Useful things to know about accelerators – part I

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Accelerators - A Window on Nature

- Particle accelerators provide the source for most high energy physics experiments
  - Provide high luminosity, high energy beams for colliders
  - Provide high brightness beams for secondary particle production
  - Also key technology for life sciences, engineering, chemistry
- How do they work?
  - How can we get to high energy?
  - How can we keep the beam in the accelerator?
  - How can we get to high luminosity?
- What are the main HEP facilities in the world today?
- What might HEP facilities look like in the future?

Accelerator Components

- Most accelerators share similar components
- Main components of an accelerator
  - Bending – dipoles
  - Focussing – quadrupoles
  - Acceleration - RF cavities
- Also
  - Vacuum
  - Diagnostics
  - Targets for secondary particle production
- First Lecture: Derive basic theory of accelerator physics
- Second Lecture: Discuss accelerator equipment and techniques

Lorentz force law

- Fundamental equation for particles moving through fields

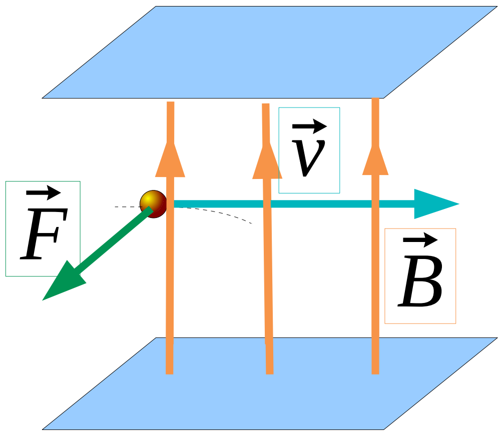
Force | Charge | Velocity | Magnetic Field | Electric Field

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E} \qquad \text{(eq. 1)}$$

- Magnetic force is perpendicular to velocity
  - Magnetic field conserves energy
- Electric force is weaker by factor velocity
  - Magnets are better for bending and focussing

Magnetic Rigidity and Bending

- Simplest magnet - “dipole”
  - Uniform magnetic field perpendicular to beam direction



Lorentz force (eq. 1) + centripetal motion:

$$qvB = \frac{pv}{\rho}$$

Radius

Rearranging:

$$\underbrace{B\rho} = \frac{p}{q}$$

Magnetic Rigidity

- Constant force → constant curvature → circular motion
- Magnetic rigidity parameterises momentum
- Charge-to-mass ratio important when accelerating multiple particle species

Worked example – LHC

- If we wanted to accelerate, say, 7 TeV/c particles, what bending radius is required?
- Maximum dipole field around 8.3 T

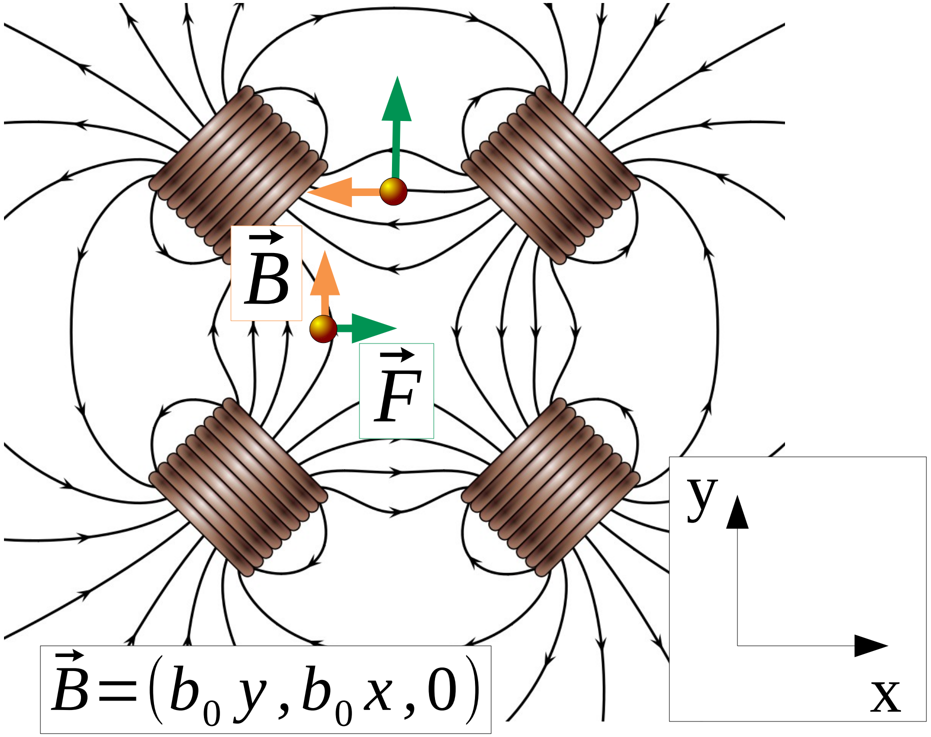
$$B\rho = \frac{p}{q}$$

$$\rho = \frac{p}{qB} = \frac{7e12}{8.3 \times 3e8} = 2800\,m$$

- Nb: LHC radius ~ 4.1 km
  - Need space for detectors, etc

Quadrupole magnets

- If we only had bending magnets, particles would soon be lost from the accelerator
- Need to keep the particles in the accelerator using focussing elements
  - Usually use quadrupoles
- Field stronger away from beam centre
  - Like a spring or pendulum
  - Simple harmonic motion
- “F” quad focuses in x and defocuses in y
- “D” quad focuses in y and defocuses in x
- Overall focussing by alternating “F” and “D”
  - Just reverse the field



Quadrupole field – horizontal (1)

- For a particle moving near to the z-axis

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

$$\vec{B} = (b_0 y, b_0 x, 0)$$

- Considering only p_x for now

$$\frac{dp_x}{dt} = q\frac{dz}{dt} B_y$$

$$\vec{v} \times \vec{B} = \begin{pmatrix} v_z B_y \\ -v_z B_x \\ 0 \end{pmatrix}$$

