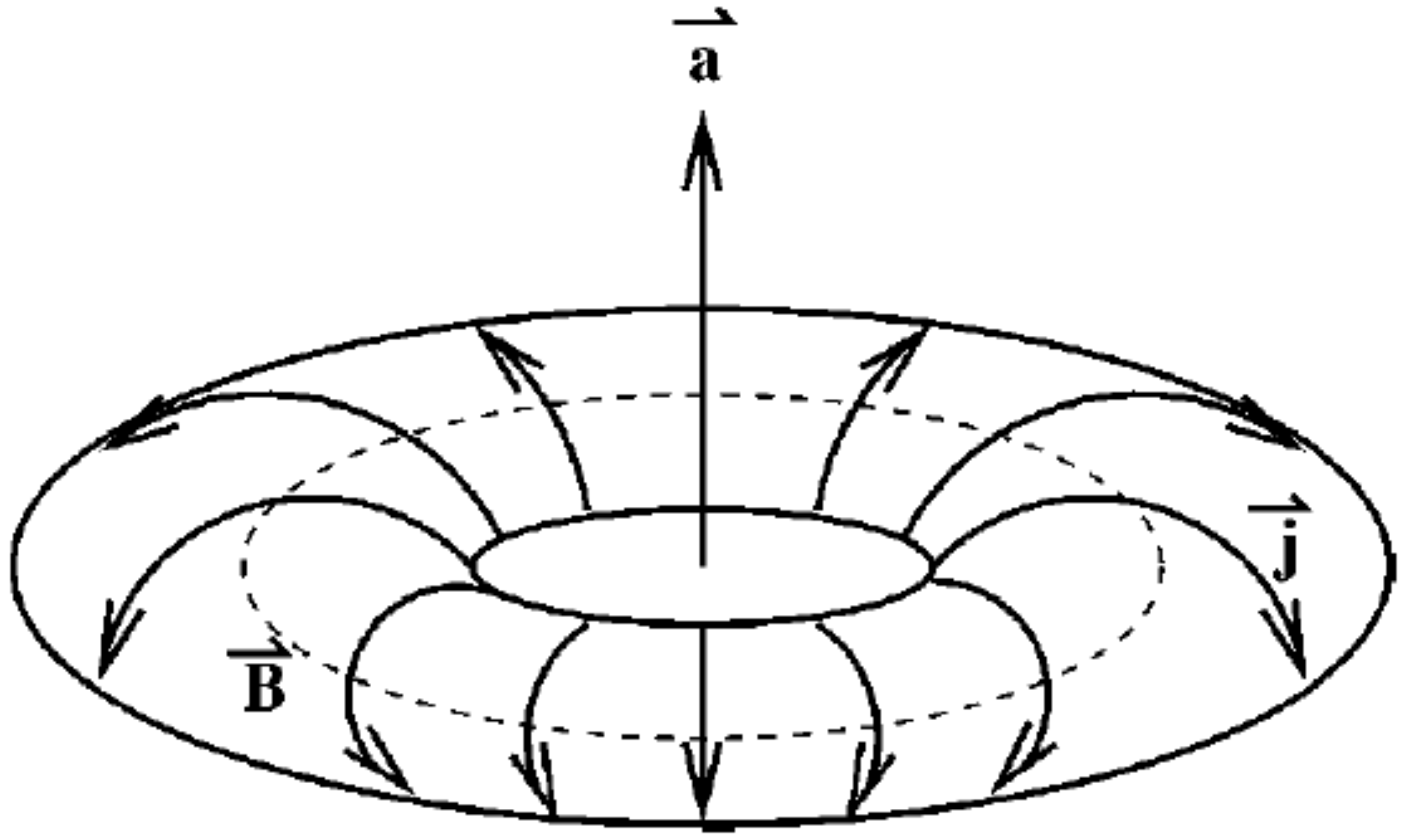
$$\mathbf{B}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{a} \times \boldsymbol{\nabla}\delta(\mathbf{r}),$$

since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f(\mathbf{r}) = 0$.

Halliday, Resnick, and Walker, *Fundamentals of Physics*



$$B = \frac{\mu_0 N I}{2\pi r} \text{ (inside)}$$
$$= 0 \text{ (outside)}$$



W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

5

Definition of Anapole Operator

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

Definition of Anapole Operator

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \mathbf{a} \delta(\mathbf{r})$ satisfies

$$\nabla^2 \mathbf{A}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{j}^{(\text{anapole})}(\mathbf{r})$$

and the anapole moment $\mathbf{a}$ can be determined via

$$\mathbf{a} = \frac{1}{q^2} \mathbf{j}^{(\text{anapole})}(\mathbf{q}).$$

Definition of Anapole Operator

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \mathbf{a} \delta(\mathbf{r})$ satisfies

$$\nabla^2 \mathbf{A}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{j}^{(\text{anapole})}(\mathbf{r})$$

and the anapole moment $\mathbf{a}$ can be determined via

$$\mathbf{a} = \frac{1}{q^2} \mathbf{j}^{(\text{anapole})}(\mathbf{q}).$$

$$\mathbf{e}_{\mathrm{q}\pm 1} \equiv \mp \frac{1}{\sqrt{2}} [\mathbf{e}_x \pm i \mathbf{e}_y]$$

$$\mathbf{e}_{\mathbf{q}0} \equiv \mathbf{e}_z = \frac{\mathbf{q}}{q}$$

Anapole operator $\hat{\mathbf{a}}$ can be determined using similar expression:

$$\hat{\mathbf{a}} = \frac{1}{q^2} \hat{\mathbf{j}}^{(\text{anapole})}(\mathbf{q}) \quad \text{or} \quad \hat{a}_\lambda = \frac{1}{q^2} \hat{j}^{(\text{anapole})}_{\mathbf{q}\lambda} \; (\lambda = 0, \pm 1).$$