- Use the chain rule

$$\frac{dp_x}{dt} = \frac{dp_x}{dz}\frac{dz}{dt}$$

- Combining these equations:

$$\frac{dp_x}{dz} = q b_0 x$$

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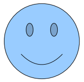
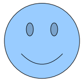
Quadrupole field – horizontal (2)

$$\frac{dp_x}{dz} = q\,b_0\,x$$ 

- Definition of x-component of momentum

$$p_x = m\,\gamma\,v_x = m\,\gamma\frac{dz}{dt}\frac{dx}{dz} = p_z\frac{dx}{dz}$$

- Substitute this definition into  gives

$$p_z\frac{d^2x}{dz^2} = q\,b_0\,x$$

- Rearrange and wrap up constant terms in focussing strength *k*

$$\frac{d^2x}{dz^2} - k\,x = 0$$

Quadrupole field – vertical

- Lorentz force law with quadrupole field definition

$$\frac{dp_y}{dt} = -q\, b_0\, v_z\, y$$

- Use chain rule and eliminate v_z

$$p_z \frac{d^2 y}{dz^2} = -q\, b_0\, y$$

- Rearrange and wrap up constant terms in defocussing strength *k*

$$\frac{d^2 y}{dz^2} + k\, y = 0$$

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Solutions

- Motion is governed by

$$\frac{d^2x}{dz^2} - kx = 0 \qquad \frac{d^2y}{dz^2} + ky = 0$$

- This is simple harmonic motion – solutions are of form

$$x = x_0 \cos\left(\sqrt{k}\,z\right) + \frac{dx_0}{dz}\frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,z\right)$$

- Taking derivative

$$\frac{dx}{dz} = -x_0\sqrt{k}\sin\left(\sqrt{k}\,z\right) + \frac{dx_0}{dz}\cos\left(\sqrt{k}\,z\right)$$

For y

$$y = y_0 \cosh\left(\sqrt{k}\,z\right) + \frac{dy_0}{dz}\frac{1}{\sqrt{k}}\sinh\left(\sqrt{k}\,z\right)$$

$$\frac{dy}{dz} = y_0\sqrt{k}\sinh\left(\sqrt{k}\,z\right) + \frac{dy_0}{dz}\cosh\left(\sqrt{k}\,z\right)$$

Transfer Matrix

- Just thinking about *x*, the particles move according to

$$x_1 = x_0 \cos\left(\sqrt{k}\,z\right) + \frac{dx_0}{dz}\sin\left(\sqrt{k}\,z\right)$$

$$\frac{dx_1}{dz} = -x_0\sqrt{k}\sin\left(\sqrt{k}\,z\right) + \frac{dx_0}{dz}\sqrt{k}\cos\left(\sqrt{k}\,z\right)$$

- We can rewrite this as a matrix

$$\begin{pmatrix} x_1 \\ \dfrac{dx_1}{dz} \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,z\right) & \dfrac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,z\right) \\ -\sqrt{k}\sin\left(\sqrt{k}\,z\right) & \cos\left(\sqrt{k}\,z\right) \end{pmatrix} \begin{pmatrix} x_0 \\ \dfrac{dx_0}{dz} \end{pmatrix}$$

- This matrix is known as the quadrupole's **transfer matrix**

$$\underline{u}_1 = \boldsymbol{M}_{\mathbf{01}}\,\underline{u}_0$$

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Questions

Questions

- Exercise – what is the transfer matrix for a drift space, that is a region with no fields at all?
  - What is the force acting on the particle?
  - What is $x(z)$ in terms of dx_0/dz and x_0
  - What is dx/dz in terms of dx_0/dz
  - Now write that as a matrix

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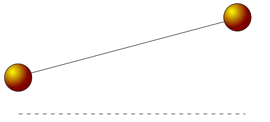
Questions

- Exercise – what is the transfer matrix for a drift space?
  - What is the force acting on the particle?
    - No force
  - What is x(z) in terms of dx_0/dz and x_0

$$x = x_0 + \frac{dx_0}{dz} z$$



  - What is dx/dz in terms of dx_0/dz

$$\frac{dx}{dz} = \frac{dx_0}{dz}$$

  - Now write that as a matrix

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$$

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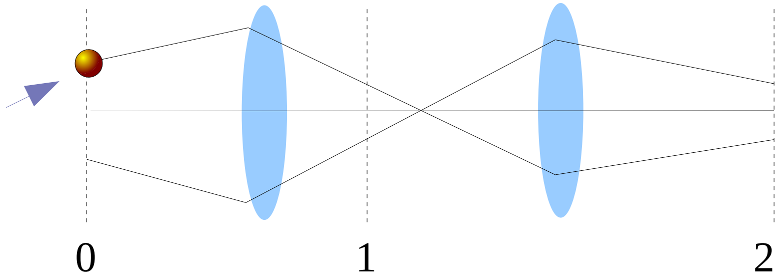
Transfer Lines

$\underline{u}_0 = \begin{pmatrix} x_0 \\ \frac{dx_0}{dz} \end{pmatrix}$



- Transfer matrix defines transport through a region
- Transfer matrices can be combined by multiplication
- Say we have transfer matrices like:

$$\underline{u}_1 = \boldsymbol{M}_{01} \underline{u}_0$$

$$\underline{u}_2 = \boldsymbol{M}_{12} \underline{u}_1$$

- Then

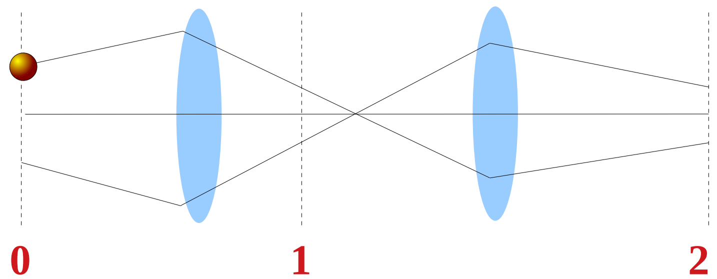
$$\underline{u}_2 = \boldsymbol{M}_{12} \boldsymbol{M}_{01} \underline{u}_0$$

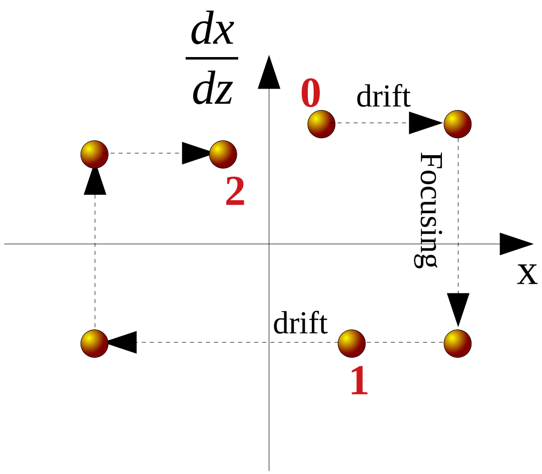
- i.e. we can define a combined transfer matrix like

$$\boldsymbol{M}_{02} = \boldsymbol{M}_{12} \boldsymbol{M}_{01}$$

Phase space

- Another instructive way to look at beam optics is by considering the phase space





Symplecticity

- There is a general rule for what transfer matrices are allowed by equations of motion
  - "Symplectic condition"
- Formally a matrix **M** is *symplectic* if it satisfies

transpose

Identity matrix

$$\mathbf{M}^{\mathrm{T}}\,\mathbf{S}\,\mathbf{M} = \mathbf{I}$$

  - Where

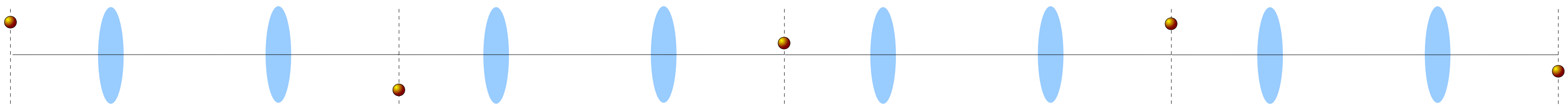
$$\boldsymbol{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- It can be shown that any *symplectic* matrix **M** can be written as

$$\boldsymbol{M} = \boldsymbol{I}\cos\mu + \boldsymbol{J}\sin\mu$$

$$\boldsymbol{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Periodic Lattices



- Following *n* identical cells or turns in a ring with one-turn matrix ***M***

$$\underline{u}_n = \boldsymbol{M}^n \underline{u}_0$$

- Rewrite

$$\boldsymbol{M} = \boldsymbol{I}\cos\mu + \boldsymbol{J}\sin\mu$$

$$\boldsymbol{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \gamma\beta - \alpha^2 = 1 \quad \text{and} \quad \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- So

$$\boldsymbol{J}^2 = -\boldsymbol{I}$$

- And

$$\boldsymbol{M}^n = \boldsymbol{I}\cos(n\mu) + \boldsymbol{J}\sin(n\mu)$$

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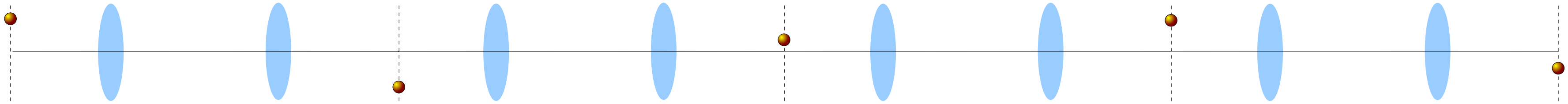
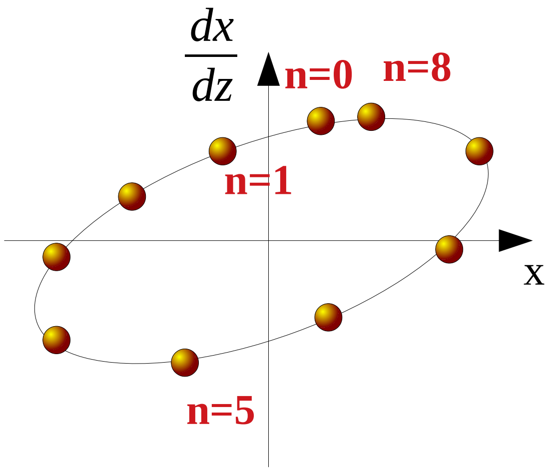
Periodic Lattices



- What does this mean?

$$M^n = I\cos(n\mu) + J\sin(n\mu)$$



- Particles move around an ellipse in phase space if Trace(**M**) < 2
- μ is the "phase advance"
  - Sometimes use "tune" ... 2π ν = μ
- α, β and γ are "Twiss parameters"
  - Tell us the alignment of the ellipse
- Each particle sits on ellipse area ε - the particle's amplitude