Definition of Anapole Operator

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

The vector potential $\mathbf{A}^{(\text{anapole})}(\mathbf{r}) = \mu_0 \mathbf{a} \delta(\mathbf{r})$ satisfies

$$\nabla^2 \mathbf{A}^{(\text{anapole})}(\mathbf{r}) = -\mu_0 \mathbf{j}^{(\text{anapole})}(\mathbf{r})$$

and the anapole moment $\mathbf{a}$ can be determined via

$$\mathbf{a} = \frac{1}{q^2} \mathbf{j}^{(\text{anapole})}(\mathbf{q}).$$

$$\mathbf{e}_{\mathrm{q}\pm1} \equiv \mp \frac{1}{\sqrt{2}} [\mathbf{e}_x \pm i \mathbf{e}_y]$$

$$\mathbf{e}_{\mathbf{q}0} \equiv \mathbf{e}_z = \frac{\mathbf{q}}{q}$$

Anapole operator $\hat{\mathbf{a}}$ can be determined using similar expression:

$$\hat{\mathbf{a}} = \frac{1}{q^2} \hat{\mathbf{j}}^{(\text{anapole})}(\mathbf{q}) \quad \text{or} \quad \hat{a}_\lambda = \frac{1}{q^2} \hat{j}_{\mathbf{q}\lambda}^{(\text{anapole})} \ (\lambda = 0, \pm 1).$$

What is the expression for the anapole current operator, $\hat{j}_{\mathbf{q}\lambda}^{(\text{anapole})}$?

Definition of Anapole Operator

Definition of Anapole Operator

The λ-th component of the current operator $\hat{\mathbf{j}}(\mathbf{q})$ is

$$\hat{j}_{\mathbf{q}\lambda} = -\sqrt{2\pi}\sum_{J=1}^{\infty}(-i)^J\sqrt{2J+1}\left[-\hat{T}^{\mathrm{el}}_{J\lambda}(q) + \lambda\hat{T}^{\mathrm{mag}}_{J\lambda}(q)\right],$$

where the transverse electric, $\hat{T}^{\mathrm{el}}_{J\lambda}(q)$, and magnetic, $\hat{T}^{\mathrm{mag}}_{J\lambda}(q)$, multipole operators are

$$\hat{T}^{\mathrm{el}}_{J\lambda}(q) = \int d^3r\,\frac{1}{q}\boldsymbol{\nabla}\times[j_J(qr)\mathbf{Y}_{JJ\lambda}(\theta,\varphi)]\cdot\hat{\mathbf{j}}(\mathbf{r}),$$

$$\hat{T}^{\mathrm{mag}}_{J\lambda}(q) = \int d^3r\,j_J(qr)\mathbf{Y}_{JJ\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r}).$$

Definition of Anapole Operator

The λ-th component of the current operator $\hat{\mathbf{j}}(\mathbf{q})$ is

$$\hat{j}_{\mathbf{q}\lambda} = -\sqrt{2\pi}\sum_{J=1}^{\infty}(-i)^J\sqrt{2J+1}\left[-\hat{T}^{\text{el}}_{J\lambda}(q) + \lambda\hat{T}^{\text{mag}}_{J\lambda}(q)\right],$$

where the transverse electric, $\hat{T}^{\text{el}}_{J\lambda}(q)$, and magnetic, $\hat{T}^{\text{mag}}_{J\lambda}(q)$, multipole operators are

$$\hat{T}^{\text{el}}_{J\lambda}(q) = \int d^3r\, \frac{1}{q}\boldsymbol{\nabla}\times[j_J(qr)\mathbf{Y}_{JJ\lambda}(\theta,\varphi)]\cdot\hat{\mathbf{j}}(\mathbf{r}),$$

$$\hat{T}^{\text{mag}}_{J\lambda}(q) = \int d^3r\, j_J(qr)\mathbf{Y}_{JJ\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r}).$$

Which multipole operator(s) should be used for $\hat{j}^{(\text{anapole})}_{\mathbf{q}\lambda}$?

Definition of Anapole Operator

Definition of Anapole Operator

J	$\hat{T}^{\mathrm{el}}_{J\lambda}(q)$	$\hat{T}^{\mathrm{mag}}_{J\lambda}(q)$
0		
1	$\not{P}T$	PT
2	$P\not{T}$	$\not{P}\not{T}$
3	$\not{P}T$	PT
$\vdots$	$\vdots$	$\vdots$

Definition of Anapole Operator

J	$\hat{T}^{\mathrm{el}}_{J\lambda}(q)$	$\hat{T}^{\mathrm{mag}}_{J\lambda}(q)$
0		
1	$\not{P}T$	PT
2	$P\not{T}$	$\not{P}\not{T}$
3	$\not{P}T$	PT
$\vdots$	$\vdots$	$\vdots$