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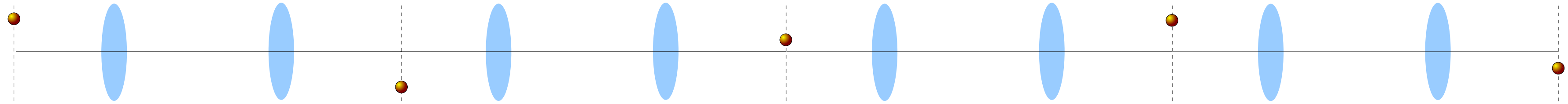
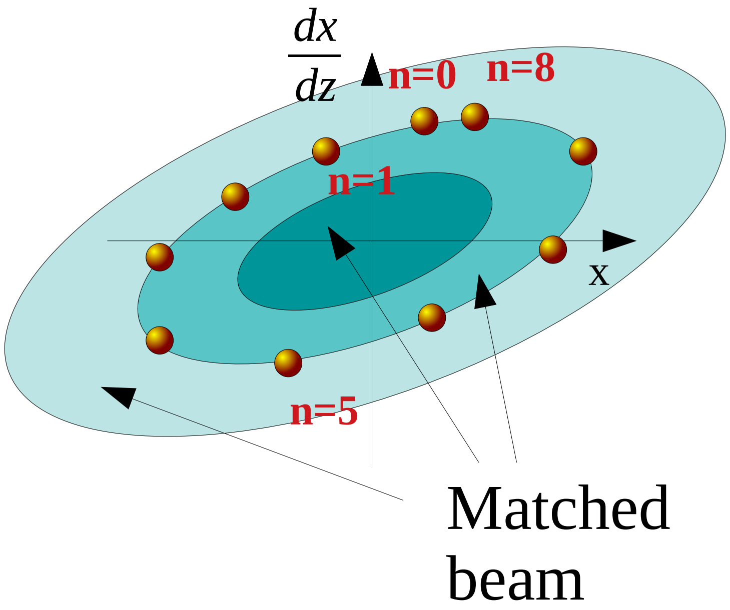
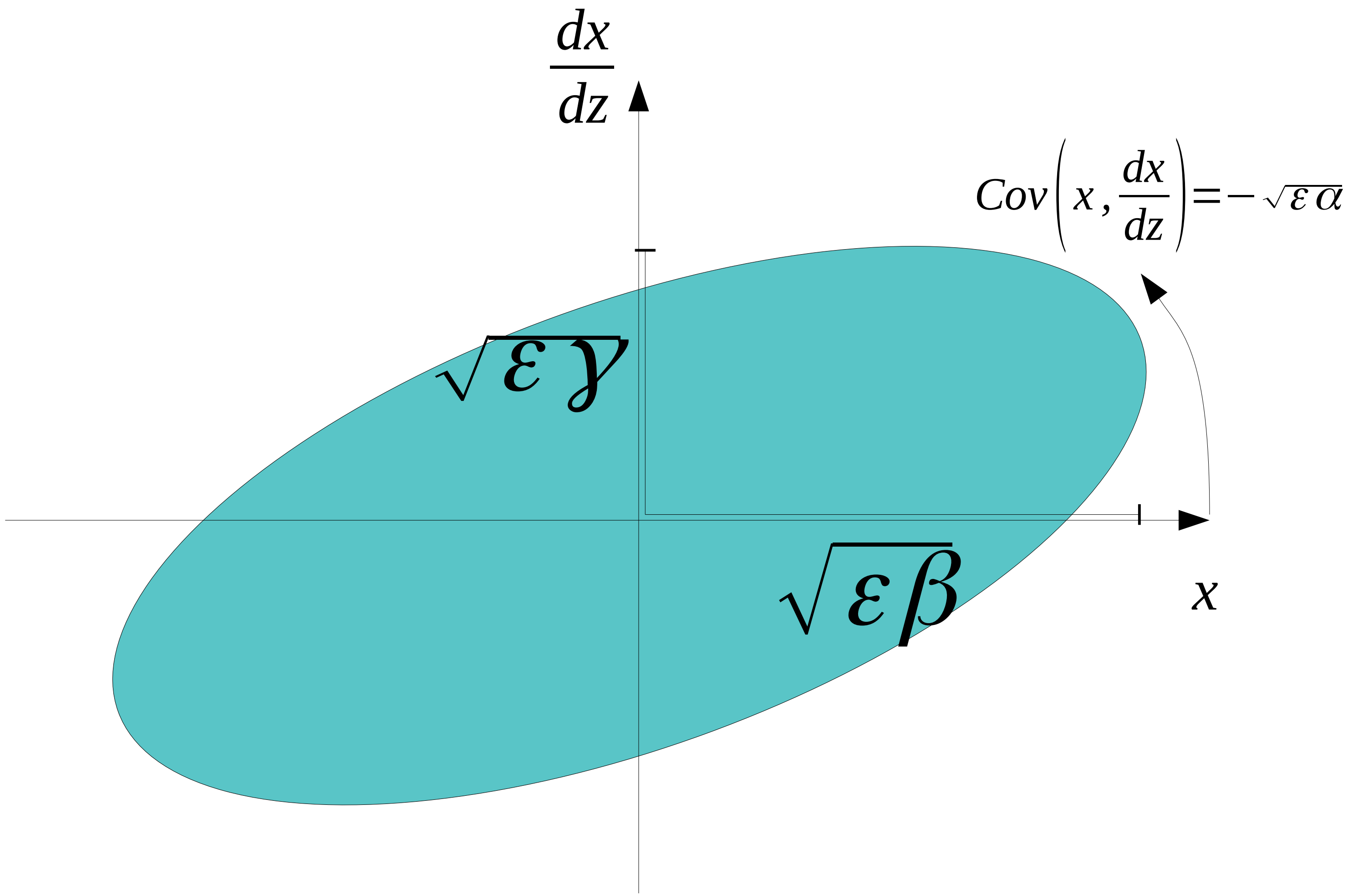
Periodic Lattices and beams



- Beam is composed of many particles
  - Particles occupy a region in phase space
- “Emittance” is area occupied by the entire beam
- Sometimes classify “RMS emittance”
  - Area occupied by ellipse 1 RMS distance from beam centre
- Low emittance is crucial for
  - High luminosity
  - Low losses



Beam ellipse



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Questions

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Questions

- What is behaviour of particles in phase space if
  - Trace(M) < 2
  - Trace(M) = 2
  - Trace(M) > 2

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Questions

- What is behaviour of particles in phase space if
  - Trace(M) < 2
    - Motion is an ellipse
  - Trace(M) = 2
    - $x \rightarrow +/- x$
  - Trace(M) > 2
    - Motion is a hyperbola

Emittance Growth

- Ideally emittance is conserved, but this is not always the case
- Long list of effects that can cause emittance growth
  - Beam mismatch
  - Scattering off residual gas
  - Scattering off particles in the same beam
  - Scattering off particles in other beams (e.g. in collider)
  - Space charge
  - Resonances

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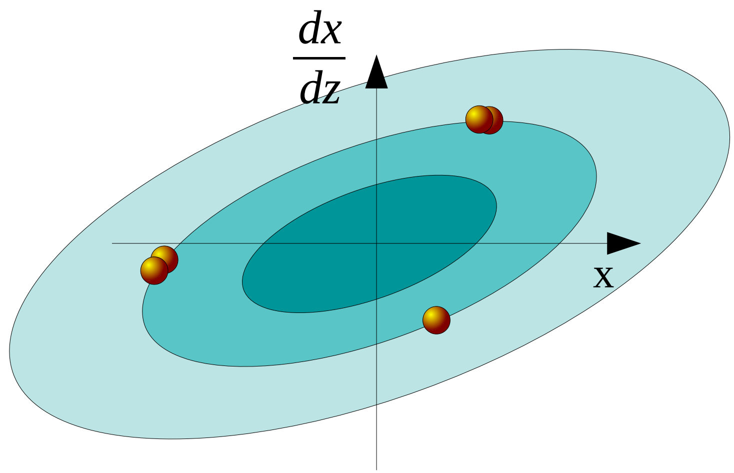
Resonances