The anapole current operator $\hat{j}^{(\mathrm{anapole})}_{\mathbf{q}\lambda}$ is then given by

$$\hat{j}^{(\mathrm{anapole})}_{\mathbf{q}\lambda} = -i\sqrt{6\pi}\,\hat{T}^{\mathrm{el}}_{1\lambda}(q)$$

and the corresponding anapole operator $\hat{a}_\lambda$ reads

$$\hat{a}_\lambda = \lim_{q^2\to 0} \frac{-i\sqrt{6\pi}}{q^2}\,\hat{T}^{\mathrm{el}}_{1\lambda}(q).$$

8 Current Conservation and Siegert's Theorem

Current Conservation and Siegert's Theorem

The transverse electric multipole operator, $\hat{T}^{\text{el}}_{J\lambda}(q)$, is given by

$$\hat{T}^{\text{el}}_{J\lambda}(q) = -\frac{i}{q}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr) Y_{J\lambda}(\theta,\varphi)\boldsymbol{\nabla}\cdot\hat{\mathbf{j}}(\mathbf{r}) - i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r}).$$

Current Conservation and Siegert's Theorem

The transverse electric multipole operator, $\hat{T}^{\text{el}}_{J\lambda}(q)$, is given by

$$\hat{T}^{\text{el}}_{J\lambda}(q) = -\frac{i}{q}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr) Y_{J\lambda}(\theta,\varphi)\boldsymbol{\nabla}\cdot\hat{\mathbf{j}}(\mathbf{r}) - i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r}).$$

Siegert's theorem: $\boldsymbol{\nabla}\cdot\hat{\mathbf{j}}(\mathbf{r}) \doteq -\frac{1}{i\hbar}(E_i - E_f)\hat{\rho}(\mathbf{r})$

Current Conservation and Siegert's Theorem

The transverse electric multipole operator, $\hat{T}^{\text{el}}_{J\lambda}(q)$, is given by

$$\hat{T}^{\text{el}}_{J\lambda}(q) = -\frac{i}{q}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr)Y_{J\lambda}(\theta,\varphi)\boldsymbol{\nabla}\cdot\hat{\mathbf{j}}(\mathbf{r}) - i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r}).$$

Siegert's theorem:

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{j}}(\mathbf{r}) \doteq -\frac{1}{i\hbar}(E_i - E_f)\hat{\rho}(\mathbf{r})$$

$$\hat{T}^{\text{el}}_{J\lambda}(q) \doteq \frac{i}{q}\frac{E_i - E_f}{i\hbar}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr)Y_{J\lambda}(\theta,\varphi)\hat{\rho}(\mathbf{r}) - i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

Current Conservation and Siegert's theorem

$$\hat{T}^{\mathrm{el}}_{J\lambda}(q) \doteq \frac{i}{q}\frac{E_i - E_f}{i\hbar}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr) Y_{J\lambda}(\theta,\varphi)\hat{\rho}(\mathbf{r})$$

$$- i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

Current Conservation and Siegert's theorem

$$\hat{T}^{\mathrm{el}}_{J\lambda}(q) \doteq \frac{i}{q}\frac{E_i - E_f}{i\hbar}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr)Y_{J\lambda}(\theta,\varphi)\hat{\rho}(\mathbf{r})$$
$$- i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

★ At the long-wavelength limit ($qr \to 0$), the second term vanishes, and the first term gives the correct leading-order behavior for transitions.

Current Conservation and Siegert's theorem

$$\hat{T}^{\mathrm{el}}_{J\lambda}(q) \doteq \frac{i}{q}\frac{E_i - E_f}{i\hbar}\sqrt{\frac{J+1}{J}}\int d^3r\, j_J(qr) Y_{J\lambda}(\theta,\varphi)\hat{\rho}(\mathbf{r})$$
$$- i\sqrt{\frac{2J+1}{J}}\int d^3r\, j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

- ★ At the long-wavelength limit ($qr \to 0$), the second term vanishes, and the first term gives the correct leading-order behavior for transitions.
- ★ For the static moment, the first term is zero, but the second term is not constrained by current conservation.

Current Conservation and Siegert's theorem

$$\hat{T}^{\mathrm{el}}_{J\lambda}(q) \doteq \frac{i}{q}\frac{E_i - E_f}{i\hbar}\sqrt{\frac{J+1}{J}}\int d^3r\; j_J(qr) Y_{J\lambda}(\theta,\varphi)\hat{\rho}(\mathbf{r}) - i\sqrt{\frac{2J+1}{J}}\int d^3r\; j_{J+1}(qr)\mathbf{Y}_{J(J+1)\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

- ★ At the long-wavelength limit ($qr \to 0$), the second term vanishes, and the first term gives the correct leading-order behavior for transitions.
- ★ For the static moment, the first term is zero, but the second term is not constrained by current conservation.
- ★ This expression of $\hat{T}^{\mathrm{el}}_{J\lambda}(q)$ is inadequate for nuclear DFT calculations because the employed current density operator is a one-body current density operator, not a many-body current density operator, so that the current conservation is not guaranteed.

Extended Siegert's theorem

J. L. Friar and S. Fallieros, PRC **29**, 1645 (1984)

Extended Siegert's theorem

J. L. Friar and S. Fallieros, PRC **29**, 1645 (1984)

Friar and Fallieros expressed the full current operator, $\hat{\mathbf{j}}(\mathbf{q})$, in terms of charge density operator, $\hat{\rho}(\mathbf{r})$, and magnetization density operator, $\hat{\boldsymbol{\mu}}(\mathbf{r})$.