- Reminder:-
  - Tune ν is number of SHM oscillations per turn
  - Phase advance $\mu = \pi\nu$ is “angle” advanced per turn
- The beam does not behave well when

$$\nu_x = l + \frac{m}{n}$$

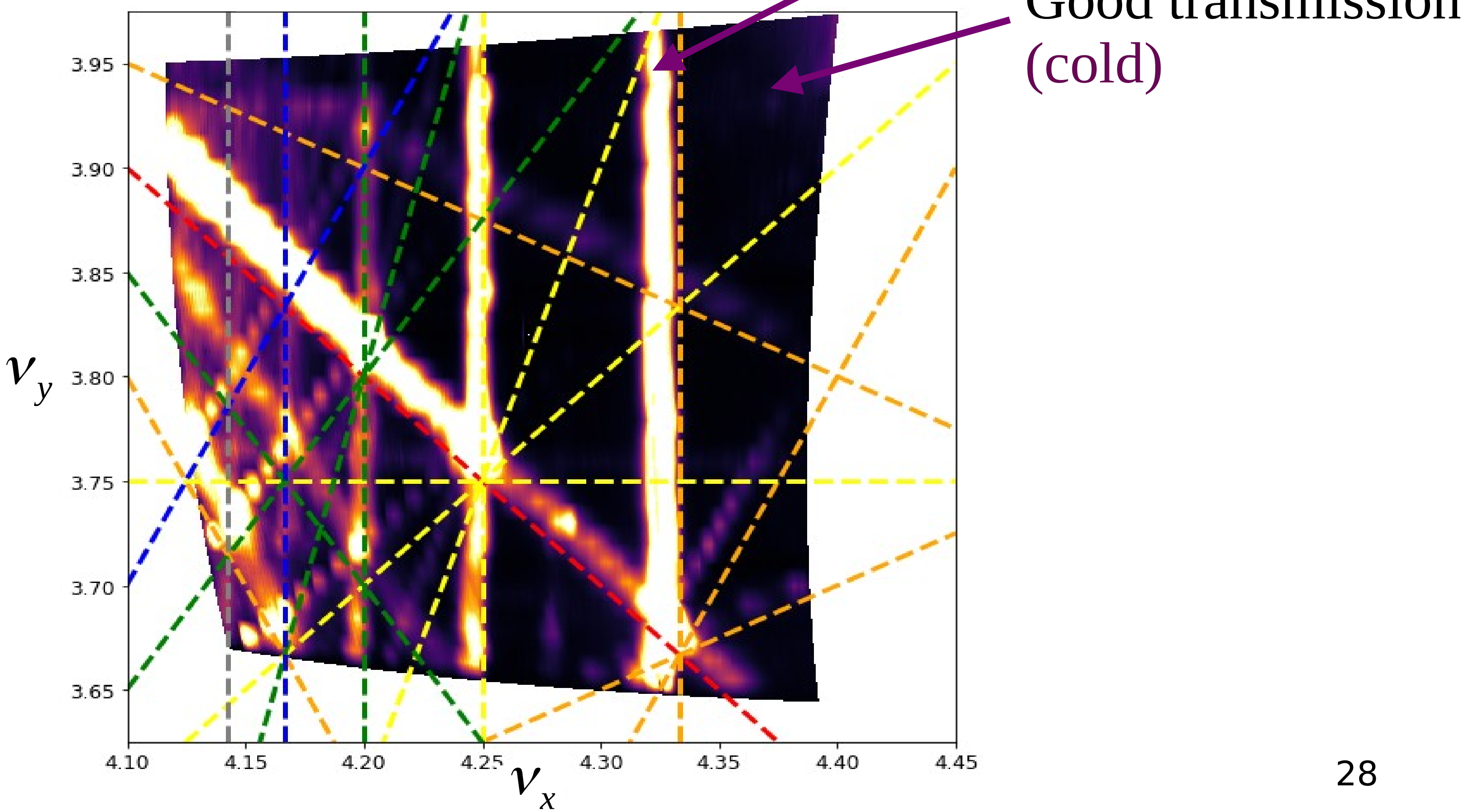
integer

  - Beam passes through the same field region every nth turn
  - Imperfections in the field get amplified
- Resonance

Resonances

- Can see poor performance for $\nu_x = 4 + \frac{1}{3} = 4.33$
- Only a very small area in phase space is transmitted
- In fact, a 2D phenomenon in (ν_x, ν_y)

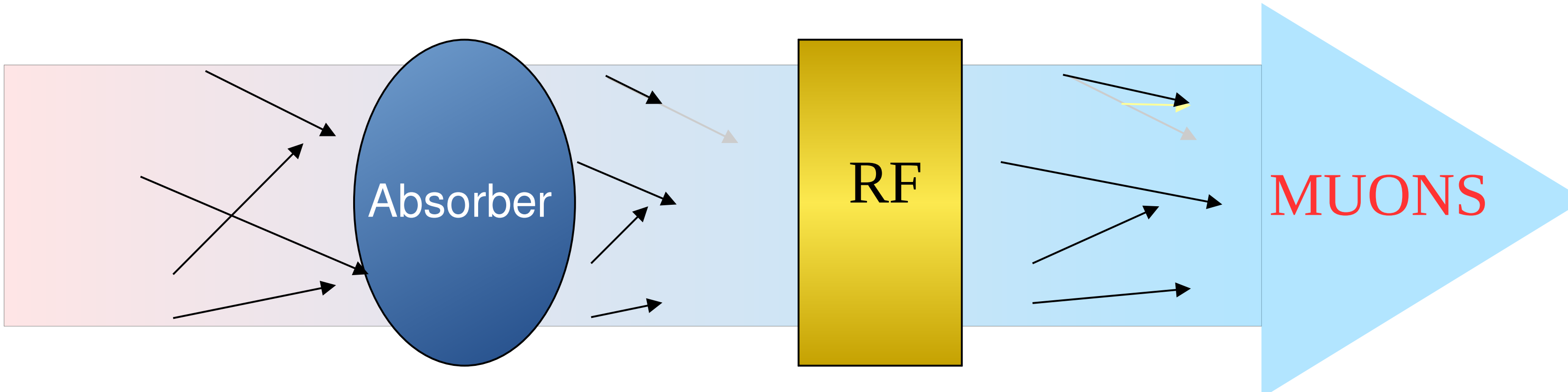
ISIS Synchrotron (measured)