Extended Siegert's theorem

J. L. Friar and S. Fallieros, PRC **29**, 1645 (1984)

Friar and Fallieros expressed the full current operator, $\hat{\mathbf{j}}(\mathbf{q})$, in terms of charge density operator, $\hat{\rho}(\mathbf{r})$, and magnetization density operator, $\hat{\boldsymbol{\mu}}(\mathbf{r})$.

$$\hat{j}_{\mathbf{q}\eta} \doteq -\sqrt{2\pi}\sum_{J=1}^{\infty} i^J\sqrt{2J+1}\left[\hat{T}^{\mathrm{el}}_{J\eta}(q)+\eta\hat{T}^{\mathrm{mag}}_{J\eta}(q)\right],$$

$$\hat{T}^{\mathrm{el}}_{J\eta}(q) \doteq \frac{E_i-E_f}{\hbar q}\sqrt{\frac{J+1}{J}}\frac{q^J}{(2J+1)!!}\int d^3r\ \hat{\rho}(\mathbf{r})r^J g_J(qr)Y_{J\eta}(\theta,\varphi)$$

$$+\frac{2q^{J+1}}{(J+2)(2J+1)!!}\int d^3r\ r^J\hat{\boldsymbol{\mu}}(\mathbf{r})\cdot\mathbf{Y}_{JJ\eta}(\theta,\varphi)h_J(qr),$$

$$\hat{T}^{\mathrm{mag}}_{J\eta}(q) = 2i\sqrt{\frac{2J+1}{J+1}}\int d^3r\ \frac{1}{r}\ \hat{\boldsymbol{\mu}}(\mathbf{r})\cdot\mathbf{Y}_{J(J-1)\eta}(\theta,\varphi)\ j_J(qr),$$

$$g_J(z) = 1-\frac{Jz^2}{2(2J+3)(J+2)}\cdots \quad \text{and} \quad h_J(z) = 1-\frac{1}{2}\frac{(J+2)z^2}{(J+4)(2J+3)}\cdots.$$

11

Final Expression of the Anapole Operator

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

Final Expression of the Anapole Operator

For calculation of the static anapole moment, one focuses on the second term of operator $\hat{T}^{\rm el}_{J\eta}(q)$ and simply omits the first term. Set $J=1$, one gets

$$\hat{T}^{\rm el}_{1\eta}(q) = -i\frac{q^2}{9}\int d^3r\, r^2\left[\frac{1}{\sqrt{6\pi}}\hat{j}_\eta(\mathbf{r}) + \frac{1}{\sqrt{3}}\left[Y_2(\theta,\varphi)\otimes\hat{\mathbf{j}}(\mathbf{r})\right]_\eta\right]\left[1-\frac{1}{2}\frac{(J+2)(qr)^2}{(J+4)(2J+3)}\cdots\right].$$

Finally, the final expression of the anapole operator, $\hat{a}_\eta$, takes the form:

$$\begin{aligned}\hat{a}_\eta &= \lim_{q^2\to 0}\frac{-i\sqrt{6\pi}}{q^2}\hat{T}^{\rm el}_{1\eta}(q)\\ &= -\frac{1}{9}\int d^3r\, r^2\left[\hat{j}_\eta(\mathbf{r}) + \sqrt{2\pi}\left[Y_2(\theta,\varphi)\otimes\hat{\mathbf{j}}(\mathbf{r})\right]_\eta\right].\end{aligned}$$

W. C. Haxton, C.-P. Liu, and M. J. Ramsey-Musolf, PRC **65**, 045502 (2002)

Laboratory Anapole Moment (Work in Progress)

Laboratory Anapole Moment (Work in Progress)

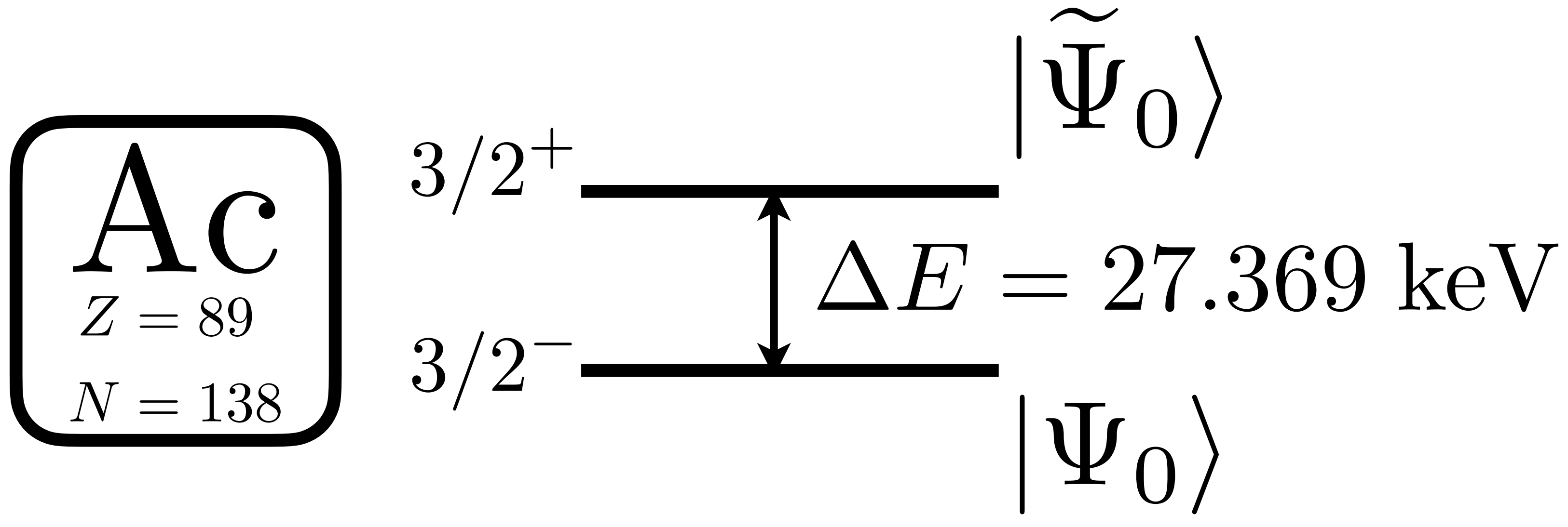
Laboratory anapole moment is calculated as

$$a_z^{\text{lab.}} \approx \sum_{i \neq 0} \frac{\langle \Psi_0 | \hat{a}_z | \Psi_i \rangle \langle \Psi_i | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{E_0 - E_i} + \text{c.c.}$$

Laboratory Anapole Moment (Work in Progress)

Laboratory anapole moment is calculated as

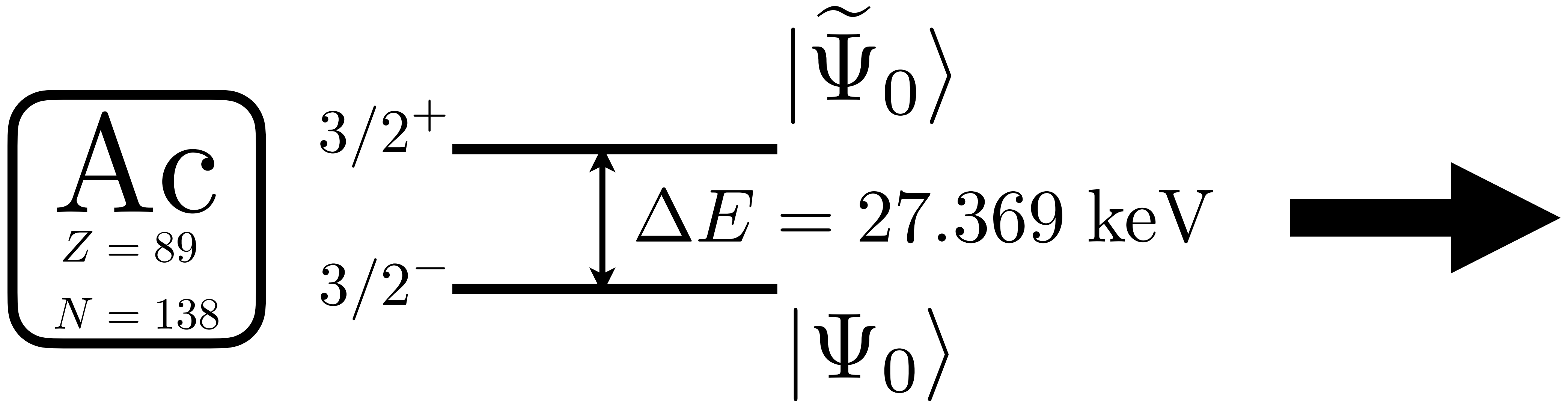
$$a_z^{\text{lab.}} \approx \sum_{i\neq 0} \frac{\langle \Psi_0 | \hat{a}_z | \Psi_i \rangle \langle \Psi_i | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{E_0 - E_i} + \text{c.c.}$$



Laboratory Anapole Moment (Work in Progress)

Laboratory anapole moment is calculated as

$$a_z^{\text{lab.}} \approx \sum_{i \neq 0} \frac{\langle \Psi_0 | \hat{a}_z | \Psi_i \rangle \langle \Psi_i | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{E_0 - E_i} + \text{c.c.}$$