Emittance Reduction (Cooling)

- Several techniques to reduce emittance
  - Synchrotron radiation cooling
  - Stochastic cooling
  - Laser cooling
  - Electron cooling
  - Ionisation cooling
- Fundamental principle is to remove “heat” from the beam using a neighbouring heat sink
  - Comoving electron beam → electron cooling
  - Comoving laser → laser cooling
  - Emission of synchrotron radiation
    - Photon emission caused by (principally) electrons bending in magnetic field

E.g. Ionisation Cooling



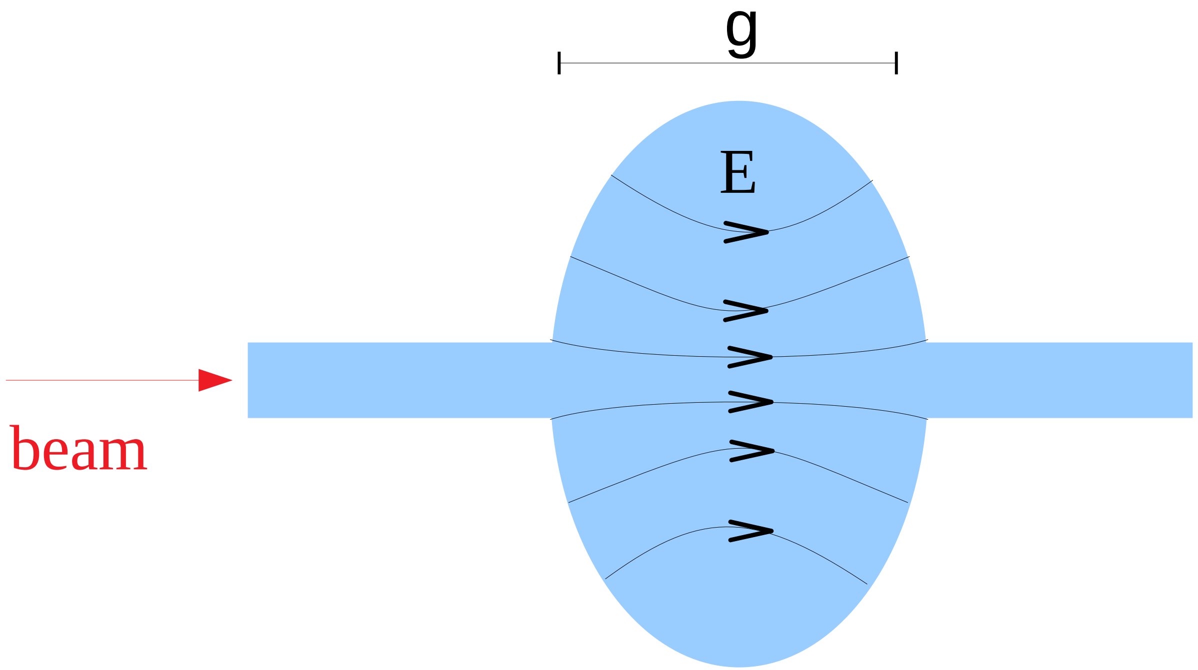
- Beam loses energy in absorbing material
  - Absorber removes momentum in all directions
  - RF cavity replaces momentum only in longitudinal direction
  - End up with beam that is more straight
- Multiple Coulomb scattering from nucleus ruins the effect
  - Mitigate with tight focussing
  - Mitigate with low-Z materials
  - Equilibrium emittance where MCS completely cancels the cooling

Longitudinal Dynamics and Acceleration

Longitudinal Dynamics

- So much for transverse motion (i.e. *x* and *y* planes)
- What about energy and acceleration?
- Electrostatic acceleration limited by breakdown potential
  - Change in energy is given by voltage differential
  - High voltage differentials cause breakdown (sparks)
  - Practically limits electrostatic acceleration to few MeV
- To accelerate beyond MeV require oscillating electric field
- RF Cavities

RF cavity field

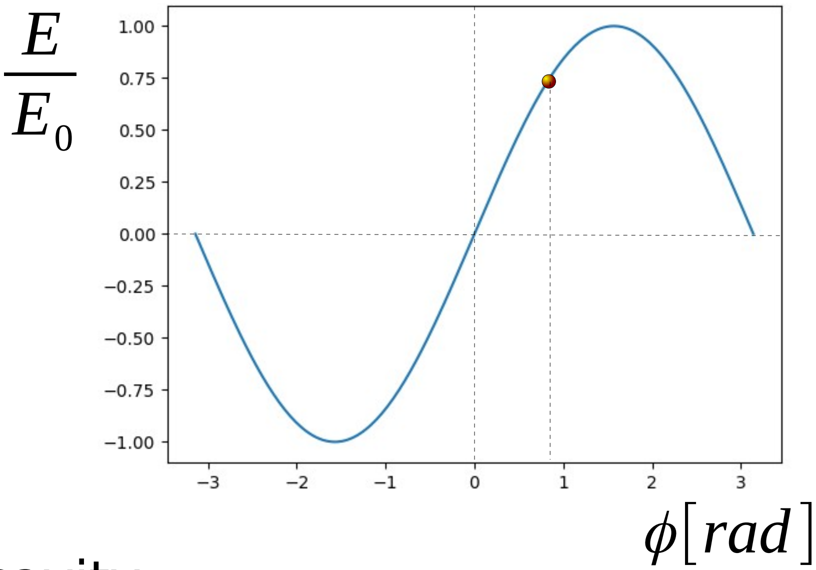


- RF cavity holds a resonating EM wave
- Recall Lorentz force law

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$$

- Force is in direction of motion - energy changes!