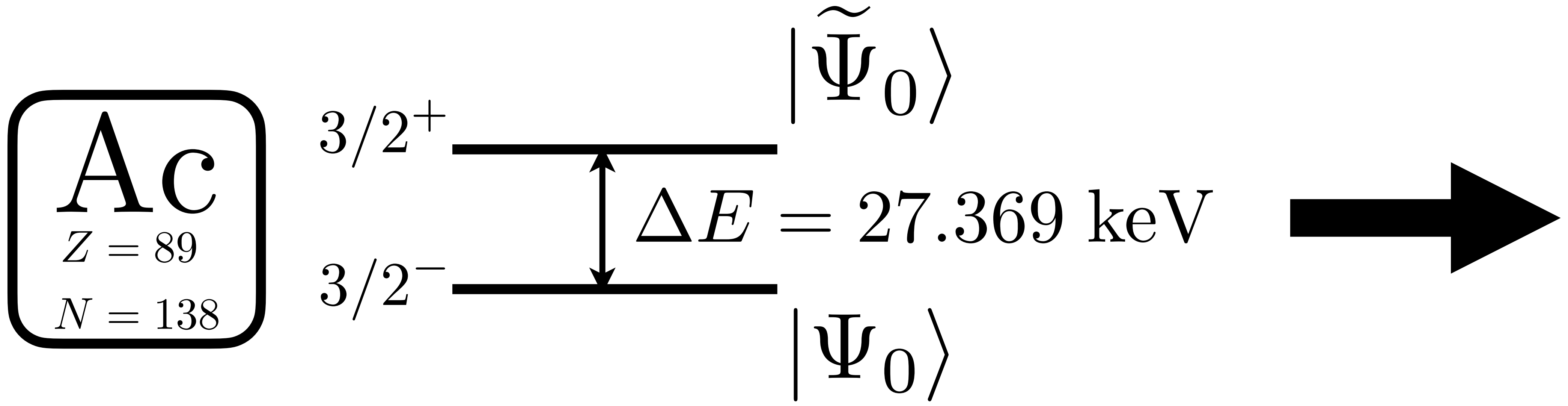


$$a_z^{\text{lab.}} \approx -2\Re \left\{ \frac{\langle \Psi_0 | \hat{a}_z | \widetilde{\Psi}_0 \rangle \langle \widetilde{\Psi}_0 | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{\Delta E} \right\}$$

Laboratory Anapole Moment (Work in Progress)

Laboratory anapole moment is calculated as

$$a_z^{\text{lab.}} \approx \sum_{i \neq 0} \frac{\langle \Psi_0 | \hat{a}_z | \Psi_i \rangle \langle \Psi_i | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{E_0 - E_i} + \text{c.c.}$$



$$a_z^{\text{lab.}} \approx -2\Re \left\{ \frac{\langle \Psi_0 | \hat{a}_z | \widetilde{\Psi}_0 \rangle \langle \widetilde{\Psi}_0 | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{\Delta E} \right\}$$

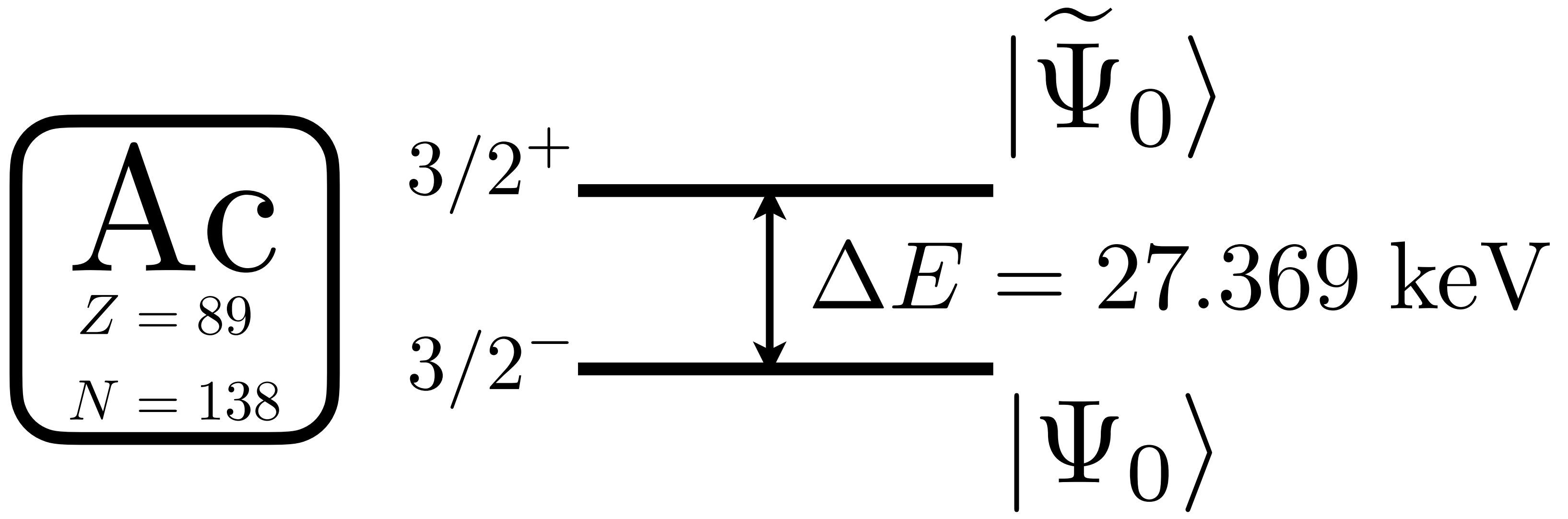
Anapole operator:

$$\hat{a}_z = -\frac{1}{9} \int d^3 r \, r^2 \left\{ \hat{j}_z(\mathbf{r}) + \sqrt{2\pi} \left[Y_2(\theta, \varphi) \otimes \hat{\mathbf{j}}(\mathbf{r}) \right]_{10} \right\}.$$

Laboratory Anapole Moment (Work in Progress)

Laboratory anapole moment is calculated as

$$a_z^{\text{lab.}} \approx \sum_{i \neq 0} \frac{\langle \Psi_0 | \hat{a}_z | \Psi_i \rangle \langle \Psi_i | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{E_0 - E_i} + \text{c.c.}$$