RF cavity field



- In RF cavity

$$\vec{E} = E_0 \sin(\omega t + \phi)\,\vec{\hat{z}}$$

- Energy change of particle crossing at ϕ

$$\delta W = \int F\, dz = q\, g\, E_0 \sin(\phi)$$

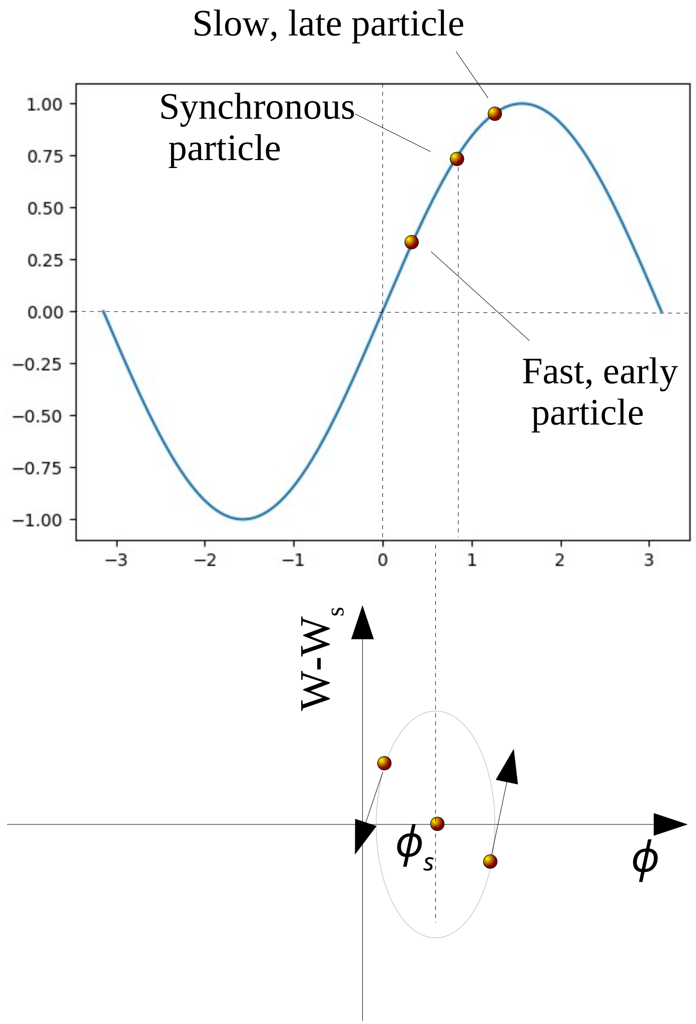
  - g is the gap length
  - Assumes gap is short so electric field doesn't change much
    - For longer gaps, can introduce an effective gap length *g T*
    - *T* is the "transit time factor" → reduces the effective gap length

Phase stability

- Phase cavities so that a “synchronous” particle always crosses at phase ϕ_s
- Particle crossing at phase ϕ relative to synchronous particle

$$\delta W = q\,g\,T\,E_0 \sin(\phi + \phi_s)$$

- Particle arriving early
  - Fast
  - t negative
  - Gets smaller energy kick
  - Ends up relatively slower
- Particle arriving late
  - Slow
  - t positive
  - Gets bigger energy kick
  - Ends up relatively faster
- Phase stability!



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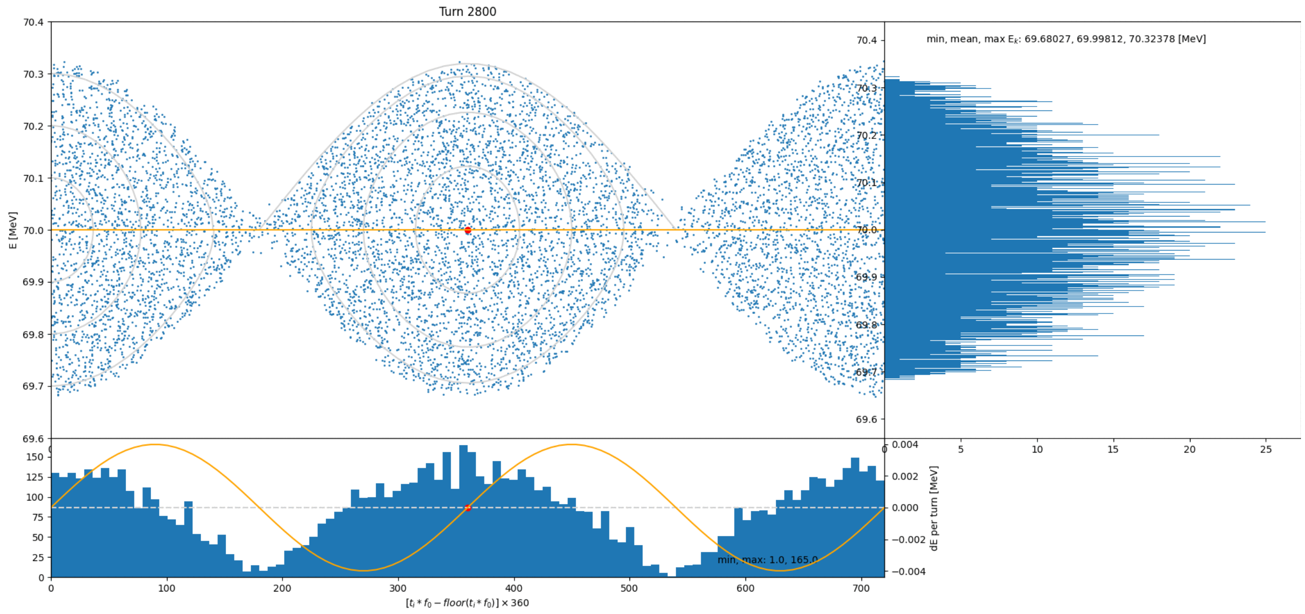
RF Bucket



- RF Bucket – region where phase is stable
- Harmonic number – number of RF oscillations per ring turn

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Dealing with momentum spread

- Momentum spread introduces a few effects
  - Dispersion
  - Chromaticity
  - Momentum compaction
- Dispersion:
  - Off-momentum particles follow a different trajectory
- Momentum compaction (rings):
  - Different path length yields different time of flight
- Chromaticity:
  - Off-momentum particles get a different focussing strength

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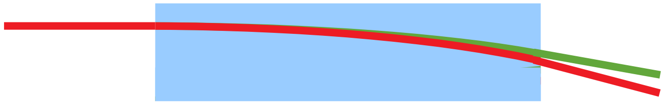
Dispersion



- Recall the definition of magnetic rigidity

$$B\rho = \frac{p}{q}$$

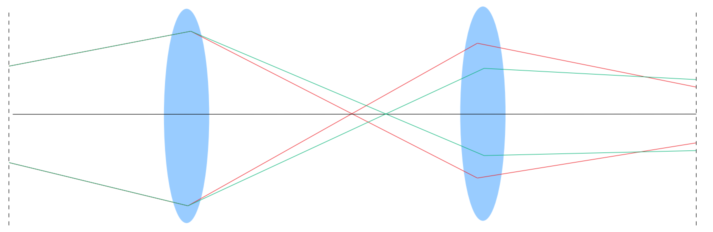
- Particles having different momentum (p) get different radius of curvature
  - Introduce dispersion *D*

$$D = p\frac{dx}{dp}$$

  - Which is another optical function that we must make periodic

Chromaticity

- Chromaticity arises because quadrupoles focus differently for different momenta



$$k = q\frac{b_0}{p}$$

- This often limits the degree of focussing at a collision point
  - Limits luminosity
- Can deliberately enhance/reduce chromaticity by
  - Introduce a dispersion
  - Using a magnet with variable focussing strength across the aperture - “sextupole”

Questions

Review

- Dipoles are used to bend a beam – rigidity is $B\rho = \frac{p}{q}$
- Quadrupoles are used to focus a beam: $k = q\frac{b_0}{p}$
- Beam in each of *x* and y can be characterised by 3 Twiss parameters and an emittance
- Lattices can be characterised by a phase advance
- RF cavities are used to accelerate the beam
- Introducing momentum spread, one can also define a dispersion (and its derivative with respect to z)

Finally… luminosity

- Luminosity defines the number of interactions in a collider per unit time for a given cross section
- Luminosity will increase if
  - Beam is narrower
  - Current is higher

$$\tilde{L} = \frac{N_1 N_2 f N_b}{4\pi \sigma_x \sigma_y}$$

Number of particles in each bunch ($N_1 N_2$)

Revolution frequency (f)

Number of bunches (N_b)

Width of Each bunch ($\sigma_x \sigma_y$)

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What dictates luminosity?

$$\tilde{L} = \frac{N_1 N_2 f N_b}{4\pi\sigma_x \sigma_y}$$

- Typically
  - Number of particles → space charge
  - Revolution frequency → ring circumference
  - Number of bunches → RF frequency
  - Beam width → $\sqrt{\varepsilon\beta}$
    - Emittance (cooling?)
    - Twiss beta (final focus and chromaticity)

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Next lecture...

- Accelerator equipment
- Types of accelerator
- Current facilities
- Future facilities

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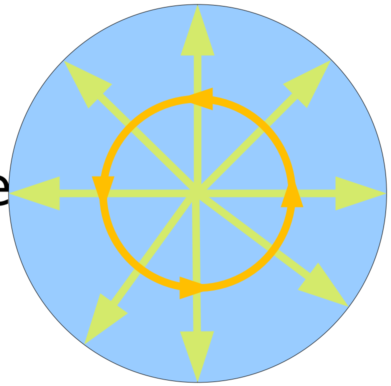
Backup

Transverse Space Charge 1

- Consider a circular beam of radius *a* having uniform density

$$\rho(r) = q\frac{I}{\beta_{rel}\, c\, \pi a^2} \qquad \boxed{r < a}$$



- Quote field around a cylinder of charge/curre

$$E(r) = \frac{1}{2\pi\varepsilon_0}\frac{I}{\beta_{rel}\, c}\frac{r}{a^2}$$

$$B_\phi(r) = \frac{1}{2\pi\varepsilon_0}\frac{I}{c^2}\frac{r}{a^2}$$

- Apply Lorentz force law

$$\vec{F} = q\vec{v}\times\vec{B} + q\vec{E}$$

$$F_r = q\vec{v}\times\vec{B} + q\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{r}{a^2}\left(\frac{I}{\beta_{rel}\, c} - \frac{I}{\beta_{rel}\, c}\beta_{rel}^2\right) = \frac{1}{2\pi\varepsilon_0}\frac{r}{a^2}\left(\frac{I}{\gamma^2\beta_{rel}\, c}\right)$$

Transverse Space Charge 2

- Force is defocusing

$$\frac{d^2x}{dz^2} - (k - K_{sc})x = 0 \quad \text{with} \quad K_{sc} = \frac{1}{2\pi\varepsilon_0}\frac{1}{a^2}\left(\frac{I}{\gamma_{rel}^3\beta_{rel}^2 c}\right)$$

- Treat SC as a perturbation

$$\boldsymbol{M}_p = \boldsymbol{M}\,\boldsymbol{M}_{sc}$$

$$\boldsymbol{M} = \boldsymbol{I}\cos\mu + \boldsymbol{J}\sin\mu$$

$$\boldsymbol{M}_{SC} = \begin{pmatrix} 1 & 0 \\ -K_{SC} & 1 \end{pmatrix}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

- Change of beam size (β)
- **Change of phase advance**
  - Drive the beam onto resonances → ruin the acceptance
- Phase advance → look at Trace of M_p

$$Tr(\boldsymbol{M}_p) = 2\cos(\mu) + \alpha\sin(\mu) - \alpha\sin(\mu) + \beta K\sin(\mu)$$

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Transverse Space Charge 3

- Consider just the *trace(**M**$_p$)*

$$Tr\left(\boldsymbol{M}_p\right) = 2\cos\left(\mu\right) + \beta K \sin\left(\mu\right)$$

- Consider compound angle formula

$$\cos\left(\mu+\delta\mu\right) = \cos\left(\mu\right)\cos\left(\delta\mu\right) + \sin\left(\mu\right)\sin\left(\delta\mu\right)$$

$$\cos\left(\mu+\delta\mu\right) \simeq \cos\left(\mu\right) + \sin\left(\mu\right)\sin\left(\delta\mu\right)$$

- Looking at the tune

$$\delta\nu = \frac{\delta\mu}{2\pi} = \frac{\beta K}{4\pi}$$

$$K_{sc} = \frac{1}{2\pi\varepsilon_0}\frac{1}{a^2}\left(\frac{I}{\gamma_{rel}^3\beta_{rel}^2 c}\right)$$

$$\delta\nu = \frac{r_0 N}{2\pi\varepsilon\beta_{rel}^2\gamma_{rel}^3}$$

$$\sigma\left(x\right) = \sqrt{\beta\epsilon}$$

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