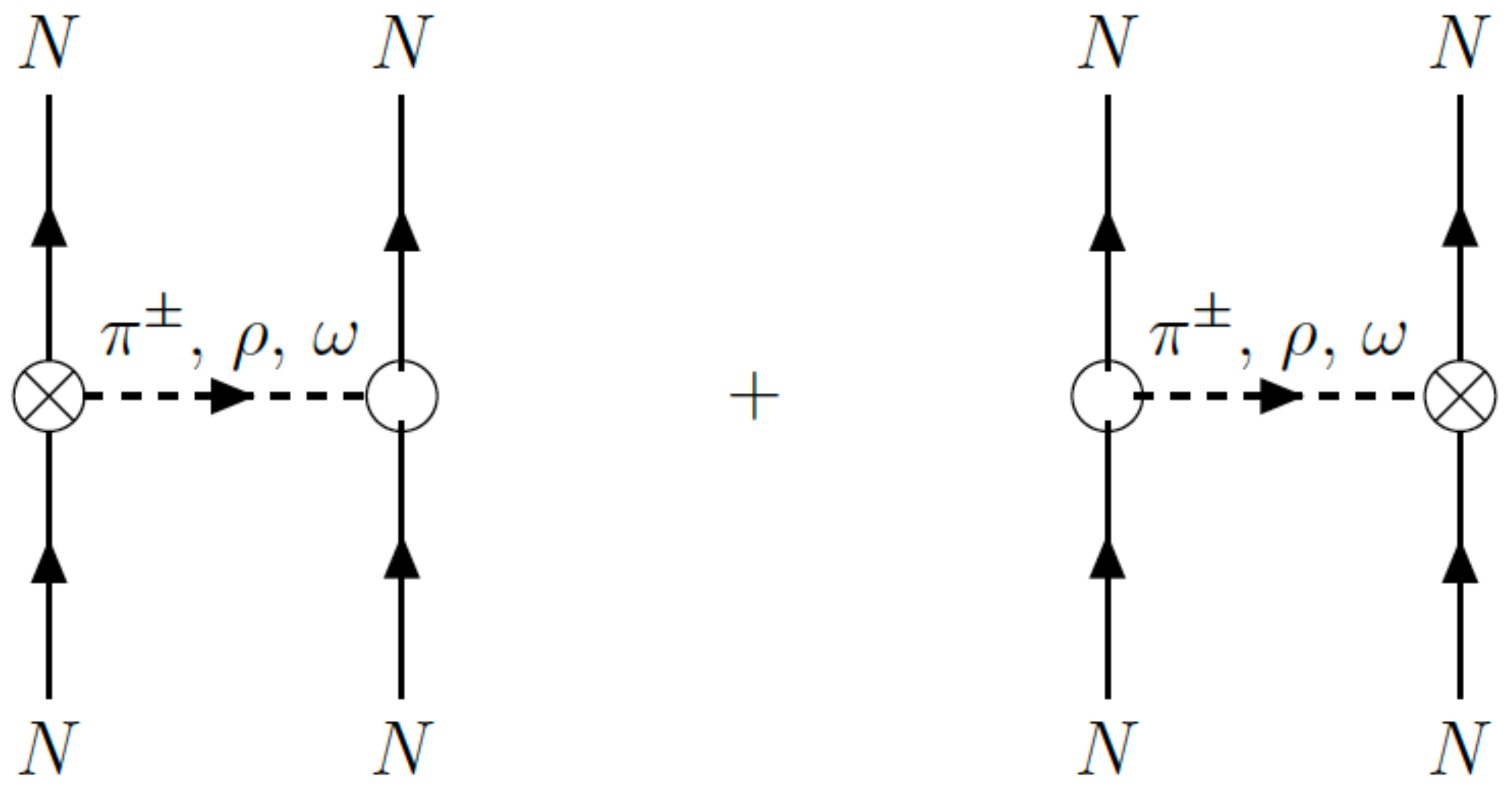


$$a_z^{\text{lab.}} \approx -2\Re \left\{ \frac{\langle \Psi_0 | \hat{a}_z | \widetilde{\Psi}_0 \rangle \langle \widetilde{\Psi}_0 | \hat{V}_{\text{PNC}} | \Psi_0 \rangle}{\Delta E} \right\}$$

Anapole operator:

$$\hat{a}_z = -\frac{1}{9} \int d^3r \, r^2 \left\{ \hat{j}_z(\mathbf{r}) + \sqrt{2\pi} \left[Y_2(\theta, \varphi) \otimes \hat{\mathbf{j}}(\mathbf{r}) \right]_{10} \right\}.$$

Parity-nonconserving potential (PNC):



B. Desplanques, J. F. Donoghue, B. R. Holstein, Ann. Phys. **124**, 449 (1980)

Summary and Outlook

Summary

- ★ Nuclear anapole moments are the parity-odd and time-reversal-even E1 moments of the electromagnetic current operator.
- ★ The anapole operator derived from the extended Siegert's theorem has been chosen because there is no prescription for constructing the many-body current that guarantees current conservation.
- ★ The laboratory anapole moment is calculated using the perturbation theory, where the DDH potential is the parity-nonconserving (PNC) potential.

Outlook

- ★ To determine the intrinsic anapole moments for the isotopic chain of Actinium using different energy density functionals.
- ★ To calculate the laboratory anapole moments for octupole deformed nuclei.

Acknowledgements

We acknowledge the support from a **Leverhulme Trust Research Project Grant**. This work was partially supported by the **STFC Grant** Nos. ST/P003885/1 and ST/V001035/1 and by the **Polish National Science Centre** under Contract No. 2018/31/B/ST2/02220. We acknowledge the **CSC-IT Center for Science Ltd., Finland**, for the allocation of computational resources. This project was partly undertaken on the **Viking Cluster**, which is a high performance compute facility provided by the **University of York**. We are grateful for computational support from the **University of York High Performance Computing service, Viking and the Research Computing team**.

Back Up Slides

Parity and Time Reversal

$$\hat{T}^{\text{el}}_{J\lambda}(q) = \int d^3r \, \frac{1}{q} \boldsymbol{\nabla} \times [j_J(qr) \mathbf{Y}_{JJ\lambda}(\theta, \varphi)] \cdot \hat{\mathbf{j}}(\mathbf{r})$$

Under the parity transformation:

$$\hat{\mathbf{j}}(\mathbf{r}) \to -\hat{\mathbf{j}}(\mathbf{r})$$

$$\boldsymbol{\nabla} \to -\boldsymbol{\nabla}$$

$$Y_{J\lambda}(\theta, \varphi) \to (-1)^J Y_{J\lambda}(\theta, \varphi)$$

$$\mathbf{Y}_{JJ\lambda}(\theta, \varphi) \equiv \frac{(-i\mathbf{r} \times \boldsymbol{\nabla}) \, Y_{J\lambda}(\theta, \varphi)}{\sqrt{J(J+1)}} \to (-1)^J \mathbf{Y}_{JJ\lambda}(\theta, \varphi)$$

Therefore: $$\hat{T}^{\text{el}}_{J\lambda}(q) \to (-1)^J \hat{T}^{\text{el}}_{J\lambda}(q)$$

B2

$$\hat{T}^{\text{mag}}_{J\lambda}(q) = \int d^3r\ j_J(qr)\mathbf{Y}_{JJ\lambda}(\theta,\varphi)\cdot\hat{\mathbf{j}}(\mathbf{r})$$

Under the parity transformation:

$$\hat{T}^{\text{mag}}_{J\lambda}(q) \to (-1)^{J+1}\hat{T}^{\text{mag}}_{J\lambda}(q)$$

Under the time-reversal transformation:

$$\begin{aligned}
\hat{\mathbf{j}}(\mathbf{r}) &\to -\hat{\mathbf{j}}(\mathbf{r}) \\
Y_{J\lambda}(\theta,\varphi) &\to (-1)^{\lambda}Y_{J(-\lambda)}(\theta,\varphi) \\
\mathbf{Y}_{JJ\lambda}(\theta,\varphi) &\to \frac{1}{\sqrt{J(J+1)}}(+i\mathbf{r}\times\boldsymbol{\nabla})(-1)^{\lambda}Y_{J(-\lambda)}(\theta,\varphi) \\
&= (-1)^{\lambda+1}\mathbf{Y}_{JJ(-\lambda)}(\theta,\varphi)
\end{aligned}$$

B3

To determine the properties of the multipole operators under the time-reversal transformation, we will adhere to the convention introduced by Bohr and Mottelson, that is:

$$\hat{\mathcal{T}}\hat{T}_{J\lambda}\hat{\mathcal{T}}^{-1} = c_{\mathcal{T}}(-1)^{J+\lambda}\hat{T}_{J(-\lambda)}$$

where $c_{\mathcal{T}}$ is the additional phase factor introduced by the time-reversal transformation.

$$\begin{aligned}\hat{T}^{\text{el}}_{J\lambda}(q) &\to \frac{1}{q}\int d^3r\, \boldsymbol{\nabla}\times\left[j_J(qr)(-1)^{\lambda+1}\mathbf{Y}_{JJ(-\lambda)}(\theta,\varphi)\right]\cdot(-1)\hat{\mathbf{j}}(\mathbf{r})\\ &= (-1)^{\lambda}\hat{T}^{\text{el}}_{J(-\lambda)}(q)\\ &= (-1)^{J+1}(-1)^{J+\lambda+1}\hat{T}^{\text{el}}_{J(-\lambda)}(q) \rightsquigarrow c_{\mathcal{T}} = (-1)^{J+1}\end{aligned}$$

B4

$$\hat{T}^{\text{mag}}_{J\lambda}(q) \to \int d^3r\ j_J(qr)(-1)^{\lambda+1}\mathbf{Y}_{JJ(-\lambda)}(\theta,\varphi)\cdot(-1)\hat{\mathbf{j}}(\mathbf{r})$$
$$= (-1)^{\lambda}\hat{T}^{\text{mag}}_{J(-\lambda)}(q)$$
$$= (-1)^{J+1}(-1)^{J+\lambda+1}\hat{T}^{\text{mag}}_{J(-\lambda)}(q) \rightsquigarrow c_{\mathcal{T}} = (-1)^{J+1}$$

B4

$$\hat{T}^{\text{mag}}_{J\lambda}(q) \to \int d^3r \; j_J(qr)(-1)^{\lambda+1}\mathbf{Y}_{JJ(-\lambda)}(\theta,\varphi)\cdot(-1)\hat{\mathbf{j}}(\mathbf{r})$$
$$= (-1)^{\lambda}\hat{T}^{\text{mag}}_{J(-\lambda)}(q)$$
$$= (-1)^{J+1}(-1)^{J+\lambda+1}\hat{T}^{\text{mag}}_{J(-\lambda)}(q) \rightsquigarrow c_{\mathcal{T}} = (-1)^{J+1}$$

	$\hat{T}^{\text{el}}_{J\lambda}(q)$	$\hat{T}^{\text{mag}}_{J\lambda}(q)$
Parity	$(-1)^J$	$(-1)^{J+1}$
Time reversal	$(-1)^{J+1}$	$(-1)^{J+1}$